The complex quadric

from the standpoint of Riemannian geometry

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Vorwort

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Contents

Chapter 0

Introduction and Preliminaries

0.1 Abstract

The complex hypersurfaces of a complex projective space $\mathbb{P}^n$ which are of least complexity (apart from the projective subspaces, whose geometry is known completely) are those which are determined by a non-degenerate quadratic equation, the *complex quadrics*. From the algebraic point of view, all complex quadrics are equivalent. However, if one regards $\mathbb{P}^n$ as a Riemannian manifold (with the Fubini-Study metric), it turns out that only certain complex quadrics are adapted to the Riemannian metric of $\mathbb{P}^n$ in the sense that they are symmetric submanifolds of $\mathbb{P}^n$. These quadrics are also singled out by the fact that they are (again apart from the projective subspaces) the only complex hypersurfaces in $\mathbb{P}^n$ which are Einstein manifolds (see Smyth, [Smy67]). In the sequel the term "complex quadric" always refers to these adapted complex quadrics.

While the algebraic behaviour of complex quadrics Q is well-known, there still remains a lot to be said about their intrinsic and extrinsic Riemannian geometry; the present dissertation provides a contribution to this subject. Specifically, the following results are obtained:

- The classification of the totally geodesic submanifolds of Q.
- The investigation of certain congruence families of totally geodesic submanifolds in Q; these families are equipped with the structure of a naturally reductive homogeneous space in a general setting, and it is investigated in which cases this structure is induced by a symmetric structure.
- It is shown that the set of the k-dimensional "subquadrics" contained in Q (which are all isometric to one another) is composed of a one-parameter-series of congruence classes; moreover the extrinsic geometry of these subquadrics is studied.
- It is well known that the following isomorphies hold between complex quadrics of low dimension and members of other series of Riemannian symmetric spaces:

  $$Q^1 \cong \mathbb{S}^2, \quad Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1, \quad Q^3 \cong \mathrm{Sp}(2)/\mathrm{U}(2), \quad Q^4 \cong G_2(\mathbb{C}^4) \quad \text{and} \quad Q^6 \cong \mathrm{SO}(8)/\mathrm{U}(4)\ .$$

  These isomorphisms are constructed explicitly in a rather geometric way.

In what follows I describe the strategies involved in obtaining these results, and discuss the results in more detail.

In the study of the geometry of any Riemannian manifold its curvature tensor plays a significant role. This is, for example, apparent from the fact that, at least in the case of the curvature tensor being parallel, it already contains all information about the local structure of the Riemannian manifold concerned (as the local version of the theorem of Cartan/Ambrose/Hicks shows). Another reason is that the curvature tensor induces an additional structure on the tangent spaces of the manifold, which is of interest in particular for the submanifold geometry of the manifold. For this reason, the algebraic structure of the curvature tensor is of importance for the study of the geometry of the manifold.

This idea is carried out for the complex quadric in the paper [Rec95] by Prof. H. Reckziegel, which was the starting point for the present dissertation. The chapters 1–3 are (with the exception of Section 3.4) an extended, more detailed exposition of the cited paper.

The following concept, which was introduced in [Rec95], is fundamental throughout the dissertation: Let $\mathbb{V}$ be a unitary space and A a conjugation[1] on $\mathbb{V}$. Following [Rec95], we then call the "circle of conjugations" $\mathfrak{A} := \{\, \lambda A \,|\, \lambda \in \mathbb{S}^1 \,\}$ a $\mathbb{C}$Q-structure and the pair $(\mathbb{V}, \mathfrak{A})$ a $\mathbb{C}$Q-space.

There are two causes for the great importance of the concept of a $\mathbb{C}$Q-structure for the study of complex quadrics. One cause is that the set of $\mathbb{C}$Q-structures on a unitary space $\mathbb{V}$ is in one-to-one correspondence with the set of complex quadrics in $\mathbb{P}(\mathbb{V})$ which are adapted to the metric of $\mathbb{P}(\mathbb{V})$ (in the sense explained above).

The second, even more fundamental cause is derived from the following result, which is already of central importance in [Rec95]: For a complex quadric $Q \subset \mathbb{P}(\mathbb{V})$ and $p \in Q$ we denote by $\perp^1_p Q$ the set of unit normal vectors to Q at p, and for $\eta \in \perp^1_p Q$ by A_η the shape operator of Q with respect to η. Then the set $\mathfrak{A}(Q,p) := \{\, A_\eta \,|\, \eta \in \perp^1_p Q \,\}$ is a $\mathbb{C}$Q-structure on the tangent space T_pQ. As the Gauss equation of second order shows, the curvature tensor of Q at p can be described via this $\mathbb{C}$Q-structure $\mathfrak{A}(Q,p)$ (and the Riemannian metric and complex structure of Q). Therefore the $\mathbb{C}$Q-spaces $(T_pQ, \mathfrak{A}(Q,p))_{p \in Q}$ describe the local information on the complex quadric in totality, and thus it appears to be reasonable to regard the Riemannian metric of Q, the complex structure of Q and the family $(\mathfrak{A}(Q,p))_{p \in Q}$ of $\mathbb{C}$Q-structures as the "fundamental geometric objects" of the complex quadric Q. This point of view had a formative influence on the present dissertation.

Two $\mathbb{C}$Q-spaces of the same dimension are isomorphic to each other. For this reason much information about the two situations described above can be obtained by the abstract study of $\mathbb{C}$Q-spaces. Such studies are carried out in Chapter 2 of the dissertation. Two of the facts obtained there are of particular importance for the further use of $\mathbb{C}$Q-spaces:

[1] Suppose that $\mathbb{V}$ is a unitary space, whose complex structure we denote by $J : \mathbb{V} \to \mathbb{V},\ v \mapsto i \cdot v$ and whose complex inner product we denote by $\langle \cdot, \cdot \rangle_{\mathbb{C}}$. Then an $\mathbb{R}$-linear map $A : \mathbb{V} \to \mathbb{V}$ is called a *conjugation* on $\mathbb{V}$, if it is self-adjoint and orthogonal with respect to the real inner product $\mathrm{Re}(\langle \cdot, \cdot \rangle_{\mathbb{C}})$, and moreover $A \circ J = -J \circ A$ holds.

(1) The group $\mathrm{Aut}(\mathfrak{A})$ of the $\mathbb{C}$Q-automorphisms of $\mathbb{V}$ (i.e. of those unitary transformations $B : \mathbb{V} \to \mathbb{V}$ for which $B \circ A \circ B^{-1} \in \mathfrak{A}$ holds for every $A \in \mathfrak{A}$) does not act transitively on the unit sphere $\mathbb{S}(\mathbb{V})$ (thus we see that in a $\mathbb{C}$Q-space, unlike in a unitary space, not all unit vectors are geometrically equivalent). More specifically, there exists a surjective, continuous function $\varphi_{\mathfrak{A}} : \mathbb{S}(\mathbb{V}) \to [0, \frac{\pi}{4}]$, which is submersive on $\varphi_{\mathfrak{A}}^{-1}(]0, \frac{\pi}{4}[)$, so that the orbits of the action of $\mathrm{Aut}(\mathfrak{A})$ on $\mathbb{S}(\mathbb{V})$ are exactly the niveau surfaces of $\varphi_{\mathfrak{A}}$. This fact is already found in [Rec95], however the simple description of $\varphi_{\mathfrak{A}}$ via the equation $2\cos(\varphi_{\mathfrak{A}}(v)) = |\langle v, Av\rangle_{\mathbb{C}}|$ with an arbitrary $A \in \mathfrak{A}$ (see Theorem 2.28(a)) is new.

(2) As it has already been said above, the curvature tensor of a complex quadric Q at $p \in Q$ can be described via the quantities of the $\mathbb{C}$Q-space $(T_pQ, \mathfrak{A}(Q,p))$ alone. For this reason, one can introduce a tensor which corresponds to the curvature tensor of Q on any $\mathbb{C}$Q-space $(\mathbb{V}, \mathfrak{A})$; we call this tensor the curvature tensor R of the $\mathbb{C}$Q-space. We describe the eigenspaces and eigenvalues of the Jacobi operator $R(\,\cdot\,, w)w : \mathbb{V} \to \mathbb{V}$ and the R-flat subspaces of $\mathbb{V}$. These data, which are already found in [Rec95], are of great importance for the study of the complex quadric as a symmetric space.

In Chapter 3, the results about $\mathbb{C}$Q-spaces are applied to complex quadrics. Section 3.1 shows in what way $\mathbb{C}$Q-(anti-)isomorphisms of a $\mathbb{C}$Q-space $(\mathbb{V}, \mathfrak{A})$ give rise to (anti-)holomorphic isometries of the complex quadric $Q(\mathfrak{A}) \subset \mathbb{P}(\mathbb{V})$ defined by the $\mathbb{C}$Q-structure $\mathfrak{A}$. The basic result already found in [Rec95] is here enhanced by a description of the "mobility" of bases in T_pQ in terms of the $\mathbb{C}$Q-theory (Theorem 3.5). Therefrom also the well-known fact that an m-dimensional complex quadric is a Hermitian symmetric space isomorphic to $\mathrm{SO}(m+2)/(\mathrm{SO}(2) \times \mathrm{SO}(m))$ follows; moreover the splitting $\mathfrak{o}(m+2) = \mathfrak{k} \oplus \mathfrak{m}$ induced by the symmetric structure is described explicitly. The information concerning the curvature tensor from Sections 2.7 and 2.8 can now be interpreted as a description of the Cartan subalgebras, the roots and the root spaces of the symmetric space Q; this viewpoint is here used in a much stronger way than in [Rec95]. Whereas the structure of the root system of Q is of course well-known, the present explicit description of the Cartan subalgebras and the root spaces in terms of the $\mathbb{C}$Q-space $(T_pQ, \mathfrak{A}(Q,p))$ alone (without use of any "artificial coordinates") cannot be found elsewhere, and is fundamental for the following investigations.

The results described up to this point constitute the fundament of the present investigation of the geometry of complex quadrics.

As a first application, the isometries of the complex quadric Q are classified in Section 3.3. Although the main result on this topic, that (a) every (anti-)holomorphic isometry $f : Q \to Q$ is induced by a $\mathbb{C}$Q-(anti-)automorphism, and that (b) for $\dim Q \neq 2$, every isometry $f : Q \to Q$ is either holomorphic or anti-holomorphic (Theorem 3.23), is already found in [Rec95], I can here provide a far shorter proof based on the fact that for every isometry $f : Q \to Q$ and every $p \in Q$ we have $\varphi_{\mathfrak{A}(Q,f(p))} \circ (f_*|\mathbb{S}(T_pQ)) = \varphi_{\mathfrak{A}(Q,p)}$ (as follows from the equivariance of the curvature operator under f_*).

The subject of the Chapters 4 and 5 is the classification of the totally geodesic submanifolds of the complex quadric Q.

Already Chen and Nagano were concerned with the classification of the totally geodesic submanifolds in symmetric spaces in their papers [CN77] and [CN78]. The paper [CN77] gives a classification of the totally geodesic submanifolds of the complex quadric by “ad hoc methods”. However, it contains several faults, which cause two types of totally geodesic submanifolds to be missed. Also in other regards, not all arguments in [CN77] are convincing. — While the paper [CN77] studied the complex quadric exclusively, the (M_+, M_-)-method introduced in [CN78] pertains to finding totally geodesic submanifolds in general Riemannian symmetric spaces of compact type. However, it is only a necessary criterion for the existence of totally geodesic embeddings of one symmetric space into another. Thus the (M_+, M_-)-method provides neither proofs for the existence of totally geodesic submanifolds in a symmetric space nor information about their position. Therefore the cited papers do not give a satisfactory investigation of the totally geodesic submanifolds of the complex quadric, and I also do not know of a treatment of the problem elsewhere.

For a more detailed discussion of the papers [CN77] and [CN78], and of the older paper [CL75] by Chen and Lue concerning the real-2-dimensional totally geodesic submanifolds of Q, refer to Remark 4.13.

In the classification of the totally geodesic submanifolds of Q performed in this dissertation, I use neither the methods of [CN77] nor the (M_+, M_-)-method. Rather I proceed as follows: As is well-known, the connected, complete, totally geodesic submanifolds of the symmetric space Q are exactly its symmetric subspaces, and the symmetric subspaces of Q running through some point $p \in Q$ are in bijective correspondence with the curvature-invariant subspaces of the tangent space T_pQ. Therefore, the problem of classifying the totally geodesic submanifolds of Q decomposes into two subproblems: (1) The classification of the curvature-invariant subspaces of T_pQ and (2) The description of the global isometry type and of the position in Q of the totally geodesic submanifolds of Q corresponding to the curvature-invariant subspaces found in the solution of the first subproblem.

The solution of subproblem (1) is based on the combination of the root space theory of symmetric spaces with the specific description of the roots and root spaces of the complex quadric obtained via the theory of $\mathbb{C}$Q-spaces. First, in Section 4.2 I derive relations between the roots resp. root spaces of a general symmetric space M of compact type and the roots resp. root spaces of its symmetric subspaces. Via the explicit description of the roots and root spaces of Q one obtains conditions for the possible position of curvature-invariant subspaces in T_pQ from these relations. These conditions permit a classification of the curvature-invariant subspaces, which is carried out in Sections 4.3 and 4.4. The proof of the classification is simplified and structured by the use of symmetry properties of the root systems; the use of these symmetry properties has been suggested by Prof. J.-H. Eschenburg (Augsburg).

Subproblem (2) is tackled in Chapter 5: For every curvature-invariant subspace $U \subset T_pQ$ found in Chapter 4 (with exception of one specific congruence type of 2-dimensional subspaces), a totally geodesic, injective, isometric immersion into Q is described whose image runs tangential to U. This completes the classification of the totally geodesic submanifolds of Q.

In particular, we obtain the following totally geodesic submanifolds in an m-dimensional complex quadric $Q \subset \mathbb{P}(\mathbb{V})$ (for a complete list, see Theorem 5.1): (1) For every $k < m$ there exist totally geodesic submanifolds Q' of Q which are isometric to a k-dimensional complex quadric. They are "subquadrics" of Q in the sense that for each such Q' there exists a complex-$(k+1)$-dimensional projective subspace $\Lambda \subset \mathbb{P}(\mathbb{V})$ so that Q' is a complex quadric in Λ in the previous sense. (2) For every $k \leq \frac{m}{2}$ there exist complex-k-dimensional projective subspaces of $\mathbb{P}(\mathbb{V})$ which are entirely contained in Q and therefore totally geodesic submanifolds of Q. (3) For $m \geq 3$ there are totally geodesic submanifolds of Q which are isometric to a 2-sphere of radius $\frac{1}{2}\sqrt{10}$; these submanifolds are neither complex nor totally real. Their diameter $\frac{\pi}{2}\sqrt{10}$ is strictly larger than the diameter $\frac{\pi}{\sqrt{2}}$ of the ambient quadric Q.

The question arises whether there are other k-dimensional subquadrics of Q besides the totally geodesic ones mentioned in (1). As I show in Chapter 6, this question is to be answered in the positive for $k \leq \frac{m}{2} - 1$. For these k there exist infinitely many congruence classes of k-dimensional subquadrics of Q, the set of these congruence classes is parametrized by an "angle" $t \in [0, \frac{\pi}{4}]$ (which is related strongly to the function $\varphi_{\mathfrak{A}} : \mathbb{S}(\mathbb{V}) \to [0, \frac{\pi}{4}]$), and a subquadric Q' of Q is a totally geodesic submanifold if and only if it belongs to the congruence class with $t = 0$. I also show that the inclusion $Q' \hookrightarrow Q$ has parallel second fundamental form if and only if Q' belongs either to the congruence class with $t = 0$ or to the congruence class with $t = \frac{\pi}{4}$. The members of the latter congruence class are exactly those subquadrics of Q whose ambient projective space $\Lambda \subset \mathbb{P}(\mathbb{V})$ is entirely contained in Q.

If Q' is a subquadric of Q belonging to the congruence class with the parameter $t \in [0, \frac{\pi}{4}]$, then this entire congruence class is by definition given by $\{\, f(Q'_t) \,|\, f \in I(Q) \,\}$, where $I(Q)$ denotes the isometry group of Q. In the general setting, where M is any Riemannian symmetric space and N_0 a submanifold of M, I call the set $\mathfrak{F}(N_0, M) := \{\, f(N_0) \,|\, f \in I(M) \,\}$ the "family of congruent submanifolds" or the "congruence family" induced by N_0 in M. I carried out the study of such congruence families, here found in Chapter 7, after Prof. M. Rapoport (Bonn) indicated the investigation of the projective subspaces in a complex quadric found in [GH78], p. 735f. to me. However, [GH78] is not concerned with the metric point of view, on which the present study is focused. My results on this subject have now been published as [KR05].

In Section 7.1 I first show in a general setting how to equip congruence families with the structure of a Riemannian manifold in such a way that it becomes a naturally reductive Riemannian homogeneous space. The remainder of Chapter 7 is concerned with the study of specific examples of congruence families. In Section 7.2 I study two examples in the complex projective space $\mathbb{P}(\mathbb{V})$: the congruence family induced by a projective subspace and the congruence family induced by a k-dimensional complex quadric. In Section 7.3 I study two examples in a complex quadric $Q \subset \mathbb{P}(\mathbb{V})$: the congruence family induced by a totally geodesic subquadric, and the congruence family induced by a projective subspace of dimension $\leq \frac{m}{2}$ contained entirely in Q. (The latter congruence family is the one considered in [GH78].) It turns out that in some, but not all of the cases considered the reductive structure of the congruence family is induced by a symmetric structure. For example, in the case of the congruence family $\mathfrak{F}(\mathbb{P}^k, Q)$ induced by a k-dimensional projective subspace (with $k \leq \frac{m}{2}$) contained in the quadric Q the following result holds true (see Theorem 7.11): If $2k = m$ holds, then $\mathfrak{F}(\mathbb{P}^k, Q)$ has exactly two connected components, which can be equipped with the structure of a Hermitian symmetric space isomorphic to $\mathrm{SO}(m+2)/\mathrm{U}(k+1)$ in such a way that the symmetric structure induces the original naturally reductive structure. On the other hand, if $2k < m$ holds, then $\mathfrak{F}(\mathbb{P}^k, Q)$ is connected, and the naturally reductive structure of $\mathfrak{F}(\mathbb{P}^k, Q)$ is not induced by a symmetric structure.

As was first noted by E. CARTAN and as is well-known, the complex quadrics Q^m of dimension $m \in \{1, 2, 3, 4, 6\}$ (and no others) are as Riemannian symmetric spaces isomorphic to members of other series of Riemannian symmetric spaces (see also [Hel78], p. 519f.). It can be read off the Dynkin diagrams of the irreducible symmetric spaces (see [Loo69], Theorem VII.3.9(a), p. 145 and Table 4 on p. 119) that the following isomorphies hold:

$$Q^1 \cong \mathbb{S}^2, \quad Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1, \quad Q^3 \cong \mathrm{Sp}(2)/\mathrm{U}(2), \quad Q^4 \cong G_2(\mathbb{C}^4) \quad \text{and} \quad Q^6 \cong \mathrm{SO}(8)/\mathrm{U}(4)\ .$$

(It follows from the fact that all the mentioned spaces are simply connected that the isomorphies given are indeed global.) This consideration does not provide a method for the construction of isomorphisms between the respective spaces. However, in the dissertation (Section 3.4 and Chapter 8), I am successful in constructing these isomorphisms explicitly in a rather geometric way: The Segre embedding gives rise to an isomorphism between Q^2 and $\mathbb{P}^1 \times \mathbb{P}^1$; in particular Q^2 is (unlike the complex quadrics of every other dimension) reducible. — The Plücker embedding provides an isomorphism between the complex Grassmannian $G_2(\mathbb{C}^4)$ and a 4-dimensional complex quadric $Q(*) \subset \mathbb{P}(\bigwedge^2\mathbb{C}^4)$; here the quadric $Q(*)$ is described by the Hodge star operator $* : \bigwedge^2\mathbb{C}^4 \to \bigwedge^2\mathbb{C}^4$. — By restricting the mentioned isomorphism $G_2(\mathbb{C}^4) \to Q(*)$ to a suitable, totally geodesic $\mathrm{Sp}(2)$-orbit in $G_2(\mathbb{C}^4)$, one obtains an isomorphism between the Hermitian symmetric space $\mathrm{Sp}(2)/\mathrm{U}(2)$ and a 3-dimensional, totally geodesic subquadric of $Q(*)$. — Via the theory of spin groups, their representations and the principle of triality I can construct an isomorphism between Q^6 and the Hermitian symmetric space $\mathrm{SO}(8)/\mathrm{U}(4)$. The latter space has several geometric realizations. For example, it is isomorphic to each of the two connected components of the congruence family $\mathfrak{F}(\mathbb{P}^3, Q^6)$ of the 3-dimensional projective subspaces contained in Q^6; this fact is used in the construction of the isomorphism $Q^6 \to \mathrm{SO}(8)/\mathrm{U}(4)$. A more well-known geometric realization of $\mathrm{SO}(8)/\mathrm{U}(4)$ is as the space of orthogonal complex structures on $\mathbb{R}^8$ with a fixed orientation, and this realization can be used to establish the mentioned isomorphism between $\mathrm{SO}(8)/\mathrm{U}(4)$ and the connected components of $\mathfrak{F}(\mathbb{P}^3, Q^6)$.

It should be mentioned that we were first pointed to the existence of the isomorphy $Q^4 \cong G_2(\mathbb{C}^4)$ by Prof. M. GUEST (Metropolitan University of Tokyo). The insights gained during the construction of this isomorphy were very fruitful also for the general understanding of complex quadrics.

The appendices contain mostly reproductive expositions of certain subjects which are of importance in the dissertation. The sources on which they are based are mentioned here and in the introduction of the respective appendix. Where appropriate we also give sources in the individual theorems and proofs.

Appendix A describes the aspects of the theory of symmetric spaces which are of importance here. For the point of view on the theory of symmetric spaces taken in Sections A.1, A.2 and

A.3, a non-published script by Prof. H. Reckziegel has furthered my understanding greatly; for the description of the root space theory for symmetric spaces in Section A.4, the script of a lecture by Prof. G. Thorbergsson has been of help.

The subject of Appendix B is the theory of Clifford algebras, spin groups, their representations and the principle of triality. These subjects play an important role in the construction of the isomorphism between Q^6 and the connected components of $\mathfrak{F}(\mathbb{P}^3, Q^6)$. The principal sources here were the book [LM89] by Lawson/Michelsohn (for Clifford algebras, spin groups and their representations), and the book [Che54] by Chevalley (for the principle of triality). Moreover, the discussions with Prof. H. Reckziegel on these subjects, which also gave rise to the script [Rec04], were very helpful.

0.2 Conventions and Notations

We describe the notations and conventions which are used throughout the dissertation.

Elementary objects.

symbol	meaning
$\mathbb{N}$	$\{1, 2, 3, \dots\}$ (natural numbers)
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{Z}$	ring of integers
$\delta_{k\ell} \quad (k, \ell \in \mathbb{Z})$	$\delta_{k\ell} = 1$ for $k = \ell$; $\delta_{k\ell} = 0$ for $k \neq \ell$ (Kronecker symbol)
$\mathfrak{S}_n$	permutation group of $\{1, \dots, n\}$
$\mathrm{sign}(\sigma) \quad (\sigma \in \mathfrak{S}_n)$	the signum of σ
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\mathbb{Q}^\times,\ \mathbb{R}^\times,\ \mathbb{C}^\times$	$\mathbb{Q} \setminus \{0\},\ \mathbb{R} \setminus \{0\},\ \mathbb{C} \setminus \{0\}$ (the multiplicative groups of these fields)
$\mathbb{R}_+,\ \mathbb{R}_-$	$\{\, t \in \mathbb{R} \,\vert\, t > 0 \,\},\ \{\, t \in \mathbb{R} \,\vert\, t < 0 \,\}$
i	$= (0, 1) \in \mathbb{C}$ (the imaginary unit of $\mathbb{C}$)
$\mathrm{Re}\, z,\ \mathrm{Im}\, z \quad (z \in \mathbb{C})$	the real resp. imaginary part of z
$\overline{z} \quad (z \in \mathbb{C})$	$= \mathrm{Re}\, z - i\, \mathrm{Im}\, z$ (the complex conjugate of z)
$\lvert z\rvert \quad (z \in \mathbb{C})$	absolute value of z
$\mathbb{S}^1$	$= \{\ z \in \mathbb{C} \ \vert\ \lvert z\rvert = 1\ \}$ (the unit circle)
$\mathrm{id}_M \quad (M$ a set$)$	the identity map $M \to M,\ p \mapsto p$
$M' \hookrightarrow M \quad (M' \subset M)$	the inclusion map $M' \to M,\ p \mapsto p$
$g \circ f \quad (f : L \to M,\ g : M \to N)$	the composition map $L \to N,\ p \mapsto g(f(p))$
$\mathrm{Fix}(f) \quad (f : M \to M)$	$= \{\, p \in M \,\vert\, f(p) = p \,\}$ (the fixed point set of f)
$\mathrm{Fix}(\mathfrak{F}) \quad (\mathfrak{F}$ a set of maps $M \to M)$	$= \bigcap_{f \in \mathfrak{F}} \mathrm{Fix}(f)$

Linear spaces. We will consider finite-dimensional *linear spaces* over the fields $\mathbb{R}$ and $\mathbb{C}$. Let V, W be linear spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$; let n be the dimension of V. In the case $\mathbb{K} = \mathbb{C}$ we call an $\mathbb{R}$-linear map $B : V \to V$ *anti-linear*, if it satisfies $B(\lambda v) = \overline{\lambda} \cdot Bv$ for every $v \in V$ and $\lambda \in \mathbb{C}$.

symbol	meaning
$\dim_{\mathbb{K}} V$, $\dim V$	the dimension of V
$\operatorname{span}_{\mathbb{K}} M$, $\operatorname{span} M$ ($M \subset V$ a subset)	the span of M in V
$V_1 \oplus V_2$ ($V_1, V_2 \subset V$ linear subspaces)	the direct sum of V_1 and V_2
$L^r(V, W)$ ($r \in \mathbb{N}$)	the space of r-linear maps $V \times \ldots \times V \to W$
$L(V, W)$	$= L^1(V, W)$ (space of linear maps $V \to W$)
V^*	$= L(V, \mathbb{K})$ (dual space of V)
$\operatorname{End}(V)$	$= L(V, V)$ (space of endomorphisms of V)
$\det(B)$ ($B \in \operatorname{End}(V)$)	the determinant of B
$\operatorname{tr}(B)$ ($B \in \operatorname{End}(V)$)	the trace of B
$\mathrm{GL}(V)$	$= \{ B \in \operatorname{End}(V) \mid \det(B) \neq 0 \}$ (general linear group)
$\mathrm{SL}(V)$	$= \{ B \in \operatorname{End}(V) \mid \det(B) = 1 \}$ (special linear group)
$\ker(B)$ ($B \in L(V, W)$)	the kernel of B
$\operatorname{Eig}(B, \lambda)$ ($B \in \operatorname{End}(V)$, $\lambda \in \mathbb{K}$)	$= \ker(B - \lambda \operatorname{id}_V)$ (if $\neq \{0\}$, this is an eigenspace of B)
$n(B, \lambda)$ ($B \in \operatorname{End}(V)$, $\lambda \in \mathbb{K}$)	$= \dim \operatorname{Eig}(B, \lambda)$ (the multiplicity of λ)
$\operatorname{Spec}(B)$ ($B \in \operatorname{End}(V)$)	$= \{ \lambda \in \mathbb{K} \mid n(B, \lambda) > 0 \}$ (the spectrum of B)
$\operatorname{Alt}^k(V)$ ($k \leq n$)	the space of alternating k-forms on V
$[v]$ ($v \in V \setminus \{0\}$)	the 1-dimensional subspace $\mathbb{K}v$ of V
$\mathbb{K}\mathrm{P}(V)$	$= \{ [v] \mid v \in V \setminus \{0\} \}$ (the projective space over V)
$\mathbb{P}(V)$	$= \mathbb{C}\mathrm{P}(V)$ in the case $\mathbb{K} = \mathbb{C}$
$\underline{B}$ ($B : V \to W$ linear isomorphism)	the (in the case $\mathbb{K} = \mathbb{C}$ holomorphic) diffeomorphism $\underline{B} : \mathbb{K}\mathrm{P}(V) \to \mathbb{K}\mathrm{P}(W)$ characterized by $\underline{B}([v]) = [Bv]$ for all $v \neq 0$
$\underline{A}$ ($\mathbb{K} = \mathbb{C}$, $A : V \to W$ anti-linear isomorphism)	the anti-holomorphic diffeomorphism $\underline{A} : \mathbb{C}\mathrm{P}(V) \to \mathbb{C}\mathrm{P}(W)$ characterized by $\underline{A}([v]) = [Av]$ for all $v \neq 0$
$G_k(V)$ ($k \leq n$)	the k-Grassmannian of V, i.e. the set of k-dimensional linear subspaces of V

For $\omega \in \operatorname{Alt}^n(V) \setminus \{0\}$ we call the equivalence class $[\omega] := \mathbb{R}_+ \cdot \omega \subset \operatorname{Alt}^n(V) \setminus \{0\}$ an *orientation* on V; we call V an *oriented linear space* if an orientation is fixed on V. If V is an oriented linear space, we call a basis $(b_1, \ldots, b_n)$ of V *positively oriented* if $\omega(b_1, \ldots, b_n) \in \mathbb{R}_+$ holds for some (and then for every) representative volume form ω of the orientation of V. It should be noted that we use this terminology even in the case where V is a complex linear space (in extension of the usual conventions).

We now suppose that V is a euclidean (for $\mathbb{K} = \mathbb{R}$) or unitary (for $\mathbb{K} = \mathbb{C}$) space; we denote its real resp. complex inner product by $\langle \cdot, \cdot \rangle$.

symbol	meaning
$V_1^{\perp, V}$, $V_1^{\perp}$ ($V_1 \subset V$ linear subspace)	ortho-complement of V_1 in V
$V_1 \obot V_2$ ($V_1, V_2 \subset V$ linear subspaces)	orthogonal direct sum of V_1 and V_2
$\mathrm{O}(V)$ ($\mathbb{K} = \mathbb{R}$)	$= \{ B \in \mathrm{GL}(V) \mid \forall v, w \in V : \langle Bv, Bw \rangle = \langle v, w \rangle \}$ (orthogonal group)
$\mathrm{SO}(V)$ ($\mathbb{K} = \mathbb{R}$)	$= \mathrm{O}(V) \cap \mathrm{SL}(V)$ (special orthogonal group)
$\mathrm{O}(n)$, $\mathrm{SO}(n)$	$\mathrm{O}(\mathbb{R}^n)$, $\mathrm{SO}(\mathbb{R}^n)$
$\mathrm{U}(V)$ ($\mathbb{K} = \mathbb{C}$)	$= \{ B \in \mathrm{GL}(V) \mid \forall v, w \in V : \langle Bv, Bw \rangle = \langle v, w \rangle \}$ (unitary group)
$\mathrm{SU}(V)$ ($\mathbb{K} = \mathbb{C}$)	$= \mathrm{SU}(V) \cap \mathrm{SL}(V)$ (special unitary group)

symbol	meaning
$\overline{\mathrm{U}}(V)$ $(\mathbb{K} = \mathbb{C})$	the set of maps $B : V \to V$ which are anti-unitary, i.e. which are anti-linear and orthogonal with respect to the real inner product $\mathrm{Re}(\langle\cdot,\cdot\rangle)$
$\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\overline{\mathrm{U}}(n)$	$\mathrm{U}(\mathbb{C}^n)$, $\mathrm{SU}(\mathbb{C}^n)$, $\overline{\mathrm{U}}(\mathbb{C}^n)$
$\mathrm{End}_+(V)$	$= \{ B \in \mathrm{End}(V) \,\vert\, \forall v, w \in V : \langle Bv, w\rangle = \langle v, Bw\rangle \}$
$\mathrm{End}_-(V)$	$= \{ B \in \mathrm{End}(V) \,\vert\, \forall v, w \in V : \langle Bv, w\rangle = -\langle v, Bw\rangle \}$
$\Vert v\Vert$ $(v \in V)$	$= \sqrt{\langle v, v\rangle}$ (the norm of v)
$\mathbb{S}_r(V)$ $(r \in \mathbb{R}_+)$	$= \{ v \in V \,\vert\, \Vert v\Vert = r \}$ (the sphere of radius r in V)
$\mathbb{S}(V)$	$= \mathbb{S}_1(V)$ (the unit sphere in V)
$\mathbb{S}^n_r$, $\mathbb{S}^n$	$\mathbb{S}_r(\mathbb{R}^{n+1})$, $\mathbb{S}(\mathbb{R}^{n+1})$
$\lambda^\sharp$ $(\lambda \in V^*)$	the Riesz vector of λ; $\lambda^\sharp \in V$ is characterized by $\lambda = \langle\cdot, \lambda^\sharp\rangle$
$\beta^\sharp$ $(\beta \in L^2(V, \mathbb{K}))$	the Riesz endomorphism of β; $\beta^\sharp : V \to V$ is characterized by $\beta(\cdot, w) = \langle\cdot, \beta^\sharp(w)\rangle$ for all $w \in V$.

Topological spaces. If X is a topological space, we denote the *topology* of X (i.e. the set of all open sets of X) by $\mathrm{Top}(X)$. For $p \in X$ we call an open set $U \in \mathrm{Top}(X)$ with $p \in U$ an *open neighbourhood* of p in X; $\mathfrak{U}^o(p, X) := \{ U \in \mathrm{Top}(X) \,\vert\, p \in U \}$ is the set of all open neighbourhoods of p in X.

Manifolds. All *manifolds* considered here are differentiable, Hausdorff, paracompact and without boundary; the term *differentiable* always means C^∞. We suppose all objects defined on manifolds (maps, tensor fields, etc.) to be differentiable, unless noted otherwise. Let M be a manifold. Then a subset $N \subset M$, which is equipped with the structure of a manifold in such a way that the inclusion map $N \hookrightarrow M$ is an immersion, is called a *submanifold* of M, see [Var74], p. 18. If additionally the intrinsic topology of N coincides with the topology inherited from M, we call N a *regular submanifold* of M. Differentiable maps $\alpha : J \to M$, where $J \subset \mathbb{R}$ is an interval, are called *curves*.

If M is a manifold and $p \in M$, we denote the *tangent space* of M in p by T_pM. If N is another manifold and $f : M \to N$ a differentiable map, we denote by $T_pf : T_pM \to T_{f(p)}N$ or by $f_* : T_pM \to T_{f(p)}N$ the *tangential* of f in p. If $\alpha : J \to M$ is a curve, we denote by $\dot\alpha(t) \in T_{\alpha(t)}M$ the *tangent vector* of α in $t \in J$.

If V is a (real or complex) linear space, which we here also regard as a manifold, and $p \in V$, there is a canonical linear isomorphism $T_pV \to V,\ u \mapsto \overrightarrow{u}$ characterized by

$$\forall u \in T_pV \ : \ (t \mapsto p + t \cdot \overrightarrow{u})^\cdot(0) = u \ ,$$

called the *arrow map*. We denote by ∂ the *canonical vector field of* $\mathbb{R}$, it is characterized by $\overrightarrow{\partial_t} = 1$ for every $t \in \mathbb{R}$. If $\alpha : J \to \mathbb{R}$ is a curve, we have $\dot\alpha(t) = \alpha_* \partial_t$ for any $t \in J$.

Let M and N be manifolds, $f : M \to N$ be a map, and denote the tangent bundle of N by $\pi : TN \to N$. Then we call the maps $X : M \to TN$ with $X \circ \pi = f$ *vector fields along* f, and denote the space of such fields by $\mathfrak{X}_f(N)$. We also put $\mathfrak{X}(N) := \mathfrak{X}_{\mathrm{id}_N}(N)$, this is the space of usual vector fields on N. If a covariant derivative ∇ on N is given, we consider $\nabla_v X$ also for vector fields $X \in \mathfrak{X}_f(N)$ and $v \in TM$ in the way described in [Poo81]. The application of the covariant derivative to vector fields along f, which greatly extends the flexibility in handling vector fields on manifolds, is due to P. Dombrowski.

Lie groups and Lie algebras. Let G and G' be Lie groups.

symbol	meaning
e_G	the neutral element of G
G_0	the neutral component of G
$\mathfrak{g}$	the Lie algebra corresponding to G
f_L ($f : G \to G'$ a Lie group homomorphism)	the linearization $f_L : \mathfrak{g} \to \mathfrak{g}'$ of f
L_g $(g \in G)$	the left translation $G \to G,\ x \mapsto g \cdot x$
I_g $(g \in G)$	the inner automorphism $G \to G,\ x \mapsto g \cdot x \cdot g^{-1}$
$\mathrm{Ad}_G(g),\ \mathrm{Ad}(g)$ $(g \in G)$	$= (I_g)_L \in \mathrm{GL}(\mathfrak{g})$ (the adjoint representation of G)

Let V be a (real or complex) linear space. Then $\mathrm{GL}(V)$ is a Lie group, whose Lie algebra $\mathfrak{gl}(V)$ is isomorphic to $\mathrm{End}(V)$ via the map

$$\mathfrak{gl}(V) \to \mathrm{End}(V),\ X \mapsto \overrightarrow{X_{\mathrm{id}_V}}$$

(where the Lie algebra structure on $\mathrm{End}(V)$ is given by the commutator $[A, B] \mapsto A \circ B - B \circ A$). In the sequel we identify $\mathfrak{gl}(V)$ with $\mathrm{End}(V)$ via this isomorphism. We also use this identification for the classical Lie subgroups of $\mathrm{GL}(V)$; thereby we obtain the following Lie algebras:

Lie algebra	requirement on V	corresponding Lie group	explicit description of the Lie algebra
$\mathfrak{sl}(V)$	V a $\mathbb{K}$-linear space	$\mathrm{SL}(V)$	$\{ X \in \mathrm{End}(V) \mid \mathrm{tr}(X) = 0 \}$
$\mathfrak{o}(V)$	V a euclidean space	$\mathrm{O}(V)$ or $\mathrm{SO}(V) = \mathrm{O}(V)_0$	$\mathrm{End}_-(V)$
$\mathfrak{u}(V)$	V a unitary space	$\mathrm{U}(V)$	$\mathrm{End}_-(V)$
$\mathfrak{su}(V)$	V a unitary space	$\mathrm{SU}(V)$	$\{ X \in \mathrm{End}_-(V) \mid \mathrm{tr}(X) = 0 \}$

Riemannian and Hermitian manifolds. If M is a Riemannian manifold, we denote the Lie group of isometries $M \to M$ by $I(M)$. If M is a Hermitian manifold, we denote the Lie subgroup of holomorphic isometries $M \to M$ by $I_h(M)$. Also, we then denote the set of anti-holomorphic isometries $M \to M$ by $I_{ah}(M)$; in the case $I_{ah}(M) \neq \varnothing$, this is a coset in $I(M)$.

If M and M' are Riemannian manifolds, $f : M \to M'$ a differentiable map and $p \in M$, we call

$$\perp_p f := \{ v \in T_{f(p)}M' \mid \forall w \in T_pM : \langle v, f_* w \rangle_{M'} = 0 \}$$

the *normal space* of f at p, also we call $\perp^1_p f := \mathbb{S}(\perp_p f)$ the sphere of *unit normal vectors* of f at p. If f is an immersion, then f gives rise to a subbundle of the tangent bundle of M' along f in this way, which we call the *normal bundle* of f and denote by $\perp f$. We then also consider the sphere bundle $\perp^1 f$ of unit spheres in $\perp f$. If N is a submanifold of M, we define the normal spaces and the normal bundle of N in terms of the inclusion map $N \hookrightarrow M$: For $p \in N$ we put $\perp_p N := \perp_p(N \hookrightarrow M)$, $\perp^1_p N := \perp^1_p(N \hookrightarrow M)$, $\perp N := \perp(N \hookrightarrow M)$ and $\perp^1 N := \perp^1(N \hookrightarrow M)$.

Chapter 1

Complex quadrics

In this chapter the intrinsic and extrinsic geometry of complex quadrics as complex hypersurfaces of the complex projective space is studied.

At first, we take the viewpoint of algebraic geometry, where complex quadrics are defined as the zero locus of a non-degenerate quadratic equation in a complex projective space $\mathbb{P}^n$ (without any binding to a Riemannian metric on $\mathbb{P}^n$). But then it turns out that among the complex quadrics of algebraic geometry there are certain ones which are particularly well-adapted to the Fubini-Study metric of the complex projective space $\mathbb{P}^n$. From that point on, we will only consider complex quadrics of the latter kind, and we will call these simply *complex quadrics*. In Sections 1.3 and 1.4 we calculate the shape operator of such a quadric Q (as a complex hypersurface of $\mathbb{P}^n$) and the curvature tensor and the Ricci tensor of Q. In particular we find that the structure of the shape operator in a point $p \in Q$ is very simple – it is that of a "circle of conjugations" on the unitary space T_pQ. This observation is fundamental for all the subsequent studies of complex quadrics.

As was already mentioned in the Introduction, the methods developed in [Rec95] for the study of the complex quadric had a very strong influence on the present dissertation. The approach to the complex quadric taken here in the first three chapters are modeled on [Rec95]. In the present chapter, in particular the calculations of the shape operator and the curvature tensor of the complex quadric closely follow [Rec95].

1.1 Complex quadrics in algebraic geometry

Let $m \in \mathbb{N}$ and a complex linear space $\mathbb{V}$ of dimension $n := m + 2$ be given. We consider the complex projective space $\mathbb{P}(\mathbb{V})$ of $\mathbb{V}$, this is an $(m + 1)$-dimensional projective variety.

In the context of algebraic geometry, one calls any subvariety $Q(\beta)$ of $\mathbb{P}(\mathbb{V})$ which is defined via a non-degenerate symmetric bilinear form $\beta : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ by

$$Q(\beta) = \{ [z] \in \mathbb{P}(\mathbb{V}) \mid z \in \mathbb{V} \setminus \{0\},\ \beta(z, z) = 0 \}$$

a complex quadric. To emphasize the absence of reference to any metric structure, we will call such quadrics *algebraic complex quadrics*. For any such quadric, we also consider the corresponding *quadratic cone*

$$\widehat{Q}(\beta) := \{\, z \in \mathbb{V} \setminus \{0\} \,|\, \beta(z,z) = 0 \,\} .$$

1.1 Example. Consider the non-degenerate symmetric bilinear form $\beta : \mathbb{C}^{m+2} \times \mathbb{C}^{m+2} \to \mathbb{C}$, $(v,w) \mapsto \sum_{k=1}^{m+2} v_k\, w_k$. Then we call the algebraic complex quadric

$$Q^m := Q(\beta) = \left\{ [z_1, \dots, z_{m+2}] \in \mathbb{P}^{m+1} \;\middle|\; \textstyle\sum_{k=1}^{m+2} z_k^2 = 0 \right\}$$

the *standard complex quadric of dimension* m .

1.2 Proposition. *Let* $\beta : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ *be a non-degenerate symmetric bilinear form.*

(a) $Q(\beta)$ *is a regular complex hypersurface*[2] *of* $\mathbb{P}(\mathbb{V})$.

(b) $\widehat{Q}(\beta)$ *is a regular complex hypersurface of* $\mathbb{V}$ *with*

$$\forall z \in \widehat{Q}(\beta) \;:\; \overrightarrow{T_z \widehat{Q}(\beta)} = \{\, v \in \mathbb{V} \,|\, \beta(v,z) = 0 \,\} .$$

Proof. (b) is a direct consequence of the complex version of the theorem on equation-defined manifolds (see for example [Nar68], Corollary 2.5.5, p. 81). For (a), we note that the map $\widehat{\pi} : \mathbb{V} \setminus \{0\} \to \mathbb{P}(\mathbb{V}),\ z \mapsto [z]$ is a surjective holomorphic submersion and that $\widehat{Q}(\beta)$ is saturated with respect to $\widehat{\pi}$. Therefore $Q = \widehat{\pi}(\widehat{Q}(\beta))$ is a complex hypersurface of $\mathbb{P}(\mathbb{V})$ and the codimension of $Q(\beta)$ in $\mathbb{P}(\mathbb{V})$ is equal to the codimension of $\widehat{Q}(\beta)$ in $\mathbb{V}$. □

1.3 Proposition. *Let* β, β' *be two non-degenerate symmetric bilinear forms on* $\mathbb{V}$ *. Then we have*

$$Q(\beta) = Q(\beta') \iff \exists\, \lambda \in \mathbb{C}^\times \;:\; \beta' = \lambda \cdot \beta .$$

Proof. The implication "$\Longleftarrow$" is obvious. Conversely, let non-degenerate symmetric bilinear forms β , β' on $\mathbb{V}$ be given so that $Q(\beta) = Q(\beta')$ holds. The relation $\beta(v,w) = 0$ can be characterized geometrically by properties of the set $\widehat{Q}(\beta)$ alone, see [Wal85], Satz 6.2.F, p. 189 and the remark following it. Therefore $Q(\beta) = Q(\beta')$ implies

$$\forall v,w \in \mathbb{V} \;:\; (\beta(v,w) = 0 \iff \beta'(v,w) = 0) . \tag{1.1}$$

[Wal85], Lemma 6.2.G, p. 190 shows that (1.1) implies the existence of $\lambda \in \mathbb{C}$ so that $\beta' = \lambda \cdot \beta$ holds; because β' is non-zero, we have $\lambda \neq 0$. □

1.4 Proposition. *Let* β *be a non-degenerate symmetric bilinear form on* $\mathbb{V}$ *. Then there exists a basis* $(b_1, \dots, b_n)$ *of* $\mathbb{V}$ *so that*

$$\forall v,w \in \mathbb{V} \;:\; \beta(v,w) = \sum_k \lambda_k(v) \cdot \lambda_k(w) \tag{1.2}$$

holds, where $(\lambda_1, \dots, \lambda_n)$ *denotes the basis of* $\mathbb{V}^*$ *which is dual to* $(b_1, \dots, b_n)$ *. We call any such basis* $(b_1, \dots, b_n)$ *an* adapted basis *for* β *.*

[2] A *complex hypersurface* is a submanifold of complex codimension 1 .

Proof. A basis $(b_1, \dots, b_n)$ satisfies (1.2) if and only if it is an orthonormal basis with respect to the non-degenerate, symmetric complex-bilinear form β. For the existence of such bases, see for example [Bri85], Satz 12.44, p. 420. □

1.5 Proposition. *Let β be a non-degenerate symmetric bilinear form on $\mathbb{V}$, and let $\mathbb{V}'$ be another n-dimensional complex linear space.*

(a) Let $B : \mathbb{V} \to \mathbb{V}'$ be a linear isomorphism. Then

$$\beta' : \mathbb{V}' \times \mathbb{V}' \to \mathbb{C},\ (v, w) \mapsto \beta(B^{-1}v, B^{-1}w) \tag{1.3}$$

is a non-degenerate symmetric bilinear form on $\mathbb{V}'$ and we have $\widehat{Q}(\beta') = B(\widehat{Q}(\beta))$. Moreover, with the biholomorphic map $\underline{B} : \mathbb{P}(\mathbb{V}) \to \mathbb{P}(\mathbb{V}')$ induced by B (in the way described in Section 0.2) we have $Q(\beta') = \underline{B}(Q(\beta))$.

(b) If β' is any non-degenerate symmetric bilinear form on $\mathbb{V}'$, then there exists a linear isomorphism $B : \mathbb{V} \to \mathbb{V}'$ so that β' is described by (1.3)*.*

This proposition shows in particular that any two m-dimensional algebraic complex quadrics are biholomorphically equivalent.

Proof. (a) is obvious. For (b), choose adapted bases $(b_1, \dots, b_n)$ for β and $(b_1', \dots, b_n')$ for β', and consider the linear map $B : \mathbb{V} \to \mathbb{V}'$ characterized by $Bb_k = b_k'$ for $k \in \{1, \dots, n\}$. Then we have

$$\forall v, w \in \mathbb{V} \ : \ \beta'(Bv, Bw) = \beta(v, w) \ ,$$

therefore β' is described by (1.3) with this choice of B. □

1.2 Symmetric complex quadrics

In the situation of the previous section, we now suppose that $\mathbb{V}$ is a unitary space. We denote its inner product by $\langle \cdot, \cdot \rangle_{\mathbb{C}}$. We will also consider $\mathbb{V}$ as an euclidean space via the real inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}} := \mathrm{Re}(\langle \cdot, \cdot \rangle_{\mathbb{C}})$; this euclidean space additionally carries the orthogonal complex structure $J : \mathbb{V} \to \mathbb{V},\ v \mapsto i \cdot v$. In the sequel, the orthogonal complement $W^{\perp}$ of an $\mathbb{R}$-linear subspace $W \subset \mathbb{V}$ is always constructed with respect to the real inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$. The map

$$\pi : \mathbb{S}(\mathbb{V}) \to \mathbb{P}(\mathbb{V}),\ z \mapsto [z]$$

is called the *Hopf fibration* of $\mathbb{V}$.

As is well-known, we have for any $z \in \mathbb{S}(\mathbb{V})$

$$\overrightarrow{T_z\mathbb{S}(\mathbb{V})} = (\mathbb{R}\, z)^{\perp} \tag{1.4}$$

and the vertical space $\mathcal{V}_z := \ker T_z\pi$ of π at z satisfies

$$\overrightarrow{\mathcal{V}_z} = \mathbb{R}\, iz \ . \tag{1.5}$$

Consequently, the complex linear subspace $\mathcal{H}_z := (\mathcal{V}_z)^{\perp, T_z\mathbb{S}(\mathbb{V})}$ of $T_z\mathbb{V}$ is described by

$$\overrightarrow{\mathcal{H}_z} = (\mathbb{C}\, z)^{\perp} \; . \tag{1.6}$$

$(\mathcal{H}_z)_{z\in\mathbb{S}(\mathbb{V})}$ is an Ehresmann connection for π. The group $G := \{\, \lambda \cdot \mathrm{id}_{\mathbb{S}(\mathbb{V})} \mid \lambda \in \mathbb{S}^1 \,\}$ acts on $\mathbb{S}(\mathbb{V})$ and $g_*|\mathcal{H}_z : \mathcal{H}_z \to \mathcal{H}_{g(z)}$ is a $\mathbb{C}$-linear isometry for every $g \in G$ and $z \in \mathbb{S}(\mathbb{V})$. Because the orbits of the action of G on $\mathbb{S}(\mathbb{V})$ are exactly the fibres of π, it follows that there is one and only one Riemannian metric on $\mathbb{P}(\mathbb{V})$ so that $\mathbb{P}(\mathbb{V})$ becomes a Hermitian manifold and π becomes a Hermitian submersion, meaning that the map

$$\pi_*|\mathcal{H}_z : \mathcal{H}_z \to T_{\pi(z)}\mathbb{P}(\mathbb{V})$$

is a $\mathbb{C}$-linear isometry for every $z \in \mathbb{S}(\mathbb{V})$. This Riemannian metric on $\mathbb{P}(\mathbb{V})$ is called the *Fubini-Study metric*. In this way, $\mathbb{P}(\mathbb{V})$ becomes an irreducible Hermitian symmetric space of rank 1, which has constant holomorphic sectional curvature 4 (see [KN69], Example XI.10.5, p. 273). In the sequel we always regard $\mathbb{P}(\mathbb{V})$ in this way.

Let $\mathbb{V}'$ be another n-dimensional unitary space and $B : \mathbb{V} \to \mathbb{V}'$ be a $\mathbb{C}$-linear isometry. Then the induced map $\underline{B} : \mathbb{P}(\mathbb{V}) \to \mathbb{P}(\mathbb{V}')$ satisfies $\pi \circ (B|\mathbb{S}(\mathbb{V})) = \underline{B} \circ \pi$, and because B preserves the inner product and the complex structure on $(\mathcal{H}_z)_{z\in\mathbb{S}(\mathbb{V})}$, $\underline{B}$ is a biholomorphic isometry. Similarly, any anti-unitary map $B : \mathbb{V} \to \mathbb{V}$ (i.e. B is anti-linear and orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$) induces an anti-holomorphic isometry $\underline{B} : \mathbb{P}(\mathbb{V}) \to \mathbb{P}(\mathbb{V}')$. It can be shown that any isometry $f : \mathbb{P}(\mathbb{V}) \to \mathbb{P}(\mathbb{V}')$ is either holomorphic or anti-holomorphic, and can be described as $f = \underline{B}$ with a suitable $\mathbb{C}$-linear resp. anti-linear isometry $B : \mathbb{V} \to \mathbb{V}'$.

Any algebraic complex quadric in $\mathbb{P}(\mathbb{V})$ is a complex hypersurface and as such inherits the structure of a Hermitian manifold from $\mathbb{P}(\mathbb{V})$. However, not every algebraic complex quadric is equally well-adapted to the metric structure of $\mathbb{P}(\mathbb{V})$. We will now describe a subset of the set of algebraic complex quadrics, whose members we will call symmetric complex quadrics, and which are particularly well-behaved with respect to the Fubini-Study metric of $\mathbb{P}(\mathbb{V})$.

For any non-degenerate, symmetric bilinear map $\beta : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$, the Riesz endomorphism $A := \beta^\sharp : \mathbb{V} \to \mathbb{V}$ of β is characterized by

$$\forall v, w \in \mathbb{V} \;:\; \beta(v,w) = \langle v, Aw\rangle_{\mathbb{C}} \; .$$

A is anti-linear and satisfies $\langle Av, w\rangle_{\mathbb{C}} = \overline{\langle v, Aw\rangle_{\mathbb{C}}}$ for $v, w \in \mathbb{V}$ (by virtue of the symmetry of β), in particular it is $\mathbb{R}$-linear and self-adjoint with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$. Of course, A contains all information of β, so we can define $Q(A) := Q(\beta)$ and $\widehat{Q}(A) := \widehat{Q}(\beta)$ without ambiguity.

1.6 Definition. *We call an anti-linear endomorphism of $\mathbb{V}$ which is self-adjoint with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ a* conjugation *on $\mathbb{V}$, if it is also orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$.*

1.7 Proposition. *Let β be a non-degenerate, symmetric bilinear map on $\mathbb{V}$ and put $A := \beta^\sharp : \mathbb{V} \to \mathbb{V}$. Then the following statements are equivalent:*

(a) β has an adapted basis which is a unitary basis of $\mathbb{V}$.

(b) A is a conjugation on $\mathbb{V}$.

If these statements hold, we call $Q(\beta)$ a symmetric complex quadric.

Proof. For (a) $\Rightarrow$ (b). Let $(b_1, \dots, b_n)$ be an adapted basis for β which is also a unitary basis of $\mathbb{V}$ and denote by $(\lambda_1, \dots, \lambda_n)$ the dual basis of $\mathbb{V}^*$. By Proposition 1.4, we have for any $v, w \in \mathbb{V}$

$$\begin{aligned}\langle v, Aw\rangle_{\mathbb{C}} = \beta(v,w) &= \textstyle\sum_k \lambda_k(v)\cdot\lambda_k(w) = \sum_{k,\ell}\lambda_k(v)\cdot\lambda_\ell(w)\cdot\langle b_k, b_\ell\rangle_{\mathbb{C}} \\ &= \big\langle \textstyle\sum_k \lambda_k(v) b_k\,,\, \sum_\ell \overline{\lambda_\ell(w)} b_\ell\big\rangle_{\mathbb{C}} = \big\langle v\,,\, \sum_\ell \overline{\lambda_\ell(w)} b_\ell\big\rangle_{\mathbb{C}}\end{aligned}$$

and consequently

$$\forall w \in \mathbb{V} \;:\; Aw = \sum_\ell \overline{\lambda_\ell(w)}\cdot b_\ell \;.$$

It follows that $A \circ A = \mathrm{id}_{\mathbb{V}}$ and hence

$$\forall v, w \in \mathbb{V} \;:\; \langle Av, Aw\rangle_{\mathbb{R}} = \langle v, A(Aw)\rangle_{\mathbb{R}} = \langle v, w\rangle_{\mathbb{R}}$$

holds. Thus A is orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ and therefore a conjugation on $\mathbb{V}$.

For (b) $\Rightarrow$ (a). A is self-adjoint with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ and therefore real diagonalizable; as a conjugation, A is also orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$, and therefore 1 and -1 are the only possible eigenvalues, whence we have $\mathbb{V} = \mathrm{Eig}(A,1) \oplus \mathrm{Eig}(A,-1)$. Because A is anti-linear, we have $A \circ J = -J \circ A$ and therefore $\mathrm{Eig}(A,-1) = J(\mathrm{Eig}(A,1))$. It follows that

$$\mathbb{V} = \mathrm{Eig}(A,1) \oplus J(\mathrm{Eig}(A,1)) \tag{1.7}$$

holds; in particular, $\mathrm{Eig}(A,1)$ and $\mathrm{Eig}(A,-1)$ are totally real[3] subspaces of $\mathbb{V}$.

Equation (1.7) shows that any orthonormal basis $(b_1, \dots, b_n)$ of $\mathrm{Eig}(A,1)$ is a unitary basis of $\mathbb{V}$ and we have for $v, w \in \mathbb{V}$

$$\beta(v,w) = \langle v, Aw\rangle_{\mathbb{C}} = \sum_{k,\ell} \lambda_k(v)\,\lambda_\ell(w)\,\langle b_k, \underbrace{Ab_\ell}_{=b_\ell}\rangle_{\mathbb{C}} = \sum_k \lambda_k(v)\,\lambda_k(w)\;,$$

showing that $(b_1, \dots, b_n)$ is an adapted basis for β. ☐

1.8 Example. The standard complex quadric Q^m of Example 1.1 is a symmetric quadric; it corresponds to the usual conjugation $z \mapsto \overline{z}$ on $\mathbb{C}^{m+2}$, which also is a conjugation in the sense of Definition 1.6.

[3]We call an $\mathbb{R}$-linear subspace $W \subset \mathbb{V}$ *totally real*, if $JW \subset W^\perp$ holds.

The algebraic properties of conjugations will be further studied in Chapter 2; for the purposes of the present chapter, we only extract the facts which were proved during the proof of Proposition 1.7, (b) $\Rightarrow$ (a):

1.9 Proposition. *Let $A : \mathbb{V} \to \mathbb{V}$ be a conjugation. Then A (seen as an $\mathbb{R}$-linear map) is real diagonalizable, its spectrum is $\{1, -1\}$, the corresponding eigenspaces $V(A) := \mathrm{Eig}(A, 1)$ and $\mathrm{Eig}(A, -1) = JV(A)$ are totally-real subspaces of $\mathbb{V}$ of real dimension n and we have $\mathbb{V} = V(A) \oplus JV(A)$.*

1.10 Proposition. *Let A, A' be two conjugations on $\mathbb{V}$. Then we have*

$$Q(A) = Q(A') \iff \exists\, \lambda \in \mathbb{S}^1 \;:\; A' = \lambda \cdot A\,.$$

Proof. For any anti-linear map $A : \mathbb{V} \to \mathbb{V}$ and $\lambda \in \mathbb{C}$, both A and λA can be conjugations only if $\lambda \in \mathbb{S}^1$ holds. Using this fact, this proposition follows from Proposition 1.3. □

1.11 Proposition. *Let $A : \mathbb{V} \to \mathbb{V}$ be a conjugation on $\mathbb{V}$ and $\mathbb{V}'$ be another n-dimensional unitary space.*

(a) Let $B : \mathbb{V} \to \mathbb{V}'$ be a $\mathbb{C}$-linear isometry. Then $A' := B \circ A \circ B^{-1}$ is a conjugation on $\mathbb{V}'$ and we have $\widehat{Q}(A') = B(\widehat{Q}(A))$ and $Q(A') = \underline{B}(Q(A))$.

(b) If A' is any conjugation on $\mathbb{V}'$, then there exists a $\mathbb{C}$-linear isometry $B : \mathbb{V} \to \mathbb{V}'$ so that $A' = B \circ A \circ B^{-1}$ holds.

Proof. For (a). Obvious. *For (b).* We consider the bilinear forms

$$\beta : \mathbb{V} \times \mathbb{V} \to \mathbb{C},\ (v, w) \mapsto \langle v, Aw \rangle_{\mathbb{C}} \quad \text{and} \quad \beta' : \mathbb{V}' \times \mathbb{V}' \to \mathbb{C},\ (v, w) \mapsto \langle v, A'w \rangle_{\mathbb{C}}\,.$$

Let $(b_1, \dots, b_n)$ and $(b_1', \dots, b_n')$ be adapted bases of β resp. β' which are also unitary bases of $\mathbb{V}$ resp. $\mathbb{V}'$ (see Proposition 1.7). Then the linear map $B : \mathbb{V} \to \mathbb{V}'$ characterized by $B(b_k) = b_k'$ for $k \in \{1, \dots, n\}$ is a linear isometry and satisfies $A' = B \circ A \circ B^{-1}$. □

1.12 Remarks. (a) I stated above that the symmetric complex quadrics are better-adapted to the Fubini-Study metric of $\mathbb{P}(\mathbb{V})$ than algebraic complex quadrics in general. This claim is justified by the following observations:

(i) As Proposition 1.11 shows, the set $\mathfrak{Q}(\mathbb{V})$ of symmetric quadrics in $\mathbb{P}(\mathbb{V})$ is a *holomorphic congruence class* of submanifolds of $\mathbb{P}(\mathbb{V})$, this means: $\mathfrak{Q}(\mathbb{V})$ is one orbit of the canonical action of the group $I_h(\mathbb{P}(\mathbb{V}))$ of holomorphic isometries of $\mathbb{P}(\mathbb{V})$ on the set of all algebraic quadrics in $\mathbb{P}(\mathbb{V})$.

(ii) Among the algebraic complex quadrics, the symmetric quadrics are exactly those which are extrinsically symmetric submanifolds of $\mathbb{P}(\mathbb{V})$ (see [NT89], p. 171), i.e. which are invariant with respect to the reflections in their normal spaces in $\mathbb{P}(\mathbb{V})$. This is the reason for naming these quadrics “symmetric”. It is a consequence that the shape operator of the inclusion $Q \hookrightarrow \mathbb{P}(\mathbb{V})$, where Q is a symmetric quadric, is parallel (see [Nai86], Corollary 1.4, p. 218). We will give a direct proof of the latter fact in Section 1.3 below.

(iii) Symmetric complex quadrics are also distinguished among the algebraic complex quadrics by the fact that they are Einstein manifolds (see Proposition 1.23 below).

(b) Let $\mathbb{V}$ be a "bare" complex linear space and $Q = Q(\beta)$ an algebraic complex quadric of $\mathbb{V}$. If we choose any adapted basis of β and denote by $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ the inner product on $\mathbb{V}$ for which this basis is a unitary basis, then Q is a symmetric complex quadric with respect to $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ (see Proposition 1.7).

We fix a conjugation $A : \mathbb{V} \to \mathbb{V}$ and consider the corresponding bilinear form

$$\beta : \mathbb{V} \times \mathbb{V} \to \mathbb{C},\ (v, w) \mapsto \langle v, Aw\rangle_{\mathbb{C}}$$

and the corresponding quadric $Q(A)$. We also consider the pre-image of $Q(A)$ under the Hopf fibration $\pi : \mathbb{S}(\mathbb{V}) \to \mathbb{P}(\mathbb{V})$

$$\widetilde{Q}(A) := \widetilde{Q}(\beta) := \{\, z \in \mathbb{S}(\mathbb{V}) \,|\, \beta(z,z) = 0 \,\} = \widehat{Q}(\beta) \cap \mathbb{S}(\mathbb{V})\ .$$

By applying the theorem on equation-defined manifolds, we see that $\widetilde{Q}(A)$ is a submanifold of $\mathbb{S}(\mathbb{V})$ of real codimension 2.

1.13 Proposition. *For $z \in \widetilde{Q}(A)$, we have*

(a) $\overrightarrow{T_z\widetilde{Q}(A)} = \{\, v \in \mathbb{V} \,|\, \langle v, Az\rangle_{\mathbb{C}} = 0,\ \langle v, z\rangle_{\mathbb{R}} = 0 \,\}$.

(b) If we denote the horizontal lift of $T_pQ(A)$ with respect to π at z by $\mathcal{H}_zQ(A) := (\pi_*|\mathcal{H}_z)^{-1}(T_pQ(A))$, we have $\mathcal{H}_zQ(A) = \mathcal{H}_z \cap T_z\widetilde{Q}(A)$ and

$$\overrightarrow{\mathcal{H}_zQ(A)} = \{\, v \in \mathbb{V} \,|\, \langle v, z\rangle_{\mathbb{C}} = \langle v, Az\rangle_{\mathbb{C}} = 0 \,\}\ . \tag{1.8}$$

Proof. For (a). This is easily verified using the theorem on equation-defined manifolds. *For (b).* The equality $\mathcal{H}_zQ(A) = \mathcal{H}_z \cap T_z\widetilde{Q}(A)$ follows easily from the fact that $\widetilde{Q}(A) = \pi^{-1}(Q(A))$ holds, and Equation (1.8) then follows from (a) and Equation (1.6). □

The symmetric complex quadrics are the central object of study in this work. For this reason, we shall henceforth adopt the following terminology:

Throughout the entire dissertation, the term *complex quadric* always refers to a symmetric complex quadric, unless noted otherwise by the use of the attribute "algebraic".

1.3 The shape operator of $Q \hookrightarrow \mathbb{P}(\mathbb{V})$

In this section we calculate the shape operator of the inclusion map $Q \hookrightarrow \mathbb{P}(\mathbb{V})$. As was already mentioned, these calculations closely follow those of [Rec95].

In the situation of the previous section, let us fix a conjugation $A : \mathbb{V} \to \mathbb{V}$ and abbreviate $Q := Q(A)$, $\widetilde{Q} := \widetilde{Q}(A)$ and $\widehat{Q} := \widehat{Q}(A)$. In the sequel, we will take the liberty of denoting by $\langle\cdot,\cdot\rangle$ the real inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ of $\mathbb{V}$, the induced Riemannian metric on the manifold $\mathbb{V}$ and the Fubini-Study metric of $\mathbb{P}(\mathbb{V})$; similarly, we will denote by J the complex structure of the euclidean space $\mathbb{V}$ and the complex structure of the Hermitian manifold $\mathbb{P}(\mathbb{V})$.

We denote for any $p \in Q$ and $\zeta \in \perp_p(Q \hookrightarrow \mathbb{P}(\mathbb{V}))$ the shape operator of the inclusion map $Q \hookrightarrow \mathbb{P}(\mathbb{V})$ with respect to ζ by $A^Q_\zeta : T_pQ \to T_pQ$.

1.14 Proposition. *The map* $\perp_p(Q \hookrightarrow \mathbb{P}(\mathbb{V})) \to \mathrm{End}_{\mathbb{R}}(T_pQ)$, $\zeta \mapsto A^Q_\zeta$ *is* $\mathbb{C}$*-linear for any* $p \in Q$.

Proof. [KN69], Proposition IX.9.1, p. 175 shows that this statement holds because $\mathbb{P}(\mathbb{V})$ is a Kähler manifold (see [KN69], Example IX.6.3, p. 159f.) and Q is a complex submanifold of $\mathbb{P}(\mathbb{V})$. □

To obtain further information on A^Q, we introduce the following objects:

- The vector field $\eta \in \mathfrak{X}_{\mathbb{S}(\mathbb{V})\hookrightarrow\mathbb{V}}(\mathbb{V})$ characterized by
$$\forall z \in \mathbb{S}(\mathbb{V}) \;:\; \overrightarrow{\eta_z} = z\;;$$
as Equation (1.4) shows, η is a unit normal field to $\mathbb{S}(\mathbb{V})$, and by Equation (1.5) the vector field $J \circ \eta$ is tangential to $\mathbb{S}(\mathbb{V})$ and vertical with respect to π.

- The tensor field C of type (1,1) on $\mathbb{V}$ characterized by
$$\forall u \in T\mathbb{V} \;:\; \overrightarrow{Cu} = A(\overrightarrow{u})\;;$$
by Proposition 1.13(b), the conjugation C_z on the unitary space $T_z\mathbb{V}$ leaves $\mathcal{H}_zQ$ invariant for every $z \in \widetilde{Q}$. Also note that we have
$$\langle C\eta_z, \eta_z\rangle_{\mathbb{C}} = 0 \tag{1.9}$$
and therefore $C\eta_z \in \mathcal{H}_z$ by Equation (1.6); in particular $C\eta_z$ is tangential to $\mathbb{S}(\mathbb{V})$.

- The vector field $\widetilde{\xi} := -C \circ \eta|\widetilde{Q}$ along $\widetilde{Q} \hookrightarrow \mathbb{V}$; by Equation (1.6) and Proposition 1.13(b), $\widetilde{\xi}$ is a unit vector field tangential to $\mathbb{S}(\mathbb{V})$, horizontal with respect to π and normal to $\widetilde{Q}$. Consequently, $\xi := \pi_*\widetilde{\xi}$ is a unit vector field of $\mathbb{P}(\mathbb{V})$ along $\pi|\widetilde{Q}$, which is normal to Q.

1.15 Proposition. *For any* $z \in \widetilde{Q}$ *and* $\lambda \in \mathbb{S}^1$, *we have* $\xi(\lambda z) = \lambda^{-2} \cdot \xi(z)$. *It follows that for any* $p \in Q$, $\xi(z)$ *runs through* $\perp^1_p Q$, *if* z *runs through the fibre* $\pi^{-1}(\{p\})$.

Proof. For any $\lambda \in \mathbb{S}^1$, we consider the map $R_\lambda : \mathbb{S}(\mathbb{V}) \to \mathbb{S}(\mathbb{V}),\ z \mapsto \lambda z$. Because R_λ is an isometry which leaves the fibres of π invariant, we have

$$\forall z \in \mathbb{S}(\mathbb{V})\ :\ (R_\lambda)_* \mathcal{H}_z = \mathcal{H}_{\lambda z}\ . \tag{1.10}$$

We now prove

$$\forall w_1 \in \mathcal{H}_z,\ w_2 \in \mathcal{H}_{\lambda z}\ : \big(\pi_* w_2 = \pi_* w_1 \iff \overrightarrow{w_2} = \lambda \cdot \overrightarrow{w_1}\big)\ . \tag{1.11}$$

Let $w_1 \in \mathcal{H}_z$ and $w_2 \in \mathcal{H}_{\lambda z}$ be given. We have $\pi \circ R_\lambda^{-1} = \pi$ and therefore

$$\pi_* w_2 = \pi_* (R_\lambda)_*^{-1} w_2\ . \tag{1.12}$$

By Equation (1.10), we further have $w_1, (R_\lambda)_*^{-1} w_2 \in \mathcal{H}_z$ and thus

$$\begin{aligned} \pi_* w_2 = \pi_* w_1 &\overset{(1.12)}{\iff} \pi_*(R_\lambda)_*^{-1} w_2 = \pi_* w_1 \\ &\iff (R_\lambda)_*^{-1} w_2 = w_1 \\ &\iff \overrightarrow{(R_\lambda)_*^{-1} w_2} = \overrightarrow{w_1} \iff \overrightarrow{w_2} = \lambda \cdot \overrightarrow{w_1}\ , \end{aligned}$$

completing the proof of (1.11).

For any $z \in \widetilde{Q}$ and $\lambda \in \mathbb{S}^1$, we have

$$\overrightarrow{\widetilde{\xi}(\lambda z)} = -A(\lambda z) = -\lambda^{-1} \cdot Az = \lambda \cdot (-\lambda^{-2} Az) = \lambda \cdot \overrightarrow{\lambda^{-2} \widetilde{\xi}(z)}$$

By (1.11) we conclude $\xi(\lambda z) = \lambda^{-2} \xi(z)$. $\square$

1.16 Theorem. *Let $z \in \widetilde{Q}$ be given and put $p := \pi(z) \in Q$. Then the following diagrams commute:*

$$\begin{tikzcd} \mathcal{H}_z Q \arrow[r, "C_z"] \arrow[d, "\pi_*|\mathcal{H}_z Q"'] & \mathcal{H}_z Q \arrow[d, "\pi_*|\mathcal{H}_z Q"] \\ T_p Q \arrow[r, "A^Q_{\xi(z)}"'] & T_p Q \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \overrightarrow{\mathcal{H}_z Q} \arrow[r, "A"] \arrow[d, "\Phi"'] & \overrightarrow{\mathcal{H}_z Q} \arrow[d, "\Phi"] \\ T_p Q \arrow[r, "A^Q_{\xi(z)}"'] & T_p Q\ , \end{tikzcd} \tag{1.13}$$

where the map $\Phi : \overrightarrow{\mathcal{H}_z Q} \to T_p Q$ occurring in the second diagram is characterized by $\Phi(\overrightarrow{v}) = \pi_* v$ for all $v \in \mathcal{H}_z Q$.

In particular, $A^Q_{\xi(z)}$ is a conjugation on the unitary space $T_p Q$.

As Proposition 1.15 shows, this theorem fully describes the shape operator A^Q.

Proof. We denote the Levi-Civita covariant derivatives of $\mathbb{V}$, $\mathbb{S}(\mathbb{V})$, $\widetilde{Q}$, $\mathbb{P}(\mathbb{V})$ and Q by $\nabla^{\mathbb{V}}$, $\nabla^{\mathbb{S}}$, $\nabla^{\widetilde{Q}}$, $\nabla^{\mathbb{P}}$ and ∇^Q, respectively. Further, we denote the covariant derivative of the normal bundle of $Q \hookrightarrow \mathbb{P}(\mathbb{V})$ by $\nabla^{\perp Q}$. For $v \in T\mathbb{P}(\mathbb{V})$, we denote by $\widetilde{v}$ the horizontal lift of v with respect to π. Also, for $w \in T_z\mathbb{V}$ we denote by $\mathcal{H}(w)$ and $\mathcal{V}(w)$ the orthogonal projection of

w onto $\mathcal{H}_z$ resp. onto $\mathcal{V}_z$. We will use the analogous notations when a vector field takes the place of the vector v resp. w.

The fundamental instrument for the proof of the theorem is the formula of O'NEILL for the horizontal lift of a covariant derivative (see [O'N83], Lemma 7.45, p. 212). In the situation where N is a manifold, $g : N \to \mathbb{S}(\mathbb{V})$ is a differentiable map, $Y \in \mathfrak{X}_g(\mathbb{S}(\mathbb{V}))$ is a horizontal vector field, $p \in N$ and $v \in T_pN$ is such a vector that $g_*v \in \mathcal{H}_{g(p)}$ holds, it states

$$\mathcal{H}(\nabla^{\mathbb{S}}_v Y) = \widetilde{\nabla^{\mathbb{P}}_v \pi_* Y} \; . \tag{1.14}$$

On the other hand, $J\eta_{g(p)}$ spans $\mathcal{V}_{g(p)}$, therefore we have

$$\mathcal{V}(\nabla^{\mathbb{S}}_v Y) = \langle \nabla^{\mathbb{S}}_v Y, J\eta_{g(p)} \rangle \cdot J\eta_{g(p)} \; . \tag{1.15}$$

Because Y is horizontal and $J\eta$ is vertical, we have $\langle Y, J\eta \circ g \rangle \equiv 0$ and therefore

$$\langle \nabla^{\mathbb{S}}_v Y, J\eta_{g(p)} \rangle = -\langle Y_p, \nabla^{\mathbb{S}}_v (J\eta \circ g) \rangle = -\langle Y_p, \nabla^{\mathbb{S}}_{g_*v} J\eta \rangle = -\langle Y_p, J g_* v \rangle = \langle g_*v, JY_p \rangle \; ; \tag{1.16}$$

note that $\nabla^{\mathbb{S}}_w J\eta = Jw$ holds for any $w \in \mathcal{H}_{g(p)}$. By plugging Equation (1.16) into Equation (1.15), we obtain

$$\mathcal{V}(\nabla^{\mathbb{S}}_v Y) = \langle g_*v, JY_p \rangle \cdot J\eta_{g(p)} \; . \tag{1.17}$$

Equations (1.14) and (1.17) together show

$$\nabla^{\mathbb{S}}_v Y = \widetilde{\nabla^{\mathbb{P}}_v \pi_* Y} + \langle g_*v, JY_p \rangle \cdot J\eta_{g(p)} \; . \tag{1.18}$$

After these preparations, we show that the first diagram of (1.13) commutes. Let $w \in \mathcal{H}_z Q$ be given. Because C and $\nabla^{\mathbb{V}}$ commute, we get via the Gauss equation and (1.9)

$$\begin{aligned} Cw = C(\nabla^{\mathbb{V}}_w \eta) = \nabla^{\mathbb{V}}_w C\eta &= \nabla^{\mathbb{S}}_w C\eta - \langle w, C\eta_z \rangle \cdot \eta_z \\ &= -\nabla^{\mathbb{S}}_w \widetilde{\xi} + \underbrace{\langle w, \widetilde{\xi}_z \rangle}_{=0} \cdot \eta_z = -\nabla^{\mathbb{S}}_w \widetilde{\xi} \; , \end{aligned}$$

and therefore by means of Equation (1.18)

$$\widetilde{\nabla^{\mathbb{P}}_w \xi} = \nabla^{\mathbb{S}}_w \widetilde{\xi} - \underbrace{\langle w, J\widetilde{\xi}_z \rangle}_{=0} \cdot J\eta_z = \nabla^{\mathbb{S}}_w \widetilde{\xi} = -Cw \; \in \; \mathcal{H}_z Q \; . \tag{1.19}$$

The Weingarten equation $\nabla^{\mathbb{P}}_w \xi = -A^Q_{\xi(z)} \pi_* w + \nabla^{\perp Q}_w \xi$ therefore shows

$$A^Q_{\xi(z)} \pi_* w = \pi_* C w \quad \text{and} \quad \nabla^{\perp Q}_w \xi = 0 \; . \tag{1.20}$$

In particular, the commutativity of the first diagram of (1.13) is proved, and the commutativity of the second diagram of (1.13) is an immediate consequence. $\square$

We read the following lemma off the second part of (1.20):

1.17 Lemma. *ξ is a parallel unit normal field along $\pi|\widetilde{Q}$.*

1.18 Theorem. *The shape operator* A^Q *of* $Q \hookrightarrow \mathbb{P}(\mathbb{V})$ *is parallel with respect to* ∇^Q *.*

Proof. As we already noted in Remark 1.12, this theorem is a consequence of the fact (not yet proven here) that Q is an extrinsically symmetric submanifold of $\mathbb{P}(\mathbb{V})$. But now we wish to give an elementary proof.

We continue to use the notations of the proof of Theorem 1.16 and note that an analogous argument as that leading to Equation (1.18) in the proof of Theorem 1.16 shows that if $g : N \to \widetilde{Q}$ is a differentiable map, $Y \in \mathfrak{X}_g(\widetilde{Q})$ is a horizontal vector field, $p \in N$ and $v \in T_pN$ is such a vector that $g_* v \in \mathcal{H}_{g(p)}Q$ holds, then we have

$$\nabla^{\widetilde{Q}}_v Y = \widetilde{\nabla^Q_v \pi_* Y} + \langle g_* v, JY_p \rangle \cdot J\eta_{g(p)} \, . \tag{1.21}$$

Because of Lemma 1.17, it suffices to show that for any curve $c : I \to Q$, any horizontal lift $\widetilde{c} : I \to \widetilde{Q}$ of c with respect to π and any parallel field $X \in \mathfrak{X}_c(Q)$ the vector field $t \mapsto A^Q_{\xi \circ \widetilde{c}(t)} X(t)$ along c is parallel.

In this situation, let $t \in I$ be given. Then Equation (1.21) shows

$$\nabla^{\widetilde{Q}}_{\partial_t} \widetilde{X} = \underbrace{\widetilde{\nabla^Q_{\partial_t} X}}_{=0} + \langle \dot{\widetilde{c}}, J\widetilde{X} \rangle \cdot J\eta \circ \widetilde{c} \Big|_t \in \mathcal{V}_{\widetilde{c}(t)} \tag{1.22}$$

and also, because we have $A^Q_{\xi\circ\widetilde{c}} X = \pi_*(C\widetilde{X})$ by Theorem 1.16,

$$\nabla^{\widetilde{Q}}_{\partial_t} C\widetilde{X} = \widetilde{\nabla^Q_{\partial_t} A^Q_{\xi\circ\widetilde{c}} X} + \langle \dot{\widetilde{c}}, JC\widetilde{X} \rangle \cdot J\eta \circ \widetilde{c} \Big|_t \, . \tag{1.23}$$

In order to combine Equations (1.22) and (1.23), it would be nice if C and $\nabla^{\widetilde{Q}}$ would commute; but they do not. Therefore we must go back to $\mathbb{V}$: By Equation (1.22), we have $\nabla^{\widetilde{Q}}_{\partial_t} \widetilde{X} \perp \mathcal{H}_{\widetilde{c}(t)} Q$, and therefore the Gauss equation shows

$$\nabla^{\mathbb{V}}_{\partial_t} \widetilde{X} = \nabla^{\widetilde{Q}}_{\partial_t} \widetilde{X} + h^{\widetilde{Q}\hookrightarrow\mathbb{V}}(\dot{\widetilde{c}}, \widetilde{X}) \Big|_t \quad \perp \quad \mathcal{H}_{\widetilde{c}(t)} Q \, ,$$

where $h^{\widetilde{Q}\hookrightarrow\mathbb{V}}$ denotes the second fundamental form of $\widetilde{Q} \hookrightarrow \mathbb{V}$. It follows that

$$\nabla^{\mathbb{V}}_{\partial_t} C\widetilde{X} = C(\nabla^{\mathbb{V}}_{\partial_t} \widetilde{X}) \quad \perp \quad \mathcal{H}_{\widetilde{c}(t)} Q \tag{1.24}$$

holds. On the other hand, we get via the Gauss equation and Equation (1.23):

$$\begin{aligned} \nabla^{\mathbb{V}}_{\partial_t} C\widetilde{X} &= \nabla^{\widetilde{Q}}_{\partial_t} C\widetilde{X} + h^{\widetilde{Q}\hookrightarrow\mathbb{V}}(\dot{\widetilde{c}}, C\widetilde{X}) \Big|_t \\ &= \widetilde{\nabla^Q_{\partial_t} A^Q_{\xi\circ\widetilde{c}} X} + \underbrace{\langle \dot{\widetilde{c}}, JC\widetilde{X} \rangle \cdot J\eta \circ \widetilde{c} \Big|_t + h^{\widetilde{Q}\hookrightarrow\mathbb{V}}(\dot{\widetilde{c}}, C\widetilde{X}) \Big|_t}_{\perp \mathcal{H}_{\widetilde{c}(t)} Q} \, . \end{aligned} \tag{1.25}$$

By combining Equations (1.24) and (1.25), we see that $\widetilde{\nabla^Q_{\partial_t} A^Q_{\xi\circ\widetilde{c}}} X \perp \mathcal{H}_{\widetilde{c}(t)} Q$ holds, whereas on the other hand, we have by definition $\widetilde{\nabla^Q_{\partial_t} A^Q_{\xi\circ\widetilde{c}}} X \in \mathcal{H}_{\widetilde{c}(t)} Q$. It follows that $\widetilde{\nabla^Q_{\partial_t} A^Q_{\xi\circ\widetilde{c}}} X = 0$ and hence $\nabla^Q_{\partial_t} A^Q_{\xi\circ\widetilde{c}} X = 0$ holds. □

1.19 Proposition. *We consider the Hermitian metric $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ on $\mathbb{P}(\mathbb{V})$ induced by $(\langle\cdot,\cdot\rangle, J)$:*

$$\forall v, w \in T\mathbb{P}(\mathbb{V}) \times_{\mathbb{P}(\mathbb{V})} T\mathbb{P}(\mathbb{V}) \; : \; \langle v, w\rangle_{\mathbb{C}} = \langle v, w\rangle + i \cdot \langle v, Jw\rangle \; . \tag{1.26}$$

Let M be a complex hypersurface of $\mathbb{P}(\mathbb{V})$. Then the second fundamental form h^M of M is related to the shape operator A^M of M by the equation

$$\forall p \in M,\; v, w \in T_pM,\; \zeta \in \perp^1_p(M \hookrightarrow \mathbb{P}(\mathbb{V})) \; : \; h^M(v,w) = \langle v, A^M_\zeta w\rangle_{\mathbb{C}} \cdot \zeta \; . \tag{1.27}$$

Of course, this fact is applicable in particular for $M = Q$.

Proof. The crucial point here is the fact that

$$\forall \zeta \in \perp^1_p(M \hookrightarrow \mathbb{P}(\mathbb{V})) \; : \; A^M_{J\zeta} = J \circ A^M_\zeta \tag{1.28}$$

holds. Indeed, if we let a section s in the unit normal bundle $\perp^1(M \hookrightarrow \mathbb{P}(\mathbb{V}))$ and $v \in TM$ be given, we have because of the parallelity of J

$$\nabla^{\mathbb{P}}_v Js = J\, \nabla^{\mathbb{P}}_v s$$

(where $\nabla^{\mathbb{P}}$ again denotes the covariant derivative of $\mathbb{P}(\mathbb{V})$). From this equation, (1.28) follows via the Weingarten equation.

For given $p \in M$ and $\zeta \in \perp^1_p(M \hookrightarrow \mathbb{P}(\mathbb{V}))$, $(\zeta, J\zeta)$ is an orthonormal basis of $\perp_p(M \hookrightarrow \mathbb{P}(\mathbb{V}))$, and therefore we obtain for any $v, w \in T_pM$

$$\begin{aligned} h^M(v,w) &= \langle h^M(v,w), \zeta\rangle\zeta + \langle h^M(v,w), J\zeta\rangle J\zeta \\ &= \langle v, A^M_\zeta w\rangle\zeta + \langle v, A^M_{J\zeta} w\rangle J\zeta \\ &\overset{(1.28)}{=} \langle v, A^M_\zeta w\rangle\zeta + \langle v, JA^M_\zeta w\rangle J\zeta \\ &\overset{(1.26)}{=} \langle v, A^M_\zeta w\rangle_{\mathbb{C}} \cdot \zeta \; . \end{aligned}$$

□

1.4 The curvature of a complex quadric

1.20 Proposition. *Let M be a complex hypersurface of $\mathbb{P}(\mathbb{V})$, $p \in M$, $u, v, w \in T_pM$ and $\zeta \in \perp^1_p(M \hookrightarrow \mathbb{P}(\mathbb{V}))$. Denoting the complex inner product on $T_p\mathbb{P}(\mathbb{V})$ (see Equation (1.26)) by $\langle\cdot,\cdot\rangle_{\mathbb{C}}$, the curvature tensor of M by R^M and the shape operator of $M \hookrightarrow \mathbb{P}(\mathbb{V})$ by A^M, we have*

$$\begin{aligned} R^M(u,v)w &= \langle w, v\rangle_{\mathbb{C}}\, u - \langle w, u\rangle_{\mathbb{C}}\, v - 2\,\langle Ju, v\rangle\, Jw + \langle v, A^M_\zeta w\rangle_{\mathbb{C}}\, A^M_\zeta u - \langle u, A^M_\zeta w\rangle_{\mathbb{C}}\, A^M_\zeta v \\ &= \langle v, w\rangle u - \langle u, w\rangle v + \langle Jv, w\rangle Ju - \langle Ju, w\rangle Jv - 2 \cdot \langle Ju, v\rangle Jw \\ &\qquad + \langle v, A^M_\zeta w\rangle A^M_\zeta u - \langle u, A^M_\zeta w\rangle A^M_\zeta v + \langle v, JA^M_\zeta w\rangle JA^M_\zeta u - \langle u, JA^M_\zeta w\rangle JA^M_\zeta v \; . \end{aligned} \tag{1.29}$$

Proof. We denote the second fundamental form of the inclusion $M \hookrightarrow \mathbb{P}(\mathbb{V})$ by h^M and the curvature tensor of $\mathbb{P}(\mathbb{V})$ by $R^{\mathbb{P}}$; as it is well-known, we have

$$\begin{aligned} R^{\mathbb{P}}(u,v)w &= \langle v,w\rangle u - \langle u,w\rangle v + \langle Jv,w\rangle Ju - \langle Ju,w\rangle Jv - 2\cdot\langle Ju,v\rangle Jw \\ &= \langle w,v\rangle_{\mathbb{C}}\, u - \langle w,u\rangle_{\mathbb{C}}\, v - 2\cdot\langle Ju,v\rangle Jw\ . \end{aligned} \tag{1.30}$$

Now let $u,v,w,x \in T_pM$ be given. The Gauss equation of second order states in the present situation:

$$\langle R^M(u,v)w,x\rangle = \langle R^{\mathbb{P}}(u,v)w,x\rangle + \langle h^M(u,x),h^M(v,w)\rangle - \langle h^M(u,w),h^M(v,x)\rangle\ . \tag{1.31}$$

Using Proposition 1.19, we obtain:

$$\begin{aligned} \big\langle h^M(u,x)\,,\,h^M(v,w)\big\rangle &= \big\langle \langle u,A^M_\zeta x\rangle_{\mathbb{C}}\,\zeta\,,\,\langle v,A^M_\zeta w\rangle_{\mathbb{C}}\,\zeta\big\rangle \\ &= \big\langle \zeta\,,\,\langle A^M_\zeta u,x\rangle_{\mathbb{C}}\,\langle v,A^M_\zeta w\rangle_{\mathbb{C}}\,\zeta\big\rangle \\ &= \big\langle \zeta, \langle\langle v,A^M_\zeta w\rangle_{\mathbb{C}} A^M_\zeta u,x\rangle_{\mathbb{C}}\,\zeta\big\rangle \\ &= \mathrm{Re}\big(\langle\langle v,A^M_\zeta w\rangle_{\mathbb{C}} A^M_\zeta u,x\rangle_{\mathbb{C}}\big) \\ &= \big\langle \langle v,A^M_\zeta w\rangle_{\mathbb{C}} A^M_\zeta u\,,\,x\big\rangle \end{aligned} \tag{1.32}$$

and analogously

$$\big\langle h^M(u,w)\,,\,h^M(v,x)\big\rangle = \big\langle \langle u,A^M_\zeta w\rangle_{\mathbb{C}} A^M_\zeta v\,,\,x\big\rangle\ . \tag{1.33}$$

We now obtain the first equals sign in (1.29) by plugging Equations (1.30), (1.32) and (1.33) into Equation (1.31), noting that $R^M(u,v)w \in T_pM$ holds because of (1.30), and varying $x \in T_pM$; the second equals sign then follows from Equation (1.26). □

We now return to the specific situation of the previous section, where Q is a complex quadric in $\mathbb{P}(\mathbb{V})$ (described by some conjugation on $\mathbb{V}$).

1.21 Proposition. *(a) The curvature tensor R^Q of Q is described by Equation* (1.29) *(if one replaces M by Q throughout).*

(b) Q is a locally symmetric space.

1.22 Remark. In Chapter 3 we will see that Q is in fact a Hermitian globally symmetric space.

Proof of Proposition 1.21. For (a). This is an immediate consequence of Proposition 1.20.

For (b). We only have to show that the curvature tensor R^Q is parallel. Let a curve $c : I \to Q$ and parallel vector fields $X,Y,Z \in \mathfrak{X}_c(Q)$ along c be given. It then suffices to show that the vector field $R^Q(X,Y)Z$ along c is again parallel.

For this we let $\widetilde{c} : I \to \widetilde{Q}$ be a horizontal lift of c with respect to π . Then we have by (a)

$$\begin{aligned} R^Q(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y& \\ &+ \langle JY,Z\rangle JX - \langle JX,Z\rangle JY - 2\cdot\langle JX,Y\rangle JZ \\ &+ \langle Y,A^Q_{\xi\circ\widetilde{c}}Z\rangle A^Q_{\xi\circ\widetilde{c}}X - \langle X,A^Q_{\xi\circ\widetilde{c}}Z\rangle A^Q_{\xi\circ\widetilde{c}}Y \\ &+ \langle Y,JA^Q_{\xi\circ\widetilde{c}}Z\rangle JA^Q_{\xi\circ\widetilde{c}}X - \langle X,JA^Q_{\xi\circ\widetilde{c}}Z\rangle JA^Q_{\xi\circ\widetilde{c}}Y\ . \end{aligned}$$

Because A^Q is parallel by Theorem 1.18, $\xi \circ \widetilde{c}$ is a parallel vector field along c by Lemma 1.17, and J and $\langle\cdot,\cdot\rangle$ are parallel tensor fields, it follows that $R^Q(X,Y)Z$ is parallel. $\square$

1.23 Proposition. *The Ricci tensor field* ric^Q *of* Q *of type (0,2) is given by*

$$\forall\, p \in Q,\ v,w \in T_pQ\ :\ \mathrm{ric}^Q(v,w) = 2m \cdot \langle v,w\rangle\ .$$

In particular, Q *is an Einstein manifold.*

Proof. Let $p \in Q$ and $v,w \in T_pQ$ be given. Fix $\zeta \in \perp^1_p Q$ and put $A := A^Q_\zeta$. By Proposition 1.15 there exists $z \in \pi^{-1}(\{p\})$ with $\zeta = \xi(z)$ and therefore Theorem 1.16 shows that A is a conjugation on the unitary space T_pQ. Choose an orthonormal basis $(a_1,\dots,a_m)$ of $V(A) = \mathrm{Eig}(A,1)$, then Proposition 1.9 shows that $(a_1,\dots,a_m,Ja_1,\dots,Ja_m)$ is an orthonormal basis of $(T_pQ,\langle\cdot,\cdot\rangle)$. Therefore, we have

$$\begin{aligned}\mathrm{ric}^Q(v,w) &= \mathrm{tr}(u \mapsto R^Q(u,v)w)\\ &= \sum_{k=1}^m \big(\,\langle R^Q(e_k,v)w,e_k\rangle + \langle R^Q(Je_k,v)w,Je_k\rangle\,\big)\ .\end{aligned} \tag{1.34}$$

We now calculate the summands via Equation (1.29): A and JA are self-adjoint, whereas J is skew-adjoint. Therefore, we obtain for any $k \in \{1,\dots,m\}$:

$$\begin{aligned}\langle R^Q(e_k,v)w,e_k\rangle = {} & \langle v,w\rangle \cdot \underbrace{\langle e_k,e_k\rangle}_{=1} - \langle e_k,w\rangle \cdot \langle v,e_k\rangle\\ & + \langle Jv,w\rangle \cdot \underbrace{\langle Je_k,e_k\rangle}_{=0} - \langle Je_k,w\rangle \cdot \underbrace{\langle Jv,e_k\rangle}_{=-\langle v,Je_k\rangle} - 2\cdot\langle Je_k,v\rangle \cdot \underbrace{\langle Jw,e_k\rangle}_{=-\langle w,Je_k\rangle}\\ & + \langle v,Aw\rangle \cdot \underbrace{\langle Ae_k,e_k\rangle}_{=1} - \underbrace{\langle e_k,Aw\rangle}_{=\langle w,e_k\rangle} \cdot \underbrace{\langle Av,e_k\rangle}_{=\langle v,e_k\rangle}\\ & + \langle v,JAw\rangle \cdot \underbrace{\langle JAe_k,e_k\rangle}_{=0} - \underbrace{\langle e_k,JAw\rangle}_{=\langle w,Je_k\rangle} \cdot \underbrace{\langle JAv,e_k\rangle}_{=\langle v,Je_k\rangle}\\ = {} & \langle v,w\rangle - 2\cdot\langle v,e_k\rangle\cdot\langle w,e_k\rangle + 2\cdot\langle v,Je_k\rangle\cdot\langle w,Je_k\rangle + \langle v,Aw\rangle\ ,\end{aligned} \tag{1.35}$$

and by an analogous calculation

$$\langle R^Q(Je_k,v)w,Je_k\rangle = \langle v,w\rangle + 2\cdot\langle v,e_k\rangle\cdot\langle w,e_k\rangle - 2\cdot\langle v,Je_k\rangle\cdot\langle w,Je_k\rangle - \langle v,Aw\rangle\ . \tag{1.36}$$

Plugging Equations (1.35) and (1.36) into Equation (1.34) gives the stated result. $\square$

1.24 Remarks. (a) Proposition 1.23 has interesting consequences:

- MYERS's theorem (see [Mye35], Theorem 2, p. 42) shows that the diameter of the compact manifold Q is $\leq \sqrt{1-\frac{1}{2m}} \cdot \pi$. As we will see in Proposition 5.20, the diameter of Q is in fact $\pi/\sqrt{2}$.
- By a result of KOBAYASHI ([Kob61], Theorem A), any compact Kähler manifold with positive definite Ricci tensor, and hence Q, is simply connected.
- It should also be mentioned that it is possible to retrieve some results of this chapter from Proposition 1.23 by using results of SMYTH's paper [Smy67]: Proposition 6 of [Smy67] shows that $(A^Q_\zeta)^2 = \mathrm{id}_{T_pQ}$ holds for any $\zeta \in \perp^1_p Q$, and Theorem 2 of the same paper shows that any complex hypersurface of $\mathbb{P}(\mathbb{V})$ which is an Einstein manifold, hence in particular Q, is a Riemannian locally symmetric space.

(b) SMYTH has classified those complete complex hypersurfaces of the complex space forms which are Einstein manifolds ([Smy67], Theorem 3); for $m \geq 2$, the (symmetric) complex quadrics are the only such hypersurfaces of $\mathbb{P}(\mathbb{V})$ aside from the projective hyperplanes.

Chapter 2

$\mathbb{C}$Q-spaces

In two places of the previous chapter, "circles of conjugations" $\{\,\lambda\,A\,|\,\lambda \in \mathbb{S}^1\,\}$ (where A is a conjugation) occur: First, Proposition 1.10 shows that there is an one-to-one correspondence between the set of such circles of conjugations on a unitary space $\mathbb{V}$ and the set of symmetric complex quadrics in $\mathbb{P}(\mathbb{V})$. Second, we saw in Theorem 1.16 that if Q is a complex quadric, then for any $p \in Q$, the set $\mathfrak{A}(Q,p) := \{\,A^Q_\zeta\,|\,\zeta \in \perp^1_p(Q \hookrightarrow \mathbb{P}(\mathbb{V}))\,\}$ of shape operators is a circle of conjugations on the unitary space T_pQ.

Because of these two applications, circles of conjugations (which we will call $\mathbb{C}$Q-structures from here on) play a fundamental role in the present approach to the study of complex quadrics. Indeed the structure of the curvature tensor of Q in some $p \in Q$ is completely described by the inner product of T_pQ, its complex structure, and the $\mathbb{C}$Q-structure $\mathfrak{A}(Q,p)$ induced by the shape operator. Therefore it seems reasonable to call these data the "fundamental geometric entities" of T_pQ.

The concept of a $\mathbb{C}$Q-space was introduced by H. Reckziegel in the article [Rec95]; also in this article, the importance of $\mathbb{C}$Q-structures for the study of complex quadrics is first realized. [Rec95] is an important source for the present chapter; in particular the most important concepts involved in the study of $\mathbb{C}$Q-spaces, namely those of the space $V(A) = \mathrm{Eig}(A,1)$ corresponding to a conjugation $A : \mathbb{V} \to \mathbb{V}$, of $\mathbb{C}$Q-automorphisms, principal vectors and adapted bases, of isotropic vectors, of the characteristic angle introduced in Section 2.5, of the corresponding orbits M_t of the action of the group of $\mathbb{C}$Q-automorphisms on $\mathbb{S}(\mathbb{V})$, and of the curvature tensor of a $\mathbb{C}$Q-space have already been introduced and discussed there.

In the present chapter, we explore the algebraic properties of $\mathbb{C}$Q-structures on a general unitary space $\mathbb{V}$.

Let $\mathbb{V}$ be an n-dimensional unitary space, whose (complex) inner product we denote by $\langle\cdot,\cdot\rangle_{\mathbb{C}}$. We also regard $\mathbb{V}$ as a $2n$-dimensional euclidean space via the real inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}} := \mathrm{Re}(\langle\cdot,\cdot\rangle_{\mathbb{C}})$. In the latter regard, $\mathbb{V}$ is equipped with the orthogonal complex structure $J : \mathbb{V} \to \mathbb{V},\ v \mapsto i \cdot v$. As was already mentioned in the Introduction, $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ can be reconstructed from

$\langle\cdot,\cdot\rangle_{\mathbb{R}}$ and J by the equation

$$\forall v,w \in \mathbb{V} \ : \ \langle v,w\rangle_{\mathbb{C}} = \langle v,w\rangle_{\mathbb{R}} + i\cdot\langle v,Jw\rangle_{\mathbb{R}} \ . \tag{2.1}$$

This equation also shows that for any totally-real linear subspace $W \subset \mathbb{V}$, the restriction of $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ to $W\times W$ attains only real values and is equal to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ on that space.

2.1 Conjugations

First, we call Definition 1.6 in mind again:

2.1 Definition. *A* conjugation *on $\mathbb{V}$ is an anti-linear map $A:\mathbb{V}\to\mathbb{V}$ which is self-adjoint and orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$. If A is a conjugation on $\mathbb{V}$, we put $V(A) := \mathrm{Eig}(A,1)$ and for any $v\in\mathbb{V}$*

$$\mathrm{Re}_A\, v := \tfrac{1}{2}(Av+v) \quad \textit{and} \quad \mathrm{Im}_A\, v := \tfrac{1}{2}J(Av-v) \ .$$

We denote the set of conjugations on $\mathbb{V}$ by $\mathrm{Con}(\mathbb{V})$.

2.2 Remarks. (a) An $\mathbb{R}$-linear map $A:\mathbb{V}\to\mathbb{V}$ is both orthogonal and self-adjoint with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ if and only if A is a reflection in the linear subspace $\mathrm{Eig}(A,1)$ of $\mathbb{V}$. If this is the case, then the additional hypothesis that A is anti-linear causes $\mathrm{Eig}(A,1)$ to be a maximal totally real subspace of $\mathbb{V}$.

(b) A conjugation A on $\mathbb{V}$ is already uniquely determined by the specification of the maximal totally real subspace $V(A)$ of $\mathbb{V}$. Occasionally, a maximal totally real subspace of $\mathbb{V}$ is called a *real structure* on $\mathbb{V}$; we see that the theory of unitary spaces equipped with a conjugation A is equivalent to the theory of unitary spaces equipped with a real structure $V(A)$.

2.3 Proposition. *Let $A:\mathbb{V}\to\mathbb{V}$ be a conjugation, $v,w\in\mathbb{V}$ and $\lambda\in\mathbb{S}^1$.*

(a) $V(A)$ and $\mathrm{Eig}(A,-1) = JV(A)$ are n-dimensional totally real subspaces of $\mathbb{V}$ and we have $\mathbb{V} = V(A)\oplus JV(A)$.

(b) $A^2 = \mathrm{id}_{\mathbb{V}}$.

(c) $\langle v,Aw\rangle_{\mathbb{C}} = \overline{\langle Av,w\rangle_{\mathbb{C}}} = \langle w,Av\rangle_{\mathbb{C}}$.

(d) $\langle Av,Aw\rangle_{\mathbb{C}} = \overline{\langle v,w\rangle_{\mathbb{C}}}$.

(e) $\mathrm{Re}_A\, v, \mathrm{Im}_A\, v \in V(A)$ and $v = \mathrm{Re}_A\, v + J\,\mathrm{Im}_A\, v$; the maps $\mathrm{Re}_A, \mathrm{Im}_A : \mathbb{V}\to V(A)$ are $\mathbb{R}$-linear and satisfy $\mathrm{Re}_A(Jv) = -\mathrm{Im}_A\, v$, $\mathrm{Im}_A(Jv) = \mathrm{Re}_A\, v$, $\mathrm{Re}_A(Av) = \mathrm{Re}_A\, v$ and $\mathrm{Im}_A(Av) = -\mathrm{Im}_A(v)$.

(f) $(\ v\in V(A) \iff \mathrm{Im}_A\, v = 0\ )$ and $(\ v\in JV(A) \iff \mathrm{Re}_A\, v = 0\ )$.

(g) *$\lambda^2 A$ is another conjugation on $\mathbb{V}$ and we have $V(\lambda^2 A) = \lambda V(A)$, $\mathrm{Re}_{\lambda^2 A}\, v = \lambda\, \mathrm{Re}_A(\overline{\lambda} v)$ and $\mathrm{Im}_{\lambda^2 A}\, v = \lambda\, \mathrm{Im}_A(\overline{\lambda} v)$.*

(h) *For an $\mathbb{R}$-linear map $\widetilde{A} : \mathbb{V} \to \mathbb{V}$, any two of the following properties imply the third: (i) $\widetilde{A}$ orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathbb{R}}$, (ii) $\widetilde{A}$ self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathbb{R}}$, (iii) $\widetilde{A}^2 = \mathrm{id}_{\mathbb{V}}$.*

Proof. For (a). This has already been shown in Proposition 1.9. *For (b).* This is an immediate consequence of the fact that A is real diagonalizable and its only eigenvalues are 1 and -1 (see Proposition 1.9). *For (c).* The second equality sign is obvious; for the first, we have by Equation (2.1):

$$\langle v, Aw \rangle_{\mathbb{C}} = \langle v, Aw \rangle_{\mathbb{R}} + i \langle v, JAw \rangle_{\mathbb{R}} = \langle v, Aw \rangle_{\mathbb{R}} - i \langle v, AJw \rangle_{\mathbb{R}} = \langle Av, w \rangle_{\mathbb{R}} - i \langle Av, Jw \rangle_{\mathbb{R}} = \overline{\langle Av, w \rangle_{\mathbb{C}}}\,.$$

For (d). This is an immediate consequence of (b) and (c). *For (e).* Obvious. *For (f).* We have

$$\mathrm{Im}_A(v) = 0 \iff \tfrac{1}{2} J(Av - v) = 0 \iff Av = v \iff v \in \mathrm{Eig}(A, 1) = V(A)\,;$$

the second equivalence is shown the same way. *For (g).* $\lambda^2 A \in \mathrm{Con}(\mathbb{V})$ is obvious. We have

$$v \in V(\lambda^2 A) \;\Leftrightarrow\; \lambda^2 A v = v \;\Leftrightarrow\; \lambda A(\overline{\lambda} v) = v \;\Leftrightarrow\; A(\overline{\lambda} v) = \overline{\lambda} v \;\Leftrightarrow\; \overline{\lambda} v \in V(A) \;\Leftrightarrow\; v \in \lambda V(A)$$

and

$$\mathrm{Re}_{\lambda^2 A}\, v = \tfrac{1}{2}(\lambda^2 A v + v) = \lambda \cdot \tfrac{1}{2}(\lambda A v + \overline{\lambda} v) = \lambda \cdot \tfrac{1}{2}(A(\overline{\lambda v}) + \overline{\lambda} v) = \lambda \cdot \mathrm{Re}_A(\overline{\lambda} v)\,;$$

the equality for $\mathrm{Im}_{\lambda^2 A}\, v$ is shown analogously. *For (h).* If $\widetilde{A}$ satisfies (i) and (ii), it is real diagonalizable and 1 and -1 are the only possible eigenvalues of $\widetilde{A}$, which shows (iii). If (i) and (iii) holds, then we have for any $v, w \in \mathbb{V}$: $\langle \widetilde{A} v, w \rangle_{\mathbb{R}} = \langle \widetilde{A}^2 v, \widetilde{A} w \rangle_{\mathbb{R}} = \langle v, \widetilde{A} w \rangle_{\mathbb{R}}$, which shows (ii). If (ii) and (iii) holds, we have $\langle \widetilde{A} v, \widetilde{A} w \rangle_{\mathbb{R}} = \langle v, \widetilde{A}^2 w \rangle_{\mathbb{R}} = \langle v, w \rangle_{\mathbb{R}}$, which shows the validity of (i). □

2.4 Proposition. *Let $A : \mathbb{V} \to \mathbb{V}$ be a conjugation, $v, v' \in \mathbb{V}$ and $\lambda \in \mathbb{S}^1$, represented as $\lambda = a + bi$ with $a, b \in \mathbb{R}$. Abbreviate $x := \mathrm{Re}_A\, v$, $y := \mathrm{Im}_A\, v$, $x' := \mathrm{Re}_A\, v'$, $y' := \mathrm{Im}_A\, v'$.*

(a) *The inner products $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ coincide on $V(A) \times V(A)$.*

(b) (i) $\langle v, v' \rangle_{\mathbb{C}} = \langle x, x' \rangle_{\mathbb{R}} + \langle y, y' \rangle_{\mathbb{R}} + i \cdot (\langle y, x' \rangle_{\mathbb{R}} - \langle x, y' \rangle_{\mathbb{R}})$

(ii) $\langle v, Jv' \rangle_{\mathbb{C}} = \langle y, x' \rangle_{\mathbb{R}} - \langle x, y' \rangle_{\mathbb{R}} - i \cdot (\langle x, x' \rangle_{\mathbb{R}} + \langle y, y' \rangle_{\mathbb{R}})$

(iii) $\|v\|^2 = \|x\|^2 + \|y\|^2$

(c) (i) $\langle v, Av' \rangle_{\mathbb{C}} = \langle x, x' \rangle_{\mathbb{R}} - \langle y, y' \rangle_{\mathbb{R}} + i \cdot (\langle x, y' \rangle_{\mathbb{R}} + \langle y, x' \rangle_{\mathbb{R}})$

(ii) $\langle v, JAv' \rangle_{\mathbb{C}} = \langle x, y' \rangle_{\mathbb{R}} + \langle y, x' \rangle_{\mathbb{R}} - i \cdot (\langle x, x' \rangle_{\mathbb{R}} - \langle y, y' \rangle_{\mathbb{R}})$

(iii) $\langle v, Av \rangle_{\mathbb{C}} = \|x\|^2 - \|y\|^2 + 2i \cdot \langle x, y \rangle_{\mathbb{R}}$

(iv) $\langle v, JAv \rangle_{\mathbb{C}} = 2 \langle x, y \rangle_{\mathbb{R}} - i \cdot (\|x\|^2 - \|y\|^2)$

Proof. For (a). As $V(A)$ is totally real in $\mathbb{V}$, we have $\langle x, Jy \rangle_{\mathbb{R}} = 0$ for any $x, y \in V(A)$. The statement therefore follows from Equation (2.1). *For (b) and (c).* These equations are shown by elementary calculations. □

2.2 $\mathbb{C}$Q-spaces and their isomorphisms

2.5 Definition. *Let $\mathbb{V}$ be a unitary space and $A : \mathbb{V} \to \mathbb{V}$ be a conjugation. Then we call the "circle of conjugations" $\mathbb{S}^1 \cdot A := \{\, \lambda A \,|\, \lambda \in \mathbb{S}^1 \,\}$ a* $\mathbb{C}$Q-structure *on $\mathbb{V}$. If $\mathfrak{A}$ is a $\mathbb{C}$Q-structure on $\mathbb{V}$, we call $(\mathbb{V}, \mathfrak{A})$ or (when there is no doubt about the intended $\mathbb{C}$Q-structure) simply $\mathbb{V}$ a* $\mathbb{C}$Q-space*.*

2.6 Example. The usual conjugation $A_0 : \mathbb{C}^n \to \mathbb{C}^n,\ v \mapsto \overline{v}$ on $\mathbb{C}^n$ also is is a conjugation in the sense of Definition 2.1 on this unitary space. Therefore $\mathfrak{A}_0 := \mathbb{S}^1 \cdot A_0$ is a $\mathbb{C}$Q-structure on $\mathbb{C}^n$. We call A_0 the *standard conjugation* and $\mathfrak{A}_0$ the *standard $\mathbb{C}$Q-structure* of $\mathbb{C}^n$.

2.7 Definition. *Let $(\mathbb{V}, \mathfrak{A})$ be a $\mathbb{C}$Q-space.*

(a) A vector $v \in \mathbb{V}$ is called $\mathfrak{A}$-principal *if there exists $A \in \mathfrak{A}$ so that $v \in V(A)$ holds.*

(b) An n-tuple $(b_1, \ldots, b_n)$ of vectors of $\mathbb{V}$ is called an $\mathfrak{A}$-adapted basis *of $\mathbb{V}$, if there exists an $A \in \mathfrak{A}$ so that $(b_1, \ldots, b_n)$ is an orthonormal basis of $V(A)$.*

2.8 Remark. In the case $\dim \mathbb{V} = 1$ all vectors of $\mathbb{V}$ are $\mathfrak{A}$-principal.

2.9 Proposition. *(a) $v \in \mathbb{V}$ is $\mathfrak{A}$-principal if and only if for some (and then for every) $A \in \mathfrak{A}$ there exists $\lambda \in \mathbb{S}^1$ so that $Av = \lambda v$ holds.*

(b) An n-tuple $(b_1, \ldots, b_n)$ of vectors of $\mathbb{V}$ is an $\mathfrak{A}$-adapted basis of $\mathbb{V}$ if and only if it is a unitary basis of $\mathbb{V}$ and there exists $A \in \mathfrak{A}$ so that $b_k \in V(A)$ holds for all $k \in \{1, \ldots, n\}$.

Proof. For (a). Let $v \in \mathbb{V}$ and $A \in \mathfrak{A}$ be given. Then, we have

$$\begin{aligned} v \text{ is } \mathfrak{A}\text{-principal} &\iff \exists\, \lambda \in \mathbb{S}^1 \ :\ v \in V(\overline{\lambda} A) \\ &\iff \exists\, \lambda \in \mathbb{S}^1 \ :\ \overline{\lambda} A v = v \iff \exists\, \lambda \in \mathbb{S}^1 \ :\ Av = \lambda v \,. \end{aligned}$$

For (b). Proposition 2.4(a) shows that $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ coincide on $V(A) \times V(A)$ for every $A \in \mathfrak{A}$; we also have $\mathbb{V} = V(A) \oplus JV(A)$. Via these two observations, the statement follows from Definition 2.7(b). $\square$

2.10 Definition. *Suppose $(\mathbb{V}, \mathfrak{A})$ and $(\mathbb{V}', \mathfrak{A}')$ are $\mathbb{C}$Q-spaces.*

(a) We call a $\mathbb{C}$-linear isometry $B : \mathbb{V} \to \mathbb{V}'$ a $\mathbb{C}$Q-isomorphism*, if*

$$\forall A \in \mathfrak{A} \ :\ B \circ A \circ B^{-1} \in \mathfrak{A}'$$

holds. In the case $(\mathbb{V}', \mathfrak{A}') = (\mathbb{V}, \mathfrak{A})$, we speak of a $\mathbb{C}$Q-automorphism*. We denote the set of $\mathbb{C}$Q-automorphisms of $(\mathbb{V}, \mathfrak{A})$ by* $\mathrm{Aut}(\mathfrak{A})$.

(b) We call a $\mathbb{C}$*-linear isometry* $B : \mathbb{V} \to \mathbb{V}$ *a* strict ℂQ-automorphism*, if*

$$\forall A \in \mathfrak{A} \; : \; B \circ A = A \circ B$$

holds. We denote the set of strict ℂQ*-automorphisms of* $(\mathbb{V}, \mathfrak{A})$ *by* $\mathrm{Aut}_s(\mathfrak{A})$ *.*

(c) An anti-linear map $B : \mathbb{V} \to \mathbb{V}'$ *is called a* ℂQ-anti-isomorphism*, if for every* $A' \in \mathfrak{A}'$*, the* $\mathbb{C}$*-linear map* $A' \circ B$ *is a* ℂQ*-isomorphism. In the case* $(\mathbb{V}', \mathfrak{A}') = (\mathbb{V}, \mathfrak{A})$*, we speak of a* ℂQ-anti-automorphism*. We denote the set of* ℂQ*-anti-automorphisms of* $(\mathbb{V}, \mathfrak{A})$ *by* $\overline{\mathrm{Aut}}(\mathfrak{A})$ *.*

(d) A complex linear subspace $U \subset \mathbb{V}$ *is called a* ℂQ-subspace *of* $(\mathbb{V}, \mathfrak{A})$ *if* U *is invariant under some (and then, under every)* $A \in \mathfrak{A}$ *. In this case* U *canonically becomes a* ℂQ*-space with the* ℂQ*-structure* $\{\, A|U \mid A \in \mathfrak{A} \,\}$*, which we call the* induced ℂQ-structure *of* U *.*

(e) For any subset M *of* $\mathbb{V}$*, we call the smallest* ℂQ*-subspace of* $\mathbb{V}$ *which contains* M *the* ℂQ-subspace generated by M *or the* ℂQ-span *of* M *. We denote this space by* $\mathrm{span}_{\mathfrak{A}} M$ *.*

(f) An injective $\mathbb{C}$*-linear map* $\iota : \mathbb{V} \to \mathbb{V}'$ *is called a* ℂQ-embedding *if* $\iota(\mathbb{V})$ *is a* ℂQ*-subspace of* $\mathbb{V}'$ *and* $\iota : \mathbb{V} \to \iota(\mathbb{V})$ *is a* ℂQ*-isomorphism.*

2.11 Examples. Let $(\mathbb{V}, \mathfrak{A})$ be a ℂQ-space. Then the map $\mathbb{V} \to \mathbb{V},\ v \mapsto \lambda \cdot v$ is a ℂQ-automorphism for any $\lambda \in \mathbb{S}^1$; it is a strict ℂQ-automorphism if and only if $\lambda \in \{\pm 1\}$ holds. The conjugations $A \in \mathfrak{A}$ are ℂQ-anti-automorphisms. In the case $\dim \mathbb{V} = 1$ we have $\mathrm{Aut}(\mathfrak{A}) = \mathrm{U}(\mathbb{V})$ and $\overline{\mathrm{Aut}}(\mathfrak{A}) = \overline{\mathrm{U}}(\mathbb{V})$.

2.12 Remarks. (a) To verify that some $\mathbb{C}$-linear isometry $B : \mathbb{V} \to \mathbb{V}'$ is a ℂQ-isomorphism of the ℂQ-spaces $(\mathbb{V}, \mathfrak{A})$ and $(\mathbb{V}', \mathfrak{A}')$, it suffices to check $B \circ A \circ B^{-1} \in \mathfrak{A}'$ for a single $A \in \mathfrak{A}$. Similarly, to verify that $B \in \mathrm{U}(\mathbb{V})$ is a strict ℂQ-automorphism, it suffices to check $B \circ A = A \circ B$ for a single $A \in \mathfrak{A}$.

(b) $\mathrm{Aut}(\mathfrak{A}) \cup \overline{\mathrm{Aut}}(\mathfrak{A})$ is a subgroup of the (abstract) group of $\mathbb{R}$-linear transformations of $\mathbb{V}$. $\mathrm{Aut}(\mathfrak{A})$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{A}) \cup \overline{\mathrm{Aut}}(\mathfrak{A})$ and contained in $\mathrm{U}(\mathbb{V})$; in Proposition 2.17, we will see that $\mathrm{Aut}(\mathfrak{A})$ is in fact a Lie subgroup of $\mathrm{U}(\mathbb{V})$. $\overline{\mathrm{Aut}}(\mathfrak{A})$ is a coset of $\mathrm{Aut}(\mathfrak{A})$ in $\mathrm{Aut}(\mathfrak{A}) \cup \overline{\mathrm{Aut}}(\mathfrak{A})$.

2.13 Proposition. *A linear subspace* U *of a* ℂQ*-space* $(\mathbb{V}, \mathfrak{A})$ *is a complex-*k*-dimensional* ℂQ*-subspace if and only if there exists* $A \in \mathfrak{A}$ *and a real-*k*-dimensional subspace* $W \subset V(A)$ *so that* $U = W \oplus JW$ *holds. If* U *is a* ℂQ*-subspace, then this representation can be achieved for every* $A \in \mathfrak{A}$ *.*

Proof. Suppose that U is a k-dimensional ℂQ-subspace of $\mathbb{V}$, let $A \in \mathfrak{A}$ be given and put $W := U \cap V(A)$. We will show that $U = W \oplus JW$ holds with this choice of W; it then follows that W is of real dimension k.

$W \oplus JW \subset U$ holds simply because of $W \subset U$ and U is a complex linear subspace. For the converse inclusion, let $v \in U$ be given. Because U is a ℂQ-subspace of $\mathbb{V}$, we have $x := \mathrm{Re}_A\, v \in$

$U \cap V(A) = W$ and $y := \mathrm{Im}_A v \in U \cap V(A) = W$. This shows that $v = x + Jy \in W \oplus JW$ holds.

Conversely, if U is a linear subspace of $\mathbb{V}$ so that $U = W \oplus JW$ holds, where W is a linear subspace of $V(A)$ for some $A \in \mathfrak{A}$, then U is clearly A-invariant and therefore also invariant under every $A' \in \mathfrak{A}$. Hence, U is a $\mathbb{C}$Q-subspace of $\mathbb{V}$. □

2.14 Proposition. *Suppose $(\mathbb{V}, \mathfrak{A})$ and $(\mathbb{V}', \mathfrak{A}')$ are n-dimensional $\mathbb{C}$Q-spaces. For a $\mathbb{C}$-linear map $B : \mathbb{V} \to \mathbb{V}'$ the following statements are equivalent:*

(a) B is a $\mathbb{C}$Q-isomorphism.

(b) B maps every $\mathfrak{A}$-adapted basis onto an $\mathfrak{A}'$-adapted basis.

(c) There exists an $\mathfrak{A}$-adapted basis which is mapped by B onto an $\mathfrak{A}'$-adapted basis.

Proof. For (a) $\Rightarrow$ (b). Obvious. *For (b) $\Rightarrow$ (c).* Trivial. *For (c) $\Rightarrow$ (a).* The hypothesis (c) means that there exist $A \in \mathfrak{A}$, $A' \in \mathfrak{A}'$ and an orthonormal basis $(b_1, \dots, b_n)$ of $V(A)$ so that $(Bb_1, \dots, Bb_n)$ is an orthonormal basis of $V(A')$. In particular, we have $B(V(A)) = V(A')$ and therefore also $B(JV(A)) = JV(A')$. It follows that $B \circ A = A' \circ B$ holds. Now, if $\lambda A \in \mathfrak{A}$ is an arbitrary element of $\mathfrak{A}$ ($\lambda \in \mathbb{S}^1$), we have

$$B \circ (\lambda A) \circ B^{-1} = \lambda \cdot B \circ A \circ B^{-1} = \lambda \cdot A' \in \mathfrak{A}'$$

and therefore B is a $\mathbb{C}$Q-isomorphism. □

2.15 Proposition. *Let $(\mathbb{V}, \mathfrak{A})$ and $(\mathbb{V}', \mathfrak{A}')$ be $\mathbb{C}$Q-spaces, $A \in \mathfrak{A}$, $A' \in \mathfrak{A}'$ and $L : V(A) \to V(A')$ an $\mathbb{R}$-linear map. Then there exists one and only one $\mathbb{C}$-linear map $L^{\mathbb{C}} : \mathbb{V} \to \mathbb{V}'$ and one and only one anti-linear map $L^{\overline{\mathbb{C}}} : \mathbb{V} \to \mathbb{V}'$ so that*

$$L^{\mathbb{C}}|V(A) = L^{\overline{\mathbb{C}}}|V(A) = L \tag{2.2}$$

holds, and these maps satisfy

$$L^{\mathbb{C}} \circ A = A' \circ L^{\mathbb{C}} \quad \text{and} \quad L^{\overline{\mathbb{C}}} \circ A = A' \circ L^{\overline{\mathbb{C}}}\,. \tag{2.3}$$

Furthermore, we have $\det_{\mathbb{C}}(L^{\mathbb{C}}) = \det_{\mathbb{R}}(L)$ and the following relationships between "qualities" of L and $L^{\mathbb{C}}$, $L^{\overline{\mathbb{C}}}$:

If L is ... ,	*then $L^{\mathbb{C}}$ is ...*	*and $L^{\overline{\mathbb{C}}}$ is ... .*
an $\mathbb{R}$-linear isomorphism	*a $\mathbb{C}$-linear isomorphism*	*an anti-linear isomorphism*
an $\mathbb{R}$-linear isometry	*a $\mathbb{C}$Q-isomorphism*	*a $\mathbb{C}$Q-anti-isomorphism*
self-adjoint	*Hermitian*	—
skew-adjoint	*skew-Hermitian*	—

We call $L^{\mathbb{C}}$ resp. $L^{\overline{\mathbb{C}}}$ the complexification *resp. the* anti-complexification *of L.*

Proof. Because we have $\mathbb{V} = V(A) \oplus JV(A)$, there can be at most one map $L^{\mathbb{C}}$ resp. $L^{\overline{\mathbb{C}}}$ which satisfies (2.2). To prove their existence, we define $L^{\mathbb{C}}$ and $L^{\overline{\mathbb{C}}}$ by

$$\forall v \in \mathbb{V} \ : \big(L^{\mathbb{C}}(v) := L(\mathrm{Re}_A\, v) + JL(\mathrm{Im}_A\, v) \quad \text{and} \quad L^{\overline{\mathbb{C}}}(v) := L(\mathrm{Re}_A\, v) - JL(\mathrm{Im}_A\, v) \big) \, . \tag{2.4}$$

It is obvious that the maps $L^{\mathbb{C}}$ and $L^{\overline{\mathbb{C}}}$ so defined are $\mathbb{R}$-linear, and that they satisfy Equation (2.2). Furthermore, for every $v \in \mathbb{V}$ we have (see Proposition 2.3(e))

$$L^{\mathbb{C}}(Jv) = L(\mathrm{Re}_A(Jv)) + JL(\mathrm{Im}_A(Jv)) = -L(\mathrm{Im}_A\, v) + JL(\mathrm{Re}_A\, v) = J(L^{\mathbb{C}}(v)) \, ,$$

hence $L^{\mathbb{C}}$ is in fact $\mathbb{C}$-linear; an analogous calculation shows that $L^{\overline{\mathbb{C}}}$ is anti-linear. We also have

$$L^{\mathbb{C}}(Av) = L(\mathrm{Re}_A(Av)) + JL(\mathrm{Im}_A(Av)) = L(\mathrm{Re}_A\, v) - JL(\mathrm{Im}_A\, v) = A'(L^{\mathbb{C}}(v)) \, ,$$

whence the equation for $L^{\mathbb{C}}$ in (2.3) follows; the equation for $L^{\overline{\mathbb{C}}}$ is shown the same way.

To show $\det_{\mathbb{C}}(L^{\mathbb{C}}) = \det_{\mathbb{R}}(L)$, we fix orthonormal bases $\mathcal{B} := (b_1, \dots, b_n)$ of $V(A)$ and $\mathcal{B}' := (b_1', \dots, b_{n'}')$ of $V(A')$. Then $\mathcal{B}$ and $\mathcal{B}'$ are also unitary bases of $\mathbb{V}$ resp. $\mathbb{V}'$, and the same matrix which represents the $\mathbb{R}$-linear map L with respect to the orthonormal bases $\mathcal{B}$ and $\mathcal{B}'$ also represents the $\mathbb{C}$-linear map $L^{\mathbb{C}}$ with respect to the unitary bases $\mathcal{B}$ and $\mathcal{B}'$. Consequently, $\det_{\mathbb{C}}(L^{\mathbb{C}}) = \det_{\mathbb{R}}(L)$ holds.

We now suppose that L is an $\mathbb{R}$-linear isomorphism and show that then $L^{\mathbb{C}}$ is a $\mathbb{C}$-linear isomorphism; the proof that $L^{\overline{\mathbb{C}}}$ is an anti-linear isomorphism runs analogously. The fact that L is a linear isomorphism implies in particular that $\dim V(A) = \dim V(A')$ and hence $\dim \mathbb{V} = \dim \mathbb{V}'$ holds. Therefore, it suffices to show that the kernel of $L^{\mathbb{C}}$ is trivial. Let $v \in \mathbb{V}$ be given so that

$$0 = L^{\mathbb{C}}(v) = L(\mathrm{Re}_A\, v) + JL(\mathrm{Im}_A\, v)$$

holds. Because we have $V(A') \perp JV(A')$ this equation implies $L(\mathrm{Re}_A\, v) = L(\mathrm{Im}_A\, v) = 0$ and thus, because L is injective, $\mathrm{Re}_A\, v = \mathrm{Im}_A\, v = 0$, hence $v = 0$.

If L is an $\mathbb{R}$-linear isometry, then $L^{\mathbb{C}}$ transforms any orthonormal basis of $V(A)$ into an orthonormal basis of $V(A')$ and therefore is a $\mathbb{C}$Q-isomorphism by Proposition 2.14, (c) $\Rightarrow$ (a). Also, $L^{\overline{\mathbb{C}}} = A' \circ L^{\mathbb{C}}$ is then a $\mathbb{C}$Q-anti-isomorphism.

The statement that L being self-adjoint (skew-adjoint) causes $L^{\mathbb{C}}$ to be Hermitian (skew-Hermitian) is proved by a direct calculation via Equations (2.4) and Proposition 2.4(b)(i).

□

2.16 Corollary. *Let $(\mathbb{V}, \mathfrak{A})$ and $(\mathbb{V}', \mathfrak{A}')$ be $\mathbb{C}$Q-spaces of dimension n resp. n'. Then there exists a $\mathbb{C}$Q-isomorphism $B : \mathbb{V} \to \mathbb{V}'$ if and only if $n = n'$ holds.*

Proof. Because any $\mathbb{C}$Q-isomorphism is in particular an isomorphism of linear spaces, $n = n'$ is a necessary condition for the existence of a $\mathbb{C}$Q-isomorphism $B : \mathbb{V} \to \mathbb{V}'$. Conversely, we suppose that $n = n'$ holds and fix $A \in \mathfrak{A}$ and $A' \in \mathfrak{A}'$; then $V(A)$ and $V(A')$ are both n-dimensional euclidean spaces. Therefore there exists a linear isometry $L : V(A) \to V(A')$. Proposition 2.15 shows that the complexification of L is a $\mathbb{C}$Q-isomorphism $\mathbb{V} \to \mathbb{V}'$. □

2.17 Proposition. *Let* $(\mathbb{V}, \mathfrak{A})$ *be an* n*-dimensional* $\mathbb{C}$Q*-space and* $A \in \mathfrak{A}$ *.*

(a) $\mathrm{Aut}_s(\mathfrak{A})$ *is a compact Lie subgroup of* $\mathrm{U}(\mathbb{V})$ *and*

$$\Psi_s : \mathrm{O}(V(A)) \to \mathrm{Aut}_s(\mathfrak{A}),\ L \mapsto L^{\mathbb{C}}$$

is an isomorphism of Lie groups with $\Psi_s^{-1}(B) = B|V(A)$ *for every* $B \in \mathrm{Aut}_s(\mathfrak{A})$ *. Consequently, the dimension of* $\mathrm{Aut}_s(\mathfrak{A})$ *is* $\frac{n(n-1)}{2}$ *and* $\mathrm{Aut}_s(\mathfrak{A})$ *has exactly two connected components. For* $B \in \mathrm{Aut}_s(\mathfrak{A})$ *, we have* $\det_{\mathbb{C}}(B) = \det_{\mathbb{R}}(B|V(A)) \in \{\pm 1\}$ *and*

$$B \in \mathrm{Aut}_s(\mathfrak{A})_0 \iff \det{}_{\mathbb{C}}(B) = 1 \ .$$

The Lie algebra $\mathfrak{aut}_s(\mathfrak{A}) \subset \mathfrak{u}(\mathbb{V})$ *of* $\mathrm{Aut}_s(\mathfrak{A})$ *is given by*[4]

$$\begin{aligned} \mathfrak{aut}_s(\mathfrak{A}) &= \{\, B \in \mathrm{End}_-(\mathbb{V}) \,|\, B \circ A = A \circ B \,\} \qquad (\textit{with } A \in \mathfrak{A}\,) \\ &= \{\, L^{\mathbb{C}} \,|\, L \in \mathfrak{o}(V(A)) \,\} \ . \end{aligned} \tag{2.5}$$

In the case $n \geq 2$ *, let us denote by* $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{V(A)}$ *resp.* $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathbb{V}}$ *the usual inner product*[5] *on* $\mathrm{End}(V(A))$ *resp. on* $\mathrm{End}(\mathbb{V})$ *; then the Killing form* $\varkappa$ *of* $\mathfrak{aut}_s(\mathfrak{A})$ *is given by*

$$\forall X, Y \in \mathfrak{aut}_s(\mathfrak{A}) \ : \ \varkappa(X,Y) = -(n-2) \cdot \langle\!\langle X|V(A),\ Y|V(A) \rangle\!\rangle_{V(A)} = -(n-2) \cdot \langle\!\langle X, Y \rangle\!\rangle_{\mathbb{V}} \ . \tag{2.6}$$

(b) $\mathrm{Aut}(\mathfrak{A})$ *is a compact Lie subgroup of* $\mathrm{U}(\mathbb{V})$ *and*

$$\Psi : \mathbb{S}^1 \times \mathrm{O}(V(A)) \to \mathrm{Aut}(\mathfrak{A}),\ (\lambda, L) \mapsto \lambda \cdot L^{\mathbb{C}}$$

is a two-fold covering map of Lie groups with $\ker \Psi = \{\pm(1, \mathrm{id}_{V(A)})\}$ *. Consequently, the dimension of* $\mathrm{Aut}(\mathfrak{A})$ *is* $1 + \frac{n(n-1)}{2}$ *. Moreover,* $\mathrm{Aut}(\mathfrak{A})$ *is connected if* n *is odd, whereas if* n *is even* $\mathrm{Aut}(\mathfrak{A})$ *has exactly two connected components. In both cases* $\mathrm{Aut}(\mathfrak{A})_0 = \Psi(\mathbb{S}^1 \times \mathrm{SO}(V(A))$ *holds. The Lie algebra* $\mathfrak{aut}(\mathfrak{A})$ *of* $\mathrm{Aut}(\mathfrak{A})$ *is given by*

$$\mathfrak{aut}(\mathfrak{A}) = \{\, \alpha\, J + X \,|\, \alpha \in \mathbb{R},\ X \in \mathfrak{aut}_s(\mathfrak{A}) \,\} \ . \tag{2.7}$$

Proof. For (a). Consider the differentiable map

$$f : \mathrm{U}(\mathbb{V}) \to \mathrm{U}(\mathbb{V}),\ B \mapsto B \circ A \circ B^{-1} \circ A^{-1} \ .$$

We have $\mathrm{Aut}_s(\mathfrak{A}) = f^{-1}(\{\mathrm{id}_{\mathbb{V}}\})$ and therefore the abstract group $\mathrm{Aut}_s(\mathfrak{A})$ is a closed subset of the compact Lie group $\mathrm{U}(\mathbb{V})$ and hence a compact Lie subgroup of $\mathrm{U}(\mathbb{V})$ (see [Var74], Theorem 2.12.6, p. 99).

[4] Here, as always, we identify $\mathfrak{u}(\mathbb{V})$ and $\mathfrak{o}(V(A))$ with the Lie algebra of skew-Hermitian endomorphisms on $\mathbb{V}$ resp. of skew-adjoint endomorphisms on $V(A)$; see the Introduction.

[5] With respect to any orthonormal basis $(b_1, \ldots, b_n)$ of $V(A)$, the inner product on $\mathrm{End}(V(A))$ is given by $\langle\!\langle B_1, B_2 \rangle\!\rangle_{V(A)} = \sum_{k=1}^n \langle B_1 b_k, B_2 b_k \rangle_{\mathbb{R}}$ for any $B_1, B_2 \in \mathrm{End}(V(A))$. Analogously, with respect to any unitary basis $(b_1, \ldots, b_n)$ of $\mathbb{V}$, the inner product on $\mathbb{V}$ is given by $\langle\!\langle B_1, B_2 \rangle\!\rangle_{\mathbb{V}} = \sum_{k=1}^n \langle B_1 b_k, B_2 b_k \rangle_{\mathbb{C}}$ for any $B_1, B_2 \in \mathrm{End}(\mathbb{V})$.

Ψ_s in fact maps into $\mathrm{Aut}_s(\mathfrak{A})$ by Proposition 2.15 and it obviously is a homomorphism of abstract groups. For every $L \in \mathrm{O}(V(A))$ we have $(L^{\mathbb{C}})|V(A) = L$; conversely for every $B \in \mathrm{Aut}_s(\mathfrak{A})$ we have $B|V(A) \in \mathrm{O}(V(A))$ and therefore the uniqueness statement for $L^{\mathbb{C}}$ in Proposition 2.15 shows that $(B|V(A))^{\mathbb{C}} = B$ holds. Therefore Ψ_s is bijective and Ψ_s^{-1} is as given in the proposition.

Ψ_s is differentiable, as the following argument shows: We fix an orthonormal basis $\mathcal{B} := (b_1, \dots, b_n)$ of $V(A)$, then $\mathcal{B}$ also is a unitary basis of $\mathbb{V}$. If we represent a given $L \in \mathrm{O}(V(A))$ as a matrix with respect to the orthonormal basis $\mathcal{B}$ of $V(A)$, then the same matrix represents $L^{\mathbb{C}} \in \mathrm{Aut}_s(\mathfrak{A})$ with respect to the unitary basis $\mathcal{B}$ of $\mathbb{V}$. Therefore the homomorphism Ψ_s is represented as a map of matrices with respect to the basis $\mathcal{B}$ simply by the inclusion map. It follows that Ψ_s is differentiable, hence an isomorphism of Lie groups, and we also see that its linearization is given by

$$(\Psi_s)_L : \mathfrak{o}(V(A)) \to \mathfrak{aut}_s(\mathfrak{A}),\ X \mapsto X^{\mathbb{C}} .$$

In particular, we have $\mathfrak{aut}_s(\mathfrak{A}) = (\Psi_s)_L(\mathfrak{o}(V(A)))$, whence Equation (2.5) follows.

It also follows from the above matrix consideration that $\det_{\mathbb{C}}(L^{\mathbb{C}}) = \det_{\mathbb{R}}(L) \in \{\pm 1\}$ holds for every $L \in \mathrm{O}(V(A))$.

For Equation (2.6): Let us denote the Killing form of $\mathfrak{o}(V(A))$ by $\varkappa_{\mathfrak{o}}$; as is well-known,

$$\forall L_1, L_2 \in \mathfrak{o}(V(A)) \ : \ \varkappa_{\mathfrak{o}}(L_1, L_2) = -(n-2) \cdot \langle\langle L_1, L_2 \rangle\rangle_{V(A)}$$

holds (see for example [IT91], Example II.2.4, p. 60). Because $(\Psi_s)_L^{-1} : \mathfrak{aut}_s(\mathfrak{A}) \to \mathfrak{o}(V(A)),\ X \mapsto X|V(A)$ is an isomorphism of Lie algebras, it preserves the Killing forms of the Lie algebras involved, whence the first equals sign in (2.6) follows. Moreover, if we again consider an orthonormal basis $\mathcal{B} := (b_1, \dots, b_n)$ of $V(A)$, we have for any $X, Y \in \mathfrak{aut}_s(\mathfrak{A})$: $\langle X b_k, Y b_k \rangle_{\mathbb{C}} = \langle X b_k, Y b_k \rangle_{\mathbb{R}} \in \mathbb{R}$ because X and Y leave the totally real subspace $V(A)$ of $\mathbb{V}$ invariant. Because $\mathcal{B}$ also is a unitary basis of $\mathbb{V}$, we therefore have $\langle\langle X, Y \rangle\rangle_{\mathbb{V}} = \langle\langle X|V(A), Y|V(A) \rangle\rangle_{V(A)}$, whence the second equals sign in (2.6) follows.

The remaining statements about $\mathrm{Aut}_s(\mathfrak{A})$ follow from the corresponding well-known facts about $\mathrm{O}(V(A))$.

For (b). We have $\mathrm{Aut}(\mathfrak{A}) = f^{-1}(\{\, \lambda \cdot \mathrm{id}_{\mathbb{V}} \,|\, \lambda \in \mathbb{S}^1 \,\})$, therefore the abstract subgroup $\mathrm{Aut}(\mathfrak{A})$ is a closed subset and hence a Lie subgroup of the compact Lie group $\mathrm{U}(\mathbb{V})$.

For any $(\lambda, L) \in \mathbb{S}^1 \times \mathrm{O}(V(A))$ we have $\Psi(\lambda, L) = (\lambda\, \mathrm{id}_{\mathbb{V}}) \circ L^{\mathbb{C}} \in \mathrm{Aut}(\mathfrak{A})$ by Proposition 2.15 and Example 2.11. Therefore the homomorphism of abstract groups Ψ in fact maps into $\mathrm{Aut}(\mathfrak{A})$; its differentiability follows from the differentiability of Ψ_s. To show that Ψ is surjective, let $B \in \mathrm{Aut}(\mathfrak{A})$ be given. Then we have $A' := B \circ A \circ B^{-1} \in \mathfrak{A}$, and therefore, there exists $\lambda \in \mathbb{S}^1$ so that $A' = \lambda^2 \cdot A$ holds. We have $B \circ A = \lambda^2 A \circ B$ and therefore $(\overline{\lambda} B) \circ A = A \circ (\overline{\lambda} B)$, whence $\overline{\lambda} B \in \mathrm{Aut}_s(\mathfrak{A})$ follows. We thus have $(\overline{\lambda} B)|V(A) \in \mathrm{O}(V(A))$ and $\Psi(\lambda, (\overline{\lambda} B)|V(A)) = B$.

Next we show

$$\ker \Psi = \{\pm(1, \mathrm{id}_{V(A)})\} \ ; \tag{2.8}$$

it follows that Ψ is a two-fold covering map of Lie groups. The inclusion "$\supset$" of Equation (2.8) is obvious. Conversely, let $(\lambda, L) \in \mathbb{S}^1 \times \mathrm{O}(V(A))$ be given so that $\mathrm{id}_{\mathbb{V}} = \Psi(\lambda, L) = \lambda \cdot L^{\mathbb{C}}$ holds. Then we have in particular $\lambda V(A) = V(A)$ and thus (note that $V(A) \neq \{0\}$ is totally real) $\lambda \in \{1, -1\}$, whence $L^{\mathbb{C}} = \lambda \,\mathrm{id}_{\mathbb{V}}$ follows. Thus we have shown $(\lambda, L) = \pm(1, \mathrm{id}_{V(A)})$, completing the proof of Equation (2.8).

It follows that we have

$$\dim \mathrm{Aut}(\mathfrak{A}) = \dim(\mathbb{S}^1 \times \mathrm{O}(V(A))) = 1 + \tfrac{n(n-1)}{2} \,.$$

To investigate the connectedness of $\mathrm{Aut}(\mathfrak{A})$, we note that $G := \mathbb{S}^1 \times \mathrm{O}(V(A))$ has exactly two connected components, namely $G_0 = \mathbb{S}^1 \times \mathrm{SO}(V(A))$ and $\mathbb{S}^1 \times \{\, L \in \mathrm{O}(V(A)) \mid \det L = -1 \,\} =: G_1$. Also, we have as a trivial consequence of Equation (2.8):

$$\forall\, (\lambda_1, L_1), (\lambda_2, L_2) \in G \;:\; (\; \Psi(\lambda_1, L_1) = \Psi(\lambda_2, L_2) \iff (\lambda_2, L_2) = \pm(\lambda_1, L_1) \;) \,. \tag{2.9}$$

In the case of odd n, we have $\det(-L) = -\det L$ for $L \in \mathrm{O}(V(A))$, therefore Equation (2.9) shows that every given $B \in \mathrm{Aut}(\mathfrak{A})$ has pre-images under Ψ in both connected components of G. It follows that $\Psi|G_0 : G_0 \to \mathrm{Aut}(\mathfrak{A})$ is an isomorphism of Lie groups and therefore $\mathrm{Aut}(\mathfrak{A})$ is connected. On the other hand, in the case of even n, we have $\det(-L) = \det L$ for $L \in \mathrm{O}(V(A))$, therefore Equation (2.9) shows that both pre-images of a given $B \in \mathrm{Aut}(\mathfrak{A})$ are contained in the same connected component of G. Therefore, G_0 and G_1 are mapped by Ψ onto disjoint, non-empty, connected, open subsets of $\mathrm{Aut}(\mathfrak{A})$ which together cover all of $\mathrm{Aut}(\mathfrak{A})$. This shows that $\mathrm{Aut}(\mathfrak{A})$ has exactly two connected components, namely $\Psi(G_0) = \mathrm{Aut}(\mathfrak{A})_0$ and $\Psi(G_1)$.

Finally, if we identify the Lie algebra of the Lie group $\mathbb{S}^1 \subset \mathbb{C}$ with its "arrowed" tangent space $\overrightarrow{T_1\mathbb{S}^1} = i\mathbb{R}$, the linearization of Ψ is given by

$$\Psi_L : i\mathbb{R} \oplus \mathfrak{o}(V(A)) \to \mathfrak{aut}(\mathfrak{A}),\; (i\alpha, X) \mapsto \alpha\, J + X^{\mathbb{C}} \,.$$

Because Ψ is a covering map of Lie groups, we have $\mathfrak{aut}(\mathfrak{A}) = \Psi_L(\mathbb{R} \oplus \mathfrak{o}(V(A)))$, and therefore Equation (2.7) follows. □

2.18 Proposition. *Let* $(\mathbb{V}, \mathfrak{A})$ *be a* $\mathbb{C}$Q*-space,* $A \in \mathfrak{A}$*, and*

$$\beta : \mathbb{V} \times \mathbb{V} \to \mathbb{C},\; (v, w) \mapsto \langle v, Aw \rangle_{\mathbb{C}}$$

the non-degenerate, symmetric bilinear form induced by A*. We consider the subgroups*

$$\begin{aligned} \mathrm{O}(\mathbb{V}, \beta) &:= \{\, B \in \mathrm{GL}(\mathbb{V}) \mid \forall v, w \in \mathbb{V} : \beta(Bv, Bw) = \beta(v, w) \,\} \\ \textit{and} \quad \mathrm{SO}(\mathbb{V}, \beta) &:= \{\, B \in \mathrm{O}(\mathbb{V}, \beta) \mid \det(B) = 1 \,\} \end{aligned}$$

of $\mathrm{GL}(\mathbb{V})$*. Then we have*

(a) $\mathrm{Aut}_s(\mathfrak{A}) = \mathrm{U}(\mathbb{V}) \cap \mathrm{O}(\mathbb{V}, \beta)$.

(b) $\mathrm{Aut}_s(\mathfrak{A})_0 = \mathrm{U}(\mathbb{V}) \cap \mathrm{SO}(\mathbb{V}, \beta)$.

Proof. For (a). Let $B \in \mathrm{U}(\mathbb{V})$ be given. Then we have for every $v, w \in \mathbb{V}$

$$\begin{aligned}\beta(Bv, Bw) - \beta(v, w) &= \langle Bv, ABw\rangle_{\mathbb{C}} - \langle v, Aw\rangle_{\mathbb{C}} = \langle Bv, ABw\rangle_{\mathbb{C}} - \langle Bv, BAw\rangle_{\mathbb{C}} \\ &= \langle Bv, (A \circ B - B \circ A)w\rangle_{\mathbb{C}} \;.\end{aligned}$$

This shows that $B \in \mathrm{O}(\mathbb{V}, \beta)$ holds if and only if we have $A \circ B = B \circ A$ and thus $B \in \mathrm{Aut}_s(\mathfrak{A})_0$.

For (b). This is a consequence of (a) and Proposition 2.17(a). □

2.3 Isotropic subspaces

Let $(\mathbb{V}, \mathfrak{A})$ be an n-dimensional $\mathbb{C}$Q-space.

2.19 Definition. *(a) The elements of $\widehat{Q}(A) \cup \{0\}$ (with $A \in \mathfrak{A}$) are called* isotropic vectors *of the $\mathbb{C}$Q-space $(\mathbb{V}, \mathfrak{A})$. In other words, $v \in \mathbb{V}$ is called isotropic if $\langle v, Av\rangle_{\mathbb{C}} = 0$ holds for some (and then, for every) $A \in \mathfrak{A}$.*

(b) A (real or complex) linear subspace $W \subset \mathbb{V}$ is called an isotropic subspace *of the $\mathbb{C}$Q-space $(\mathbb{V}, \mathfrak{A})$ if every $w \in W$ is isotropic in $(\mathbb{V}, \mathfrak{A})$.*

2.20 Proposition. *Let $W \subset \mathbb{V}$ be an isotropic subspace, $A \in \mathfrak{A}$ and $v, w \in W$. Then we have:*

(a) $\langle v, Aw\rangle_{\mathbb{C}} = 0$.

(b) $\langle \mathrm{Re}_A\, v, \mathrm{Re}_A\, w\rangle_{\mathbb{R}} = \langle \mathrm{Im}_A\, v, \mathrm{Im}_A\, w\rangle_{\mathbb{R}} = \frac{1}{2}\langle v, w\rangle_{\mathbb{R}}$.

(c) $\langle \mathrm{Re}_A\, v, \mathrm{Im}_A\, w\rangle_{\mathbb{R}} = -\langle \mathrm{Im}_A\, v, \mathrm{Re}_A\, w\rangle_{\mathbb{R}}$*; in particular* $\langle \mathrm{Re}_A\, v, \mathrm{Im}_A\, v\rangle_{\mathbb{R}} = 0$.

(d) The "complex closure" $\widehat{W} := W + JW$ also is an isotropic subspace of $\mathbb{V}$.

(e) The $\mathbb{R}$-linear maps $\mathrm{Re}_A | W : W \to V(A)$ and $\mathrm{Im}_A | W : W \to V(A)$ are injective, and the map $\tau := (\mathrm{Im}_A \circ (\mathrm{Re}_A | W)^{-1}) : \mathrm{Re}_A(W) \to \mathrm{Im}_A(W)$ is an $\mathbb{R}$-linear isometry so that

$$W = \{\, x + J\tau x \,|\, x \in \mathrm{Re}_A(W)\,\} \tag{2.10}$$

holds.

(f) In the situation of (e), W is a complex subspace if and only if $\mathrm{Re}_A(W) = \mathrm{Im}_A(W) =: Y$ holds and $\tau : Y \to Y$ is a complex structure on Y. W is totally real if and only if $\mathrm{Re}_A(W) \perp \mathrm{Im}_A(W)$ holds.

(g) For every $w \in W$, we have $w + Aw \in V(A)$, and if $\dim_{\mathbb{R}} W = \dim_{\mathbb{C}} \mathbb{V}$ holds, then every $x \in V(A)$ can be obtained in this way.

Proof. For (a). $\beta : W \times W \to \mathbb{C}$, $(v,w) \mapsto \langle v, Aw\rangle_{\mathbb{C}}$ is a symmetric bilinear form. The quadratic form corresponding to β vanishes because W is isotropic, and therefore we have for any $v, w \in W$: $\beta(v,w) = \frac{1}{2}\left(\beta(v+w,v+w) - \beta(v,v) - \beta(w,w)\right) = 0$.

For (b) and (c). From Proposition 2.4(b)(i), we get

$$\langle v, w\rangle_{\mathbb{R}} = \langle \mathrm{Re}_A\, v, \mathrm{Re}_A\, w\rangle_{\mathbb{R}} + \langle \mathrm{Im}_A\, v, \mathrm{Im}_A\, w\rangle_{\mathbb{R}} \;, \tag{2.11}$$

and from (a) we obtain by Proposition 2.4(c)(i)

$$0 = \langle v, Aw\rangle_{\mathbb{C}} = \langle \mathrm{Re}_A\, v, \mathrm{Re}_A\, w\rangle_{\mathbb{R}} - \langle \mathrm{Im}_A\, v, \mathrm{Im}_A\, w\rangle_{\mathbb{R}} + i \cdot \left(\langle \mathrm{Re}_A\, v, \mathrm{Im}_A\, w\rangle_{\mathbb{R}} + \langle \mathrm{Im}_A\, v, \mathrm{Re}_A\, w\rangle_{\mathbb{R}}\right)$$

and consequently

$$\langle \mathrm{Re}_A\, v, \mathrm{Re}_A\, w\rangle_{\mathbb{R}} = \langle \mathrm{Im}_A\, v, \mathrm{Im}_A\, w\rangle_{\mathbb{R}} \;, \tag{2.12}$$

$$\langle \mathrm{Re}_A\, v, \mathrm{Im}_A\, w\rangle_{\mathbb{R}} = -\langle \mathrm{Im}_A\, v, \mathrm{Re}_A\, w\rangle_{\mathbb{R}} \;. \tag{2.13}$$

By combining Equations (2.11) and (2.12) we obtain (b), whereas Equation (2.13) proves (c).

For (d). Let $\widehat{v} \in \widehat{W}$ be given, say $\widehat{v} = v_1 + Jv_2$ with $v_1, v_2 \in W$. Then we have

$$\begin{aligned} \langle \widehat{v}, A\widehat{v}\rangle_{\mathbb{C}} &= \langle v_1 + Jv_2, Av_1 - JAv_2\rangle_{\mathbb{C}} = \langle v_1, Av_1\rangle_{\mathbb{C}} - \langle v_1, JAv_2\rangle_{\mathbb{C}} + \langle Jv_2, Av_1\rangle_{\mathbb{C}} - \langle Jv_2, JAv_2\rangle_{\mathbb{C}} \\ &= \langle v_1, Av_1\rangle_{\mathbb{C}} - \langle v_1, JAv_2\rangle_{\mathbb{C}} - \langle v_2, JAv_1\rangle_{\mathbb{C}} - \langle v_2, Av_2\rangle_{\mathbb{C}} \;. \end{aligned} \tag{2.14}$$

The first and the fourth summand in (2.14) vanish because v_1 and v_2 are isotropic; the second and the third summand vanish by (a), note that $J \circ A = i\,A \in \mathfrak{A}$ holds. Thus we have shown $\langle \widehat{v}, A\widehat{v}\rangle_{\mathbb{C}} = 0$, and hence $\widehat{W}$ is isotropic.

For (e). (b) shows that for $v \in W$ either of the conditions $\mathrm{Re}_A\, v = 0$ and $\mathrm{Im}_A\, v = 0$ implies $v = 0$. Therefore the surjective $\mathbb{R}$-linear maps

$$\mathcal{R} := (\mathrm{Re}_A |W) : W \to \mathrm{Re}_A(W) \quad \text{and} \quad \mathcal{I} := (\mathrm{Im}_A |W) : W \to \mathrm{Im}_A(W)$$

are linear isomorphisms, and consequently the linear map $\tau = \mathcal{I} \circ \mathcal{R}^{-1} : \mathrm{Re}_A(W) \to \mathrm{Im}_A(W)$ also is a linear isomorphism. τ satisfies Equation (2.10) and (b) shows that τ is a linear isometry.

For (f). Let $\tau : \mathrm{Re}_A(W) \to \mathrm{Im}_A(W)$ be the linear isometry from (e). Suppose that W is a complex subspace and let $x \in \mathrm{Re}_A(W)$ be given. Then we have $v := x + J(\tau x) \in W$ and thus also $Jv \in W$. Jv can be calculated in two different ways:

$$\begin{aligned} Jv &= J(x + J(\tau x)) = -\tau x + Jx \\ &= \mathrm{Re}_A(Jv) + J\tau(\mathrm{Re}_A(Jv)) \;; \end{aligned}$$

thus, we obtain

$$\mathrm{Re}_A(Jv) = -\tau x \quad \text{and} \quad \tau(\mathrm{Re}_A(Jv)) = x \;, \tag{2.15}$$

hence, we see that $x = \tau(\mathrm{Re}_A(Jv)) \in \mathrm{Im}_A(W)$ holds. By varying x, we obtain $\mathrm{Re}_A(W) \subset \mathrm{Im}_A(W)$; because $\mathrm{Re}_A(W)$ and $\mathrm{Im}_A(W)$ have the same dimension (τ is an isomorphism between them), it follows that we have $\mathrm{Re}_A(W) = \mathrm{Im}_A(W) =: Y$. Equation (2.15) also shows that $\tau(\tau x) = -\tau(\mathrm{Re}_A(Jv)) = -x$ holds for $x \in Y$ and therefore τ is a complex structure on Y.

Conversely, we now suppose that τ is a complex structure on $\mathrm{Re}_A(W) = \mathrm{Im}_A(W) =: Y$ and let $v \in W$ be given, say $v = x + J(\tau x)$ with $x \in \mathrm{Re}_A(W)$. Then, we also have $\tau x \in \mathrm{Im}_A(W) = Y$ and therefore

$$W \ni \tau x + J\tau(\tau x) = \tau x - Jx = -J(x + J(\tau x)) = -Jv ,$$

hence $Jv \in W$. This shows that W is a complex subspace of $\mathbb{V}$.

To prove the characterization of totally-real isotropic subspaces, we calculate $\langle v, Jw\rangle_{\mathbb{R}}$ for $v, w \in W$. By Proposition 2.4(b)(ii) and part (c) of the present proposition, we obtain

$$\langle v, Jw\rangle_{\mathbb{R}} = \langle \mathrm{Im}_A v, \mathrm{Re}_A w\rangle_{\mathbb{R}} - \langle \mathrm{Re}_A v, \mathrm{Im}_A w\rangle_{\mathbb{R}} = (-2) \cdot \langle \mathrm{Re}_A v, \mathrm{Im}_A w\rangle_{\mathbb{R}} .$$

This equality shows that W is totally-real (meaning that $\langle v, Jw\rangle_{\mathbb{R}} = 0$ holds for all $v, w \in W$) if and only if $\mathrm{Re}_A(W) \perp \mathrm{Im}_A(W)$ holds (meaning that $\langle \mathrm{Re}_A v, \mathrm{Im}_A w\rangle_{\mathbb{R}} = 0$ holds for all $v, w \in W$).

For (g). Let $w \in W$ be given, then we have $A(w+Aw) = Aw+w$ and therefore $w+Aw \in V(A)$. By (a), we have $A(W) \perp W$, and therefore the $\mathbb{R}$-linear map $W \to V(A),\ w \mapsto w + Aw$ is injective; in the case $\dim_{\mathbb{R}} W = \dim_{\mathbb{C}} \mathbb{V} = \dim_{\mathbb{R}} V(A)$ it is therefore also surjective. □

2.21 Proposition. *Let $A \in \mathfrak{A}$, Y_1, Y_2 be linear subspaces of $V(A)$ and $\tau : Y_1 \to Y_2$ be a linear isometry, and put*

$$W := \{\, x + J(\tau x) \,|\, x \in Y_1 \,\} .$$

Furthermore, suppose that either of the following two cases holds:

(i) $Y_1 = Y_2 =: Y$ *and* $\tau : Y \to Y$ *is a complex structure on* Y.

(ii) $Y_1 \perp Y_2$.

Then W is an isotropic subspace of $\mathbb{V}$; it is a complex subspace in case (i) and a totally real subspace in case (ii).

Proof. Let $v \in W$ be given, say $v = x + J(\tau x)$ with $x \in Y_1$. Both in case (i) and in case (ii) we have $\langle x, \tau x\rangle_{\mathbb{R}} = 0$ and therefore by Proposition 2.4(c)(iii)

$$\langle v, Av\rangle_{\mathbb{C}} = \|x\|^2 - \underbrace{\|\tau x\|^2}_{=\|x\|^2} + 2i\,\langle x, \tau x\rangle_{\mathbb{R}} = 0 .$$

This shows that W is isotropic. The statements about W being complex resp. totally real were already shown in Proposition 2.20(f). □

2.22 Corollary. *If W is an isotropic subspace of $\mathbb{V}$, we have*

$$\dim_{\mathbb{R}} W \leq \begin{cases} n & \text{for } n \text{ even} \\ n-1 & \text{for } n \text{ odd} \end{cases}, \tag{2.16}$$

and equality can be attained. If W is a complex isotropic subspace, we have $\dim_{\mathbb{C}} W \leq \left[\frac{n}{2}\right]$, and equality can be attained.

Proof. Let an isotropic subspace W of $\mathbb{V}$ be given. Proposition 2.20(e) shows that $\dim_{\mathbb{R}} W = \dim_{\mathbb{R}}(\mathrm{Re}_A(W)) \leq \dim V(A) = n$ holds.

To complete the proof of the inequality (2.16), we have to show that $\dim_{\mathbb{R}} W = n$ is possible only if n is even. We suppose that there exists an isotropic subspace W of $\mathbb{V}$ with $\dim_{\mathbb{R}} W = n$. By Proposition 2.20(d), $\widehat{W} := W + JW$ also is an isotropic subspace of $\mathbb{V}$. On the one hand, we have $W \subset \widehat{W}$ and therefore $\dim_{\mathbb{R}} \widehat{W} \geq \dim_{\mathbb{R}} W = n$, on the other hand, we have $\dim_{\mathbb{R}} \widehat{W} \leq n$ because $\widehat{W}$ is isotropic. Therefore $\dim_{\mathbb{R}} \widehat{W} = n$ holds. Because $\widehat{W}$ is a complex linear space, we see that n is even.

The inequality in the complex case is an immediate consequence of (2.16).

To show that equality can be attained in both inequalities, we fix $A \in \mathfrak{A}$ and put $Y := V(A)$ in the case of even n, whereas we fix an $(n-1)$-dimensional subspace Y of $V(A)$ in the case of odd n. In either case, Y is of even real dimension, so there exists an orthogonal complex structure $\tau : Y \to Y$. Proposition 2.21 shows that $W := \{\, x + J(\tau x) \,|\, x \in Y \,\}$ is a complex isotropic subspace of $\mathbb{V}$. We have $2\dim_{\mathbb{C}} W = \dim_{\mathbb{R}} W = \dim_{\mathbb{R}} Y$, and therefore for this W equality is attained in both inequalities of the proposition. □

2.4 Complex quadrics and $\mathbb{C}$Q-spaces

Using the language of $\mathbb{C}$Q-spaces, we can rephrase central results of Chapter 1 more succinctly.

Let $\mathbb{V}$ be a unitary space. Proposition 1.10 shows that there is a one-to-one correspondence between $\mathbb{C}$Q-structures on $\mathbb{V}$ and complex quadrics in $\mathbb{P}(\mathbb{V})$. If $\mathfrak{A}$ is a $\mathbb{C}$Q-structure on $\mathbb{V}$, we therefore call the quadric $Q(\mathfrak{A})$ characterized by $Q(\mathfrak{A}) = Q(A)$ for all $A \in \mathfrak{A}$ the complex quadric belonging to the $\mathbb{C}$Q-structure $\mathfrak{A}$. Similarly, we define $\widehat{Q}(\mathfrak{A})$ and $\widetilde{Q}(\mathfrak{A})$.

2.23 Proposition. *Let $(\mathbb{V}, \mathfrak{A})$ be a $\mathbb{C}$Q-space and $A \in \mathfrak{A}$. Then we have*

(a) $\widehat{Q}(\mathfrak{A}) = \{\, x + Jy \,|\, x, y \in V(A),\ \|x\| = \|y\| \neq 0,\ x \perp y \,\}$

(b) $\widetilde{Q}(\mathfrak{A}) = \{\, x + Jy \,|\, x, y \in V(A),\ \|x\| = \|y\| = \frac{1}{\sqrt{2}},\ x \perp y \,\}$

Proof. For (a). Let $v \in \mathbb{V} \setminus \{0\}$ be given, say $v = x + Jy$ with $x, y \in V(A)$. Then we have by Proposition 2.4(c)(iii):

$$v \in \widehat{Q}(\mathfrak{A}) \;\Leftrightarrow\; \langle v, Av \rangle_{\mathbb{C}} = 0 \;\Leftrightarrow\; \|x\|^2 - \|y\|^2 + 2i\,\langle x, y \rangle = 0 \;\Leftrightarrow\; \big(\, \|x\|^2 = \|y\|^2 \overset{v \neq 0}{\neq} 0 \text{ and } x \perp y \,\big)\,.$$

For (b). Because we have $\widetilde{Q}(\mathfrak{A}) = \widehat{Q}(\mathfrak{A}) \cap \mathbb{S}(\mathbb{V})$, this follows from (a). □

2.24 Remark. Proposition 2.23(b) shows that $\widetilde{Q}(\mathfrak{A})$ is homothetic to the Stiefel manifold of orthonormal 2-frames in $V(A)$. Consequently, $Q(\mathfrak{A})$ is homothetic to the Grassmann manifold of oriented 2-planes in $V(A)$.

2.25 Theorem. *Let $(\mathbb{V},\mathfrak{A})$ be a $\mathbb{C}$Q-space and $p \in Q := Q(\mathfrak{A})$.*

(a) The set of shape operators

$$\mathfrak{A}(Q,p) := \{\, A^Q_\zeta \,|\, \zeta \in \perp^1_p(Q \hookrightarrow \mathbb{P}(\mathbb{V}))\,\}$$

is a $\mathbb{C}$Q-structure on the unitary space T_pQ. In the sequel, we will always regard T_pQ as a $\mathbb{C}$Q-space in this way.

(b) Let $z \in \pi^{-1}(\{p\})$ (where $\pi : \mathbb{S}(\mathbb{V}) \to \mathbb{P}(\mathbb{V})$ is the Hopf fibration) and $A' \in \mathfrak{A}(Q,p)$ be given. Then there exists one and only one $A \in \mathfrak{A}$ so that the following diagram commutes:

$$\begin{tikzcd} \overrightarrow{\mathcal{H}_zQ} \arrow[r, "A"] \arrow[d, "\Phi"'] & \overrightarrow{\mathcal{H}_zQ} \arrow[d, "\Phi"] \\ T_pQ \arrow[r, "A'"'] & T_pQ\,. \end{tikzcd}$$

Here the map $\Phi : \overrightarrow{\mathcal{H}_zQ} \to T_pQ$ is characterized by $\Phi(\overrightarrow{v}) = \pi_* v$ for all $v \in \mathcal{H}_zQ$.

We call A the lift of A' at z.

Proof. For (a). We fix $A \in \mathfrak{A}$ and consider the fields ξ and C constructed in Section 1.3 with respect to this A. Then we have by Proposition 1.15

$$\mathfrak{A}(Q,p) = \{\, A^Q_{\xi(z)} \,|\, z \in \pi^{-1}(\{p\})\,\}\,.$$

Fixing some $z_0 \in \pi^{-1}(\{p\})$, we further have by Propositions 1.15 and 1.14

$$\begin{aligned}\mathfrak{A}(Q,p) &= \{\, A^Q_{\xi(\lambda\, z_0)} \,|\, \lambda \in \mathbb{S}^1\,\} = \{\, A^Q_{\lambda^{-2}\,\xi(z_0)} \,|\, \lambda \in \mathbb{S}^1\,\} \\ &= \{\, \lambda^{-2}\, A^Q_{\xi(z_0)} \,|\, \lambda \in \mathbb{S}^1\,\} = \{\, \lambda\, A^Q_{\xi(z_0)} \,|\, \lambda \in \mathbb{S}^1\,\}\,.\end{aligned}$$

Theorem 1.16 shows that $A^Q_{\xi(z_0)}$ is a conjugation on T_pQ and therefore we see that $\mathfrak{A}(Q,p)$ is a $\mathbb{C}$Q-structure on the unitary space T_pQ.

For (b). We let $z \in \pi^{-1}(\{p\})$ and $A' \in \mathfrak{A}(Q,p)$ be given. As we saw in the proof of (a), there exists $\lambda \in \mathbb{S}^1$ so that $A' = \lambda\, A^Q_{\xi(z)}$ holds. If we replace A by $\lambda A \in \mathfrak{A}$, then with this conjugation the assertion (b) is fulfilled, see Diagrams (1.13) in Theorem 1.16. ☐

2.26 Theorem. *Let $(\mathbb{V},\mathfrak{A})$ be a $\mathbb{C}$Q-space. We put $Q := Q(\mathfrak{A})$, denote by $\pi : \mathbb{S}(\mathbb{V}) \to \mathbb{P}(\mathbb{V})$ the Hopf fibration, and for $z \in \widetilde{Q}(\mathfrak{A})$ by $\mathcal{H}_zQ \subset T_z\mathbb{V}$ the horizontal lift of $T_{\pi(z)}Q$ with respect to π at z.*

Then $\mathcal{H}_z Q$ *is a* $\mathbb{C}$Q*-subspace of the* $\mathbb{C}$Q*-space* $(T_z\mathbb{V}, \mathbb{S}^1 \cdot C_z)$, *where* C *is the endomorphism field from Section 1.3. Moreover, we have*

$$\overrightarrow{\mathcal{H}_z Q} = \mathrm{span}_{\mathfrak{A}}\{z\}^{\perp} \tag{2.17}$$
$$= \{\, v \in \mathbb{V} \mid \mathrm{Re}_A\, v, \mathrm{Im}_A\, v \;\perp\; \mathrm{Re}_A\, z, \mathrm{Im}_A\, z \,\} \tag{2.18}$$

(where $A \in \mathfrak{A}$ *is arbitrary), and* $\pi_*|\mathcal{H}_z Q : \mathcal{H}_z Q \to T_{\pi(z)}Q$ *is a* $\mathbb{C}$Q*-isomorphism.*

Proof. Let $A \in \mathfrak{A}$ be given. We have $\mathrm{span}_{\mathfrak{A}}\{z\} = \mathbb{C}z \oplus \mathbb{C}(Az)$, therefore (2.17) follows from Proposition 1.13(b). It follows that $\overrightarrow{\mathcal{H}_z Q}$ is a $\mathbb{C}$Q-subspace of $\mathbb{V}$ and hence, $\mathcal{H}_z Q$ is a $\mathbb{C}$Q-subspace of $(T_z\mathbb{V}, \mathbb{S}^1 \cdot C_z)$. To prove Equation (2.18), let $v \in \mathbb{V}$ be given. We abbreviate $x := \mathrm{Re}_A\, z$, $y := \mathrm{Im}_A\, z$, $v_x := \mathrm{Re}_A\, v$ and $v_y := \mathrm{Im}_A\, v$. Then we have by Proposition 1.13(b) and Proposition 2.4(b)(i),(c)(i):

$$\begin{aligned} v \in \overrightarrow{\mathcal{H}_z Q} &\iff \langle v, z\rangle_{\mathbb{C}} = \langle v, Az\rangle_{\mathbb{C}} = 0 \\ &\iff \langle v_x, x\rangle_{\mathbb{R}} + \langle v_y, y\rangle_{\mathbb{R}} = \langle v_y, x\rangle_{\mathbb{R}} - \langle v_x, y\rangle_{\mathbb{R}} = \langle v_x, x\rangle_{\mathbb{R}} - \langle v_y, y\rangle_{\mathbb{R}} = \langle v_x, y\rangle_{\mathbb{R}} + \langle v_y, x\rangle_{\mathbb{R}} = 0 \\ &\iff \langle v_x, x\rangle_{\mathbb{R}} = \langle v_y, y\rangle_{\mathbb{R}} = \langle v_y, x\rangle_{\mathbb{R}} = \langle v_x, y\rangle_{\mathbb{R}} = 0 \,. \end{aligned}$$

This proves Equation (2.18). Finally, the fact that $\pi_*|\mathcal{H}_z Q : \mathcal{H}_z Q \to T_{\pi(z)}Q$ is a $\mathbb{C}$Q-isomorphism follows from Theorem 1.16. $\square$

2.27 Proposition. *Let* $(\mathbb{V}, \mathfrak{A})$ *be a* $\mathbb{C}$Q*-space of dimension* $n \geq 2$. *Then* $\widetilde{Q}(\mathfrak{A})$ *(and therefore also* $\widehat{Q}(\mathfrak{A})$*) is not contained in a proper* $\mathbb{R}$*-linear subspace of* $\mathbb{V}$.

Proof. It suffices to give an $\mathbb{R}$-basis of $\mathbb{V}$ which consists entirely of elements of $\widetilde{Q}(\mathfrak{A})$. For this purpose, let us fix $A \in \mathfrak{A}$ and an orthonormal basis $(b_1, \dots, b_n)$ of $V(A)$. We put $v_k := \frac{1}{\sqrt{2}}(b_k + Jb_{k+1})$ for $1 \leq k \leq n-1$ and $v_n := \frac{1}{\sqrt{2}}(b_n + Jb_1)$. Then $(v_1, \dots, v_n, Jv_1, \dots, Jv_n)$ is an $\mathbb{R}$-basis of $\mathbb{V}$, which consists of elements of $\widetilde{Q}(\mathfrak{A})$ by Proposition 2.23(b). $\square$

2.5 The $\mathfrak{A}$-angle

In a unitary space $\mathbb{V}$, all unit vectors are geometrically equivalent in the sense that the unitary group of $\mathbb{V}$ acts transitively on $\mathbb{S}(\mathbb{V})$. However, if we equip $\mathbb{V}$ with a $\mathbb{C}$Q-structure, this additional structure provides a differentiation of the "geometric quality" of unit vectors; this is to say that $\mathrm{Aut}(\mathfrak{A})$ does not act transitively on $\mathbb{S}(\mathbb{V})$. As we will see, the orbit space of the latter action can be parameterized by a number $t \in [0, \frac{\pi}{4}]$; the explicit parametrization of the orbit space in this way is due to H. Reckziegel (see [Rec95], Proposition 3 and Proposition 4(g)). We call the parameter t corresponding to such an orbit the $\mathfrak{A}$-*angle* of that orbit, or of its elements.

Let $(\mathbb{V}, \mathfrak{A})$ be an n-dimensional $\mathbb{C}$Q-space with $n \geq 2$.

2.28 Theorem. *Let $v \in \mathbb{V} \setminus \{0\}$ be given.*

(a) There is one and only one number $\varphi(v) \in [0, \frac{\pi}{4}]$ so that

$$\forall A \in \mathfrak{A} \;:\; |\langle v, Av \rangle_{\mathbb{C}}| = \cos(2\varphi(v)) \cdot \|v\|^2 \tag{2.19}$$

holds. We call $\varphi(v)$ the $\mathfrak{A}$-angle or characteristic angle *of v.*

(b) There exists $A \in \mathfrak{A}$ so that

$$\langle v, Av \rangle_{\mathbb{C}} \text{ is real and } \geq 0 \, . \tag{2.20}$$

For $\varphi(v) \neq \frac{\pi}{4}$, A is determined uniquely by (2.20); for $\varphi(v) = \frac{\pi}{4}$, (2.20) holds for every $A \in \mathfrak{A}$. If (2.20) holds for $A \in \mathfrak{A}$, we call A adapted *to v.*

(c) If $A \in \mathfrak{A}$ is adapted to v, then we have

$$\| \mathrm{Re}_A \, v\| = \cos\varphi(v) \cdot \|v\| \, , \quad \| \mathrm{Im}_A \, v\| = \sin\varphi(v) \cdot \|v\| \quad \text{and} \quad \mathrm{Re}_A \, v \perp \mathrm{Im}_A \, v \, . \tag{2.21}$$

Therefore, there exists a representation

$$\begin{cases} v = \|v\| \, (\cos\varphi(v) \cdot x + \sin\varphi(v) \cdot Jy) \\ \text{where } x, y \in \mathbb{S}(V(A)) \text{ and } x \perp y \text{ holds.} \end{cases} \tag{2.22}$$

We call any representation of v as in (2.22) a canonical representation *of v; it is uniquely determined by v for $0 < \varphi(v) < \frac{\pi}{4}$.*

Proof. Without loss of generality, we may suppose $\|v\| = 1$.

For (a). First, we note

$$\forall A \in \mathfrak{A}, \; \lambda \in \mathbb{S}^1 \;:\; \langle v, \lambda A v \rangle_{\mathbb{C}} = \overline{\lambda} \cdot \langle v, Av \rangle_{\mathbb{C}} \, . \tag{2.23}$$

This shows that $|\langle v, Av \rangle_{\mathbb{C}}|$ is independent of the choice of $A \in \mathfrak{A}$. Let us fix $A \in \mathfrak{A}$. Then Cauchy/Schwarz's inequality for a unitary space shows

$$|\langle v, Av \rangle_{\mathbb{C}}| \leq \|v\| \cdot \|Av\| = 1 \, .$$

Therefore it follows that there is one and only one $\varphi(v) \in [0, \frac{\pi}{4}]$ so that Equation (2.19) holds.

For (b). If $\varphi(v) = \frac{\pi}{4}$ holds, then we have $\langle v, Av \rangle_{\mathbb{C}} = 0$ for every $A \in \mathfrak{A}$ by Equation (2.19), and therefore Equation (2.20) is satisfied for every $A \in \mathfrak{A}$.

Thus, we now suppose $\varphi(v) \neq \frac{\pi}{4}$. We fix $A_0 \in \mathfrak{A}$, then we have $\langle v, A_0 v \rangle_{\mathbb{C}} \neq 0$ by Equation (2.19). Put $\lambda := \frac{\langle v, A_0 v \rangle_{\mathbb{C}}}{|\langle v, A_0 v \rangle_{\mathbb{C}}|} \in \mathbb{S}^1$ and $A := \lambda A_0 \in \mathfrak{A}$. Then Equation (2.23) shows

$$\langle v, Av \rangle_{\mathbb{C}} = \overline{\lambda} \cdot \langle v, A_0 v \rangle_{\mathbb{C}} = \frac{\overline{\langle v, A_0 v \rangle_{\mathbb{C}}} \cdot \langle v, A_0 v \rangle_{\mathbb{C}}}{|\langle v, A_0 v \rangle_{\mathbb{C}}|} = |\langle v, A_0 v \rangle_{\mathbb{C}}| > 0 \, .$$

Equation (2.23) also shows that A is the only element of $\mathfrak{A}$ for which (2.20) holds.

For (c). We have by Proposition 2.4(b)(iii)

$$\|\operatorname{Re}_A v\|^2 + \|\operatorname{Im}_A v\|^2 = \|v\|^2 = 1 \ . \tag{2.24}$$

If $A \in \mathfrak{A}$ is adapted to v, we also have by Proposition 2.4(c)(iii)

$$\begin{aligned} \cos(2\varphi(v)) &\overset{(2.19)}{=} |\langle v, Av\rangle_{\mathbb{C}}| \overset{(2.20)}{=} \langle v, Av\rangle_{\mathbb{C}} \\ &= \|\operatorname{Re}_A v\|^2 - \|\operatorname{Im}_A v\|^2 + 2i\,\langle \operatorname{Re}_A v, \operatorname{Im}_A v\rangle_{\mathbb{R}} \ , \end{aligned}$$

and hence

$$\|\operatorname{Re}_A v\|^2 - \|\operatorname{Im}_A v\|^2 = \cos 2\varphi(v) \ , \tag{2.25}$$

$$\langle \operatorname{Re}_A v, \operatorname{Im}_A v\rangle_{\mathbb{R}} = 0 \ . \tag{2.26}$$

Adding Equations (2.24) and (2.25) gives

$$2\,\langle \operatorname{Re}_A v, \operatorname{Re}_A v\rangle_{\mathbb{R}} = 1 + \cos(2\varphi(v)) = 2\,(\cos\varphi(v))^2$$

and therefore

$$\|\operatorname{Re}_A v\| = \cos\varphi(v) \ . \tag{2.27}$$

By subtracting Equation (2.25) from Equation (2.24), one similarly obtains

$$\|\operatorname{Im}_A v\| = \sin\varphi(v) \ . \tag{2.28}$$

Equations (2.26), (2.27) and (2.28) prove (2.21).

For $\varphi(v) \neq 0$, Equations (2.27) and (2.28) show that $\operatorname{Re}_A v, \operatorname{Im}_A v \neq 0$ holds, and therefore we see that (2.22) holds for $x := \operatorname{Re}_A v/\|\operatorname{Re}_A v\|$ and $y := \operatorname{Im}_A v/\|\operatorname{Im}_A v\|$, and for no other choice of $x, y \in V(A)$. For $0 < \varphi(v) < \frac{\pi}{4}$ there is only one $A \in \mathfrak{A}$ which is adapted to v and therefore v then already determines the canonical representation uniquely. On the other hand, if $\varphi(v) = 0$ holds, then we have $\operatorname{Im}_A v = 0$ by Equation (2.28), and thus $v = \operatorname{Re}_A v \in V(A)$ holds. Therefore (2.22) then is satisfied with $x := v \in \mathbb{S}(V(A))$ and any $y \in \mathbb{S}(V(A))$ which is orthogonal to x. □

2.29 Proposition. *Let $v \in \mathbb{V} \setminus \{0\}$ be given.*

(a) v principal $\iff$ $\varphi(v) = 0$.

(b) v isotropic $\iff$ $\varphi(v) = \frac{\pi}{4}$.

Proof. For (a). If v is principal, there exists $A \in \mathfrak{A}$ so that $v \in V(A)$ holds. We have $\langle v, Av\rangle_{\mathbb{C}} = \langle v, v\rangle_{\mathbb{C}} > 0$, so A is adapted to v. We have $\operatorname{Im}_A v = 0$ and therefore Theorem 2.28(c) shows that $\sin\varphi(v) = \|\operatorname{Im}_A v\|/\|v\| = 0$ holds. Consequently we have $\varphi(v) = 0$. Conversely, suppose that $\varphi(v) = 0$ holds and let $A \in \mathfrak{A}$ be adapted to v. Then Theorem 2.28(c) shows $\|\operatorname{Im}_A v\| = \sin\varphi(v) \cdot \|v\| = 0$ and therefore $v = \operatorname{Re}_A v \in V(A)$. Hence v is principal.

For (b). Let $A \in \mathfrak{A}$ be fixed. Theorem 2.28(a) shows that we have

$$v \text{ isotropic} \iff \langle v, Av\rangle_{\mathbb{C}} = 0 \iff \cos(2\varphi(v)) = 0 \iff \varphi(v) = \tfrac{\pi}{4} \ .$$ □

2.30 Proposition. *The map* $\varphi : \mathbb{V} \setminus \{0\} \to [0, \frac{\pi}{4}],\ v \mapsto \varphi(v)$ *is continuous; its restriction to the set*

$$N := \{\, v \in \mathbb{V} \setminus \{0\} \,|\, 0 < \varphi(v) < \tfrac{\pi}{4} \,\}$$

is differentiable. $\varphi|(N \cap \mathbb{S}(\mathbb{V}))$ *(and therefore also* $\varphi|N$ *) is a submersion.*

Proof. We fix $A \in \mathfrak{A}$. Then Theorem 2.28(a) shows that we have

$$\forall v \in \mathbb{V} \setminus \{0\}\ :\ \varphi(v) = \tfrac{1}{2} \arccos\left(\frac{|\langle v, Av \rangle_{\mathbb{C}}|}{\|v\|^2} \right) .$$

All the maps composing φ in this representation are continuous, therefore φ also is. Furthermore, the map $\mathbb{V} \setminus \{0\} \to \mathrm{I\!R},\ v \mapsto |\langle v, Av \rangle_{\mathbb{C}}|$ is differentiable at those $v \in \mathbb{V} \setminus \{0\}$ with $\langle v, Av \rangle_{\mathbb{C}} \neq 0$, i.e. $\varphi(v) \neq \frac{\pi}{4}$, and $\arccos : [0,1] \to [0, \frac{\pi}{2}]$ is differentiable on $[0,1[$. It follows that $\varphi|N$ is differentiable.

To show that the restriction of φ to $N \cap \mathbb{S}(\mathbb{V})$ is a submersion, let $v \in N \cap \mathbb{S}(\mathbb{V})$ be given and $A \in \mathfrak{A}$ be adapted to v, put $t_0 := \varphi(v) \in]0, \frac{\pi}{4}[$, $x := \mathrm{Re}_A(v)/\cos(t_0) \in \mathbb{S}(V(A))$ and $y := \mathrm{Im}_A(v)/\sin(t_0) \in \mathbb{S}(V(A))$, and consider the differentiable curve

$$\gamma :]0, \tfrac{\pi}{4}[\to (N \cap \mathbb{S}(\mathbb{V})),\ t \mapsto \cos(t)x + \sin(t)Jy$$

with $\gamma(t_0) = v$. An easy calculation using Theorem 2.28(a) shows that $\varphi \circ \gamma(t) = t$ holds for every $t \in]0, \frac{\pi}{4}[$, and therefore we have $\overrightarrow{T_v\varphi(\dot\gamma(t_0))} = (\varphi \circ \gamma)'(t_0) = 1$. This shows φ to be submersive at v. □

We now introduce a somewhat relaxed version of the concept of adapted-ness. Its main purpose is to simplify the formulation of some statements in Section 2.7 (concerning the eigenspaces of the Jacobi operator corresponding to the curvature tensor of the complex quadric), Section 4.4 (in the classification of totally geodesic submanifolds of the complex quadric) and Section 5.4 (concerning geodesics in the complex quadric).

2.31 Definition. $A \in \mathfrak{A}$ *is called* weakly adapted *to* $v \in \mathbb{V} \setminus \{0\}$ *if* $\langle v, Av \rangle_{\mathbb{C}} \in \mathrm{I\!R}$ *holds.*

It is clear that if A is adapted to v, then it is also weakly adapted to v.

2.32 Proposition. *Let* $v \in \mathbb{V} \setminus \{0\}$ *be given.*

(a) For $A \in \mathfrak{A}$*, the following statements are equivalent:*

(i) A *is weakly adapted to* v*.*

(ii) Either A *or* $-A$ *is adapted to* v*.*

(iii) $\mathrm{Re}_A\, v \perp \mathrm{Im}_A\, v$ *holds.*

(b) *If* $A \in \mathfrak{A}$ *is weakly adapted to* v *, there exists a representation*

$$\begin{cases} v = \|v\| \, (\cos(t) \cdot x + \sin(t) \cdot Jy) \\ \text{where } t \in \mathbb{R}\,,\ x, y \in \mathbb{S}(V(A)) \text{ and } x \perp y \text{ holds.} \end{cases} \tag{2.29}$$

We call any representation of v *as in* (2.29) *a* weak canonical representation *of* v *.*

Proof. For (a). For $A \in \mathfrak{A}$ we have

$$\begin{aligned} A \text{ is weakly adapted to } v &\iff \langle v, Av\rangle_{\mathbb{C}} \in \mathbb{R} \\ &\iff \exists \varepsilon \in \{\pm 1\} \ :\ \langle v, \varepsilon\, Av\rangle_{\mathbb{C}} \geq 0 \\ &\iff \exists \varepsilon \in \{\pm 1\} \ :\ \varepsilon A \text{ is adapted to } v\,, \end{aligned}$$

and therefore the equivalence "(i) $\Leftrightarrow$ (ii)" is shown. For the equivalence "(i) $\Leftrightarrow$ (iii)" we note that Proposition 2.4(c)(iii) shows that $\langle v, Av\rangle_{\mathbb{C}} \in \mathbb{R}$ is equivalent to $\langle \mathrm{Re}_A\, v, \mathrm{Im}_A\, v\rangle_{\mathbb{R}} = 0$.

For (b). By (a), either A or $-A$ is adapted to v. If A is adapted to v, then any canonical representation of v with respect to A satisfies (2.29). On the other hand, if $-A$ is adapted to v, we consider a canonical representation of v with respect to $-A$:

$$\begin{cases} v = \|v\| \, (\cos\varphi(v) \cdot x' + \sin\varphi(v) \cdot Jy') \\ \text{where } x', y' \in \mathbb{S}(V(-A)) \text{ and } x' \perp y' \text{ holds.} \end{cases}$$

Abbreviating $\varphi := \varphi(v)$, we thus have

$$v = \|v\| \cdot (\cos(\varphi - \tfrac{\pi}{2}) \cdot Jy' + \sin(\varphi - \tfrac{\pi}{2}) J(Jx'))\,.$$

We therefore see that (2.29) is satisfied with $t := \varphi - \frac{\pi}{2}$, $x := Jy' \in JV(-A) = V(A)$ and $y := Jx' \in V(A)$. ☐

2.33 Remark. The $\mathfrak{A}$-angle of a vector $w \in \mathbb{S}(\mathbb{V})$ can be read off any weak canonical representation of w by the following fact: If $A \in \mathfrak{A}$ and an orthonormal system (x, y) in $V(A)$ are given, then the function

$$\varphi_{(x,y)} : \mathbb{R} \to [0, \tfrac{\pi}{4}],\ t \mapsto \varphi(\,\cos(t)x + \sin(t)Jy\,)$$

is $\frac{\pi}{2}$-periodic and satisfies $\varphi_{(x,y)}(t) = |t|$ for $|t| \leq \frac{\pi}{4}$.

Proof. This is an immediate consequence of the fact that we have by Theorem 2.28(a)

$$\cos(2\varphi_{(x,y)}(t)) = |\langle \cos(t)x + \sin(t)Jy\,,\, \cos(t)x - \sin(t)Jy\rangle_{\mathbb{C}}| = |\cos(t)^2 - \sin(t)^2| = |\cos(2t)|\,.$$ ☐

2.34 Proposition. *Let* $(\mathbb{V}', \mathfrak{A}')$ *be another* ℂQ*-space,* $B : \mathbb{V} \to \mathbb{V}'$ *be a* ℂQ*-isomorphism or a* ℂQ*-anti-isomorphism and* $v \in \mathbb{V} \setminus \{0\}$ *.*

(a) $\varphi(Bv) = \varphi(v)$ *.*

(b) *If* $A \in \mathfrak{A}$ *is (weakly) adapted to* v *, then* $B \circ A \circ B^{-1} \in \mathfrak{A}'$ *is (weakly) adapted to* Bv *.*

(c) If B *is a* $\mathbb{C}$Q*-isomorphism and* $v = \|v\|\,(\cos(t)\cdot x + \sin(t)\cdot Jy)$ *is a (weak) canonical representation for* v*, then* $Bv = \|v\|\,(\cos(t)\cdot Bx + \sin(t)\cdot JBy)$ *is a (weak) canonical representation for* Bv*.*

Proof. The statements are obvious for the case where B is a $\mathbb{C}$Q-isomorphism.

To show (a),(b) for the case where B is a $\mathbb{C}$Q-anti-isomorphism, it therefore suffices to consider the case where B equals some element $A_0 \in \mathfrak{A}$ because of the fact that the map $B \mapsto B \circ A_0$ is a bijection from the set of $\mathbb{C}$Q-isomorphisms $\mathbb{V} \to \mathbb{V}'$ onto the set of $\mathbb{C}$Q-anti-isomorphisms $\mathbb{V} \to \mathbb{V}'$. In this case we have for any $v \in \mathbb{V} \setminus \{0\}$

$$\langle Bv, A_0(Bv)\rangle_{\mathbb{C}} = \langle A_0 v, v\rangle_{\mathbb{C}} = \overline{\langle v, A_0 v\rangle_{\mathbb{C}}}\ .$$

This equation shows that $\varphi(Bv) = \varphi(v)$ holds, and also that if A_0 is (weakly) adapted to v, then $B \circ A_0 \circ B^{-1} = A_0$ is (weakly) adapted to Bv. □

2.35 Corollary. *Suppose* $v \in \mathbb{V} \setminus \{0\}$ *and* $\lambda \in \mathbb{S}^1$ *are given. Then* $\varphi(\lambda v) = \varphi(v)$ *holds, and if* $A \in \mathfrak{A}$ *is (weakly) adapted to* v*, then* $\lambda^2 A$ *is (weakly) adapted to* λv*.*

Proof. Apply Proposition 2.34(a),(b) to the $\mathbb{C}$Q-automorphism $B : \mathbb{V} \to \mathbb{V},\ v \mapsto \lambda v$. □

2.6 The action of Aut($\mathfrak{A}$) on $\mathbb{S}(\mathbb{V})$

Let $(\mathbb{V}, \mathfrak{A})$ be an n-dimensional $\mathbb{C}$Q-space. As Aut($\mathfrak{A}$) consists of unitary maps of $\mathbb{V}$, this group acts via isometries on $\mathbb{S}(\mathbb{V})$. In the present section, we determine the orbits of this action and show that the action is irreducible.

2.36 Proposition. *(a) For* $n \geq 2$*, the orbits of the action of* Aut($\mathfrak{A}$) *on* $\mathbb{S}(\mathbb{V})$ *are the sets*

$$M_t := \{\, v \in \mathbb{S}(\mathbb{V}) \,|\, \varphi(v) = t \,\}\ ,$$

where t *runs through* $[0, \frac{\pi}{4}]$*.*

(b) For $n > 2$*, already* Aut$(\mathfrak{A})_0$ *acts transitively on* M_t*.*

2.37 Example. $M_{\pi/4} = \widetilde{Q}(\mathfrak{A})$.

Proof of Proposition 2.36. For (a). Let $v \in \mathbb{S}(\mathbb{V})$ be given, denote by $\mathcal{O} \subset \mathbb{S}(\mathbb{V})$ the orbit through v of the action of Aut($\mathfrak{A}$) on $\mathbb{S}(\mathbb{V})$ and put $t := \varphi(v) \in [0, \frac{\pi}{4}]$. Then we have to show $\mathcal{O} = M_t$. Proposition 2.34(a) already gives $\mathcal{O} \subset M_t$. Conversely, let $v' \in M_t$ be given. Then there exist conjugations $A, A' \in \mathfrak{A}$ which are adapted to v resp. to v' and canonical representations

$$v = \cos(t)x + \sin(t)Jy \quad \text{and} \quad v' = \cos(t)x' + \sin(t)Jy' \tag{2.30}$$

with $x,y \in \mathbb{S}(V(A))$, $x',y' \in \mathbb{S}(V(A'))$, $x \perp y$ and $x' \perp y'$. Hence, there exists a linear isometry $L : V(A) \to V(A')$ with $Lx = x'$ and $Ly = y'$. By Proposition 2.15, the complexification $L^{\mathbb{C}} : \mathbb{V} \to \mathbb{V}$ is a $\mathbb{C}$Q-automorphism, and Equations (2.30) show that $v' = L^{\mathbb{C}} v \in \mathcal{O}$ holds. Thus we have proved $M_t \subset \mathcal{O}$.

For (b). For odd n we have $\mathrm{Aut}(\mathfrak{A})_0 = \mathrm{Aut}(\mathfrak{A})$ by Proposition 2.17(b), and therefore the statement then was already shown in (a).

If $n \geq 4$ is even, let $v, v' \in M_t$ be given. We have to show that there exists $B \in \mathrm{Aut}(\mathfrak{A})_0$ so that $Bv = v'$ holds. By (a), there exists $\widetilde{B} \in \mathrm{Aut}(\mathfrak{A})$ so that $\widetilde{B}v = v'$ holds; by Proposition 2.17(b) $\widetilde{B}$ can be represented as $\widetilde{B} = \lambda\,\widetilde{L}^{\mathbb{C}}$ with $\lambda \in \mathbb{S}^1$ and $\widetilde{L} \in \mathrm{O}(V(A))$. If $\widetilde{L} \in \mathrm{SO}(V(A))$ holds, we have $B := \widetilde{B} \in \mathrm{Aut}(\mathfrak{A})_0$ (again, see Proposition 2.17(b)) and $Bv = v'$. Otherwise, we represent v as in (2.30), choose $z \in \mathbb{S}(V(A))$ orthogonal to x and y and consider the orthogonal transformation $S : V(A) \to V(A)$ characterized by $Sz = -z$ and $S|(\mathbb{R}z)^{\perp} = \mathrm{id}_{(\mathbb{R}z)^{\perp}}$. We have $\det S = -1$ and therefore $L := \widetilde{L} \circ S \in \mathrm{SO}(V(A))$, hence $B := \lambda\,L^{\mathbb{C}} \in \mathrm{Aut}(\mathfrak{A})_0$. Also, the construction of B shows that $Bv = \widetilde{B}v = v'$ holds. □

2.38 Proposition. *For $0 < t < \frac{\pi}{4}$, M_t is a hypersurface of $\mathbb{S}(\mathbb{V})$. For any $v \in M_t$, we have $T_v M_t = \ker(T_v\varphi)$.*

Proof. Because $\varphi|\{\, v \in \mathbb{S}(\mathbb{V}) \,|\, 0 < \varphi(v) < \frac{\pi}{4} \,\}$ is a submersion (Proposition 2.30), this proposition is an immediate consequence of the theorem on equation-defined manifolds. □

2.39 Proposition. *(a)* $\mathrm{Aut}(\mathfrak{A})$ *acts irreducibly on* $\mathbb{V}$.

(b) For $n > 2$, already $\mathrm{Aut}(\mathfrak{A})_0$ *acts irreducibly on* $\mathbb{V}$.

Proof. Put $G := \mathrm{Aut}(\mathfrak{A})_0$ in the case $n > 2$, $G := \mathrm{Aut}(\mathfrak{A})$ in the case $n = 2$. It suffices to show that G acts irreducibly on $\mathbb{V}$. Let a G-invariant subspace $U \neq \{0\}$ of $\mathbb{V}$ be given. We fix $v_0 \in \mathbb{S}(U)$ and put $t := \varphi(v_0) \in [0, \frac{\pi}{4}]$. By Proposition 2.36, G acts transitively on M_t, and therefore the G-invariance of U implies

$$M_t \subset U \,. \tag{2.31}$$

We also fix $A \in \mathfrak{A}$. In the case $t = 0$, (2.31) shows that $V(A) \subset \mathbb{R} \cdot M_0 \subset U$ holds. Because U is a complex subspace, this implies $U = \mathbb{V}$.

Hence, we may now suppose $t > 0$. Let $w \in U^{\perp}$ be given; we will show $w = 0$. (2.31) shows that we have in particular

$$\forall\, v \in M_t \;:\; \langle w, v \rangle_{\mathbb{R}} = 0 \,. \tag{2.32}$$

Let (x,y) be any orthonormal system in $V(A)$. Then we have $v := \cos(t)x + \sin(t)Jy \in M_t$ and therefore by Equation (2.32):

$$\begin{aligned} 0 = \langle w, v \rangle_{\mathbb{R}} &= \langle \mathrm{Re}_A\, w + J\,\mathrm{Im}_A\, w,\; \cos(t)x + \sin(t)Jy \rangle_{\mathbb{R}} \\ &= \cos(t) \cdot \langle \mathrm{Re}_A\, w, x \rangle_{\mathbb{R}} + \sin(t) \cdot \langle \mathrm{Im}_A\, w, y \rangle_{\mathbb{R}} \,. \end{aligned} \tag{2.33}$$

We also have $\cos(t)x - \sin(t)Jy \in M_t$ and therefore by an analogous calculation

$$0 = \cos(t) \cdot \langle \mathrm{Re}_A\, w, x\rangle_{\mathbb{R}} - \sin(t) \cdot \langle \mathrm{Im}_A\, w, y\rangle_{\mathbb{R}} \ . \tag{2.34}$$

Because of $t \neq 0$, we have $\cos(t),\ \sin(t) \neq 0$, and therefore Equations (2.33) and (2.34) together imply

$$\langle \mathrm{Re}_A\, w, x\rangle_{\mathbb{R}} = \langle \mathrm{Im}_A\, w, y\rangle_{\mathbb{R}} = 0 \ . \tag{2.35}$$

Because we have $n \geq 2$, any given $x \in \mathbb{S}(V(A))$ can be extended to an orthonormal system (x, y) in $V(A)$, and therefore Equation (2.35) shows that we have $\langle \mathrm{Re}_A\, w, x\rangle_{\mathbb{R}} = 0$ for every $x \in \mathbb{S}(V(A))$, hence $\mathrm{Re}_A\, w = 0$. Analogously, we see that $\mathrm{Im}_A\, w = 0$ holds, and therefore we have $w = 0$.

Thus, we conclude $U^{\perp} = \{0\}$ and therefore $U = \mathbb{V}$. □

2.40 Remark. In the case $n = 2$, the action of $\mathrm{Aut}(\mathfrak{A})_0$ on $\mathbb{V}$ is in fact reducible, as the following consideration shows:

For $n = 2$, there are exactly two 1-dimensional complex, isotropic subspaces U_1, U_2 of $\mathbb{V}$, and $\mathbb{V} = U_1 \oplus U_2$ holds. If we fix $A \in \mathfrak{A}$ and an orthogonal complex structure $j : V(A) \to V(A)$ (j is unique up to a choice of sign), then these subspaces are given by

$$U_1 = \{\, x + J(jx) \,|\, x \in V(A) \,\} \quad \text{and} \quad U_2 = \{\, x - J(jx) \,|\, x \in V(A) \,\} \ .$$

U_k (with $k \in \{1, 2\}$) is invariant under the action of $\mathrm{Aut}(\mathfrak{A})_0$: For any $B \in \mathrm{Aut}(\mathfrak{A})_0$, $B(U_k)$ is a complex 1-dimensional, isotropic subspace of $\mathbb{V}$, and therefore we have $B(U_k) \in \{U_1, U_2\}$. Because $\mathrm{Aut}(\mathfrak{A})_0$ is connected, it follows that in fact $B(U_k) = U_k$ holds.

As we will see in Section 5.5, the decomposition $\mathbb{V} = U_1 \oplus U_2$ gives rise to two totally geodesic foliations of the 2-dimensional complex quadric Q^2, whose leaves are isometric to $\mathbb{P}^1$ and which intersect orthogonally in every point of Q^2.

2.41 Remark. Because the hypersurfaces M_t ($0 < t < \frac{\pi}{4}$) are orbits of an action on the sphere via isometries, $(M_t)_{0<t<\pi/4}$ is a family of *isoparametric hypersurfaces* in $\mathbb{S}(\mathbb{V})$; its focal sets are M_0 and $M_{\pi/4}$.

The general situation of a homogeneous (and hence isoparametric) hypersurface in a sphere has been investigated by Hsiang/Lawson ([HL71], Section II.1) and Takagi/Takahashi ([TT72]). Hsiang and Lawson classified in [HL71] those compact subgroups $G \subset \mathrm{O}(r+1)$ for which the principal orbits of the action of G on $\mathbb{S}^r$ are hypersurfaces in $\mathbb{S}^r$. It follows from the tables given in [HL71], Theorem 5, p. 16 that any (closed, connected) homogeneous hypersurface in $\mathbb{S}^r$ is as a homogeneous space isomorphic to a principal orbit of the isotropy representation of a symmetric space of rank 2.

As we will see in Chapter 3, complex quadrics Q of dimension $n \geq 2$ are Riemannian symmetric spaces of rank 2, and the action of $\mathrm{Aut}(\mathfrak{A}(Q, p))_0$ on T_pQ is isomorphic to the action of the isotropy representation of Q at p. Because of Proposition 2.36(b), it follows for a $\mathbb{C}$Q-space

$(\mathbb{V},\mathfrak{A})$ of dimension $n \geq 3$ that the isoparametric hypersurfaces M_t $(0 < t < \frac{\pi}{4})$ in $\mathbb{S}(\mathbb{V})$ are as homogeneous $\mathrm{Aut}(\mathfrak{A})_0$-spaces isomorphic to principal orbits of the isotropy action of an n-dimensional complex quadric.

In [TT72], Takagi and Takahashi studied homogeneous hypersurfaces in a sphere from the viewpoint of isoparametric hypersurfaces. Because of the result of Hsiang/Lawson cited above, it is sufficient to consider the orbits under the isotropy representation of symmetric spaces M of rank 2. Among other things, Takagi and Takahashi show how the root system of the symmetric space M can be used to calculate the (constant) principal curvatures of a principal orbit of the isotropy representation and their multiplicities (see [TT72], Theorem (3), p. 470 and Table II, p. 480).

Via the cited results from [TT72] and the information on the roots and the root spaces of a complex quadric given in Section 3.2, one can obtain the following facts on the family (M_t) of isoparametric hypersurfaces in $\mathbb{S}(\mathbb{V})$, where $(\mathbb{V},\mathfrak{A})$ is again a $\mathbb{C}$Q-space of dimension $n \geq 3$: M_t (with $0 < t < \frac{\pi}{4}$) has $g = 4$ principal curvatures. (In the case $n = 2$, the number of principal curvatures is reduced to 2.) For $A \in \mathfrak{A}$ and an orthonormal system (x, y) in $V(A)$, we have $v := \cos(t)x + \sin(t)Jy \in M_t$, the unit vector $u \in T_v\mathbb{S}(\mathbb{V})$ characterized by $\overrightarrow{u} = -\sin(t)x + \cos(t)Jy$ is normal to M_t, and with respect to this unit normal vector, the principal curvatures of M_t and their multiplicities are

principal curvature	$-\cot(t)$	$\tan(t)$	$\frac{\cos(t)+\sin(t)}{\cos(t)-\sin(t)}$	$-\frac{\cos(t)-\sin(t)}{\cos(t)+\sin(t)}$
multiplicity	$n-2$	$n-2$	1	1

.

2.7 The curvature tensor of a $\mathbb{C}$Q-space

In Proposition 1.21 we calculated the curvature tensor of a complex quadric Q. As that proposition shows, the curvature tensor of Q at some point $p \in Q$ can be described in terms of the inner product on T_pQ, its complex structure and the $\mathbb{C}$Q-structure $\mathfrak{A}(Q,p)$ alone, and therefore a corresponding tensor can be defined on any $\mathbb{C}$Q-space $(\mathbb{V},\mathfrak{A})$. We now turn to the study of the algebraic properties of this tensor, which we will call the *curvature tensor of the $\mathbb{C}$Q-space* $(\mathbb{V},\mathfrak{A})$.

2.42 Definition. *For $A \in \mathfrak{A}$ we call the $\mathbb{R}$-trilinear map $R : \mathbb{V}\times\mathbb{V}\times\mathbb{V} \to \mathbb{V},\ (u,v,w) \mapsto R(u,v)w$ defined by*

$$\forall u,v,w \in \mathbb{V} \ :\ R(u,v)w = \langle w,v\rangle_{\mathbb{C}}u - \langle w,u\rangle_{\mathbb{C}}v - 2\cdot\langle Ju,v\rangle_{\mathbb{R}}Jw + \langle v,Aw\rangle_{\mathbb{C}}Au - \langle u,Aw\rangle_{\mathbb{C}}Av\ ,$$

the curvature tensor of the $\mathbb{C}$Q-space $(\mathbb{V},\mathfrak{A})$. *$R$ is independent of the choice of $A \in \mathfrak{A}$.*

Proof for the independence of R from $A \in \mathfrak{A}$. For any $\lambda \in \mathbb{S}^1$ we have $\langle v,\lambda Aw\rangle_{\mathbb{C}}\lambda Au - \langle u,\lambda Aw\rangle_{\mathbb{C}}\lambda Av = \langle v,Aw\rangle_{\mathbb{C}}Au - \langle u,Aw\rangle_{\mathbb{C}}Av$. □

Let Q be a complex quadric and $p \in Q$. Then the curvature tensor of the $\mathbb{C}$Q-space $(T_pQ, \mathfrak{A}(Q,p))$ in the sense of the present section is equal to the curvature tensor of the quadric Q at p (see Proposition 1.21). Of course, this fact is the motivation for Definition 2.42. Consequently, the results we obtain on the curvature tensor of a $\mathbb{C}$Q-space imply results on the curvature tensor of a complex quadric.

2.43 Proposition. *Let R be the curvature tensor of the $\mathbb{C}$Q-space $(\mathbb{V}, \mathfrak{A})$.*

(a) R is $\mathbb{C}$-linear in w and skew-symmetric in (u,v).

(b) For $u, v, w \in \mathbb{V}$, we have

$$\begin{aligned} R(u,v)w = &\langle v,w\rangle_{\mathbb{R}} u - \langle u,w\rangle_{\mathbb{R}} v \\ &+ \langle Jv,w\rangle_{\mathbb{R}} Ju - \langle Ju,w\rangle_{\mathbb{R}} Jv - 2\cdot\langle Ju,v\rangle_{\mathbb{R}} Jw \\ &+ \langle v,Aw\rangle_{\mathbb{R}} Au - \langle u,Aw\rangle_{\mathbb{R}} Av \\ &+ \langle v,JAw\rangle_{\mathbb{R}} JAu - \langle u,JAw\rangle_{\mathbb{R}} JAv\,. \end{aligned}$$

(c) If $A \in \mathfrak{A}$ and $u, v, w \in V(A)$ holds, we have

$$R(u,v)w = 2\cdot(\langle v,w\rangle_{\mathbb{R}} u - \langle u,w\rangle_{\mathbb{R}} v)\,.$$

Thus $R|V(A)^3$ is the curvature tensor of a space of constant sectional curvature 2.

(d) If W is an isotropic subspace of $\mathbb{V}$ and $u, v, w \in W$ holds, we have

$$R(u,v)w = \langle v,w\rangle_{\mathbb{R}} u - \langle u,w\rangle_{\mathbb{R}} v + \langle Jv,w\rangle_{\mathbb{R}} Ju - \langle Ju,w\rangle_{\mathbb{R}} Jv - 2\cdot\langle Ju,v\rangle_{\mathbb{R}} Jw\,.$$

Thus if W is a complex isotropic subspace, then $R|W^3$ is the curvature tensor of a space of constant holomorphic sectional curvature 4. If W is a totally real isotropic subspace, then $R|W^3$ is the curvature tensor of a space of constant sectional curvature 1.

Proof. Obvious. (For (b), use Equation (2.1).) □

Let Q be a complex quadric, $p \in Q$ and $A \in \mathfrak{A}(Q,p)$. Then Proposition 2.43(c) shows in particular that the subspace $V(A)$ of T_pQ is *curvature-invariant* (i.e. $\forall u, v, w \in V(A) : R(u,v)w \in V(A)$ holds). Because Q is a symmetric space (see Chapter 3), it follows that there exists a totally geodesic, complete, connected submanifold M of Q with $p \in M$ and $T_pM = V(A)$. Because the restriction of the curvature tensor to $V(A)$ is the curvature tensor of a sphere of radius $1/\sqrt{2}$, M is locally isometric to such a sphere. In Section 5.3 we will explicitly construct an embedding onto M; it will turn out that M is in fact globally isometric to the sphere.

Similarly Proposition 2.43(d) shows that any isotropic subspace W of $(T_pQ, \mathfrak{A}(Q,p))$ which is either complex or totally real, is curvature-invariant and therefore gives rise to a totally geodesic, complete, connected submanifold M of Q with $p \in M$ and $T_pM = W$. If W is complex, then $R|W^3$ is the curvature tensor of a complex projective space equipped with the Fubini-Study metric, and consequently M is locally isometric to such a space; in fact we will see in Section 5.5 that this also is a global isometry. On the other hand, if W is totally real, then $R|W^3$ is the curvature tensor of a sphere of radius 1 and therefore M is locally isometric to such a sphere; in this case it turns out however that M is not globally isometric to the sphere, but rather to a real projective space.

2.44 Definition. *Let $(\mathbb{V}, \mathfrak{A})$ and $(\mathbb{V}', \mathfrak{A}')$ be $\mathbb{C}$Q-spaces, denote by R and R' their respective curvature tensors and let an $\mathbb{R}$-linear map $B : \mathbb{V} \to \mathbb{V}'$ be given. We call B* curvature-equivariant, *if*

$$\forall\, u, v, w \in \mathbb{V} \;:\; B(R(u,v)w) = R'(Bu, Bv)Bw$$

holds.

2.45 Proposition. *$\mathbb{C}$Q-isomorphisms and $\mathbb{C}$Q-anti-isomorphisms are curvature-equivariant.*

Proof. Let $(\mathbb{V}, \mathfrak{A})$ and $(\mathbb{V}', \mathfrak{A}')$ be $\mathbb{C}$Q-spaces and denote their respective curvature tensors by R and R'.

Let $B : \mathbb{V} \to \mathbb{V}'$ be a $\mathbb{C}$Q-isomorphism. Let us fix $A \in \mathfrak{A}$, then there exists $A' \in \mathfrak{A}'$ with $B \circ A = A' \circ B$. Using this equation and the fact that B is a $\mathbb{C}$-linear isometry, the curvature-equivariance of B can be read off Definition 2.42.

Any $\mathbb{C}$Q-anti-isomorphism $\overline{B} : \mathbb{V} \to \mathbb{V}'$ can be represented as $\overline{B} = B \circ A$, where $A \in \mathfrak{A}$ holds and $B : \mathbb{V} \to \mathbb{V}'$ is a $\mathbb{C}$Q-isomorphism. Because of the previous result, it therefore suffices to show that A is curvature-equivariant, and this is easily done via Proposition 2.43(b). $\square$

2.46 Remark. As we will see in Section 3.3, any curvature-equivariant $\mathbb{C}$-linear or anti-linear map already is a $\mathbb{C}$Q-isomorphism resp. a $\mathbb{C}$Q-anti-isomorphism. Moreover, for $n \geq 3$ any curvature-equivariant $\mathbb{R}$-linear map is already either $\mathbb{C}$-linear or anti-linear, and hence a $\mathbb{C}$Q-isomorphism or a $\mathbb{C}$Q-anti-isomorphism.

We now give another representation for the curvature tensor of $(\mathbb{V}, \mathfrak{A})$ which is frequently useful.

We consider for any $u, v \in \mathbb{V}$ the skew-Hermitian linear map

$$u \wedge v : \;\mathbb{V} \to \mathbb{V},\; w \mapsto \langle w, v \rangle_{\mathbb{C}}\, u - \langle w, u \rangle_{\mathbb{C}}\, v \,. \tag{2.36}$$

It should be noted that if $u, v \in V(A)$ holds for some $A \in \mathfrak{A}$, then $u \wedge v$ is the complexification of the skew-adjoint linear map

$$V(A) \to V(A),\; x \mapsto \langle x, v \rangle_{\mathbb{R}}\, u - \langle x, u \rangle_{\mathbb{R}}\, v \,,$$

whence it follows that in this case $u \wedge v \in \mathfrak{aut}_s(\mathfrak{A})$ holds.

2.47 Proposition. *Denote by R the curvature tensor of $(\mathbb{V}, \mathfrak{A})$ and fix $A \in \mathfrak{A}$. Then we have for any $u, v, w \in \mathbb{V}$*

$$R(u,v)w = \rho(u,v) \cdot Jw + C(u,v)w \tag{2.37}$$

with the functions

$$\begin{aligned} &\rho : \mathbb{V}\times\mathbb{V} \to \mathbb{R},\ (u,v) \mapsto 2\cdot\langle u, Jv\rangle_{\mathbb{R}} \\ \textit{and}\quad &C : \mathbb{V}\times\mathbb{V} \to \mathfrak{aut}_s(\mathfrak{A}),\ (u,v) \mapsto u\wedge v + Au\wedge Av \\ &\qquad = 2\cdot(\mathrm{Re}_A(u)\wedge\mathrm{Re}_A(v) + \mathrm{Im}_A(u)\wedge\mathrm{Im}_A(v)) . \end{aligned}$$

C does not depend on the choice of $A\in\mathfrak{A}$.

Proof. Let $A\in\mathfrak{A}$ and $u,v\in\mathbb{V}$ be given. Because both sides of Equation (2.37) are $\mathbb{C}$-linear in w and C also is $\mathbb{C}$-linear, it suffices to consider $w\in V(A)$. In this case, we have

$$\begin{aligned} &R(u,v)w - \rho(u,v)\cdot Jw \\ &= \langle w,v\rangle_{\mathbb{C}} u - \langle w,u\rangle_{\mathbb{C}} v + \langle v,Aw\rangle_{\mathbb{C}} Au - \langle u,Aw\rangle_{\mathbb{C}} Av \ =\ (u\wedge v + Au\wedge Av)w \\ &= \langle w,v\rangle_{\mathbb{C}} u + \langle v,w\rangle_{\mathbb{C}} Au - (\langle w,u\rangle_{\mathbb{C}} v + \langle u,w\rangle_{\mathbb{C}} Av) \\ &= \langle w,v\rangle_{\mathbb{C}} u + A(\langle w,v\rangle_{\mathbb{C}} u) - (\langle w,u\rangle_{\mathbb{C}} v + A(\langle u,w\rangle_{\mathbb{C}} v)) \\ &= 2\cdot(\mathrm{Re}_A(\langle w,v\rangle_{\mathbb{C}} u) - \mathrm{Re}_A(\langle w,u\rangle_{\mathbb{C}} v)) \\ &= 2\cdot(\mathrm{Re}(\langle w,v\rangle_{\mathbb{C}})\cdot\mathrm{Re}_A u - \mathrm{Im}(\langle w,v\rangle_{\mathbb{C}})\cdot\mathrm{Im}_A u - \mathrm{Re}(\langle w,u\rangle_{\mathbb{C}})\cdot\mathrm{Re}_A v + \mathrm{Im}(\langle w,u\rangle_{\mathbb{C}})\cdot\mathrm{Im}_A v) \\ &= 2\cdot(\langle w,v\rangle_{\mathbb{R}}\cdot\mathrm{Re}_A u - \langle w,Jv\rangle_{\mathbb{R}}\cdot\mathrm{Im}_A u - \langle w,u\rangle_{\mathbb{R}}\cdot\mathrm{Re}_A v + \langle w,Ju\rangle_{\mathbb{R}}\cdot\mathrm{Im}_A v) \\ &= 2\cdot(\langle w,\mathrm{Re}_A v\rangle_{\mathbb{R}}\cdot\mathrm{Re}_A u - \langle w,\mathrm{Re}_A u\rangle_{\mathbb{R}}\cdot\mathrm{Re}_A v + \langle w,\mathrm{Im}_A v\rangle_{\mathbb{R}}\cdot\mathrm{Im}_A u - \langle w,\mathrm{Im}_A u\rangle_{\mathbb{R}}\cdot\mathrm{Im}_A v) \\ &= 2\cdot(\mathrm{Re}_A(u)\wedge\mathrm{Re}_A(v) + \mathrm{Im}_A(u)\wedge\mathrm{Im}_A(v))w . \end{aligned}$$

This calculation shows that the equals sign in the definition of C indeed holds, and that Equation (2.37) holds. Because $R(u,v)w$ and $\rho(u,v)Jw$ are independent of the choice of $A\in\mathfrak{A}$, we see from Equation (2.37) that $C(u,v)w$ also is independent of this choice. Via the second presentation of $C(u,v)$ given in its definition we see that C indeed maps into $\mathfrak{aut}_s(\mathfrak{A})$. □

2.48 Corollary. *Let U be an $\mathbb{R}$-linear subspace of $\mathbb{V}$ which is* curvature-invariant*, meaning that*

$$\forall\, u,v,w\in U\ :\ R(u,v)w\in U$$

holds. If there exist $A\in\mathfrak{A}$ and $x,y\in V(A)\setminus\{0\}$ so that $Jx,y\in U$ and $\langle x,y\rangle_{\mathbb{R}}\neq 0$ holds, then U is in fact a complex subspace of $\mathbb{V}$.

Proof. If ρ and C denote the functions from Proposition 2.47, we have $\rho(Jx,y) = 2\,\langle Jx,Jy\rangle_{\mathbb{R}} = 2\,\langle x,y\rangle_{\mathbb{R}} \neq 0$ and $C(Jx,y) = 2\,(0\wedge y + x\wedge 0) = 0$, and therefore $R(Jx,y) = \rho(Jx,y)\cdot J$. Because U is curvature-invariant and $Jx,y\in U$ holds, U is invariant under $R(Jx,y)$ and hence also under J. □

We now suppose $n\geq 2$ and consider for any vector $w\in\mathbb{V}$ the *Jacobi operator*

$$R_w : \mathbb{V}\to\mathbb{V},\ v\mapsto R(v,w)w$$

corresponding to w.

Because the eigenvalues and eigenspaces of the Jacobi operators corresponding to the curvature tensor of a symmetric space are related to the roots and root spaces of this symmetric space (we will investigate this relationship in Section 3.2), it is of interest for the study of the complex quadric to give an explicit description of the eigenvalues and eigenspaces of the operators R_w .

This is done in the following theorem, which closely follows the description already given in [Rec95].

2.49 Theorem. *Let* $w \in \mathbb{S}(\mathbb{V})$ *be given, and suppose that*

$$w = \cos(t)x + \sin(t)Jy \tag{2.38}$$

is a weak canonical representation of w *in the sense of Proposition 2.32(b), i.e. we have* $t \in \mathrm{I\!R}$, $A \in \mathfrak{A}$ *and* $x, y \in \mathbb{S}(V(A))$ *with* $x \perp y$.

Then the Jacobi operator R_w *has the following eigenvalues* $\kappa_k(t)$, *eigenspaces* E_k *and multiplicities:*

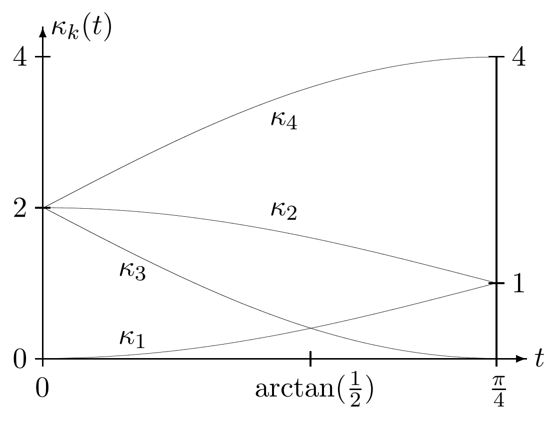
k	$\kappa_k(t) \in \mathrm{Spec}(R_w)$	$E_k = \mathrm{Eig}(R_w, \kappa_k(t))$	$n(R_w, \kappa_k(t))$
0	0	$\mathrm{I\!R}x \oplus \mathrm{I\!R}(Jy)$	2
1	$1 - \cos(2t)$	$J((\mathrm{I\!R}x \oplus \mathrm{I\!R}y)^\perp)$	$n-2$
2	$1 + \cos(2t)$	$(\mathrm{I\!R}x \oplus \mathrm{I\!R}y)^\perp$	$n-2$
3	$2(1 - \sin(2t))$	$\mathrm{I\!R}(Jx + y)$	1
4	$2(1 + \sin(2t))$	$\mathrm{I\!R}(Jx - y)$	1

Here and in the sequel, $^\perp$ *denotes the ortho-complement in* $V(A)$. *For* $n = 2$, *the eigenvalues* $1 \pm \cos(2t)$ *do not exist (their multiplicity is zero). Also, if some of the eigenvalues given in the table coincide — in the interval* $[0, \frac{\pi}{4}]$ *this happens for* $t \in \{0, \arctan(\frac{1}{2}), \frac{\pi}{4}\}$ *— then one has to add the corresponding eigenspaces and multiplicities. The eigenfunctions* $\kappa_k(t)$ *are* π*-periodic and satisfy the following symmetry equations for any* $t \in \mathrm{I\!R}$ *:*

$$\kappa_1(\tfrac{\pi}{2} - t) = \kappa_2(t) = \kappa_1(t + \tfrac{\pi}{2}) \ , \quad \kappa_2(\tfrac{\pi}{2} - t) = \kappa_1(t) = \kappa_2(t + \tfrac{\pi}{2}) \ , \tag{2.39}$$

$$\kappa_3(\tfrac{\pi}{2} - t) = \kappa_3(t) = \kappa_4(t + \tfrac{\pi}{2}) \quad \text{and} \quad \kappa_4(\tfrac{\pi}{2} - t) = \kappa_4(t) = \kappa_3(t + \tfrac{\pi}{2}) \ . \tag{2.40}$$

Therefore the following graph of the functions $\kappa_k(t)$ *for* $t \in [0, \frac{\pi}{4}]$ *conveys all information on them:*



Warning. It follows from Equations (2.39) and (2.40) that in the setting of the theorem, the Jacobi operators corresponding to $w = \cos(t)x + \sin(t)Jy$, to $w' := \cos(\frac{\pi}{2} - t)x + \sin(\frac{\pi}{2} - t)Jy$ and to $w'' := \cos(t + \frac{\pi}{2})x + \sin(t + \frac{\pi}{2})Jy$ have the same eigenvalues. However, this does not mean that these Jacobi operators are equal, because the corresponding eigenspaces differ.

Proof of Theorem 2.49. One easily verifies via Proposition 2.47 that the elements of E_k are in fact eigenvectors of R_w corresponding to the eigenvalue $\kappa_k(t)$. For example, let us check this for $k = 1$. Let $v = Jz$ with $z \in V(A)$, $z \perp x, y$ be given. Then we have

$$\rho(v, w) = 2\langle v, Jw\rangle_{\mathbb{R}} = 2\langle Jz, J(\cos(t)x + \sin(t)Jy)\rangle_{\mathbb{R}} = 2\cos(t) \cdot \langle z, x\rangle_{\mathbb{R}} = 0$$

and

$$C(v, w) = 2\,(\mathrm{Re}_A\, v \wedge \mathrm{Re}_A\, w + \mathrm{Im}_A\, v \wedge \mathrm{Im}_A\, w) = 2\sin(t) \cdot z \wedge y\ ;$$

consequently, we obtain

$$\begin{aligned} R_w(v) &= R(v, w)w = \rho(v, w)Jw + C(v, w)w \\ &= 2\sin(t) \cdot (z \wedge y)(\cos(t)x + \sin(t)Jy) \\ &= 2\sin(t)\cos(t) \cdot (z \wedge y)x + 2\sin(t)^2 \cdot J((z \wedge y)y) \\ &= 2\sin(t)^2 Jz = (1 - \cos(2t))v\ . \end{aligned}$$

Therefore v is in fact an eigenvector of R_w belonging to the eigenvalue $1 - \cos(2t)$.

Because the spaces E_k together span $\mathbb{V}$, it is clear that the table is complete. The symmetry relations (2.39) and (2.40) follow from the well-known properties of $\sin$ and $\cos$. □

2.50 Corollary. *Let $w \in \mathbb{S}(\mathbb{V})$ be given, let $A \in \mathfrak{A}$ be adapted to w and let $w = \cos(\,\varphi(w)\,)x + \sin(\,\varphi(w)\,)Jy$ be a canonical representation of w with respect to A. Then we have*

$$\ker R_w = \begin{cases} \mathbb{R}x \oplus J((\mathbb{R}x)^\perp) = \mathbb{R}w \oplus J((\mathbb{R}w)^\perp) & \text{for } \varphi(w) = 0 \\ \mathbb{R}x \oplus \mathbb{R}Jy = \mathbb{R}w \oplus \mathbb{R}Aw & \text{for } 0 < \varphi(w) < \frac{\pi}{4} \\ \mathbb{R}x \oplus \mathbb{R}Jy \oplus \mathbb{R}(Jx + y) = \mathbb{R}w \oplus \mathbb{C}Aw & \text{for } \varphi(w) = \frac{\pi}{4} \end{cases} .$$

Proof. The statement is an immediate consequence of Theorem 2.49. □

2.51 Corollary. *Let $w \in \mathbb{S}(\mathbb{V})$ be given. Then the $\mathfrak{A}$-angle $\varphi(w)$ of w is determined by* $\mathrm{Spec}(R_w)$*; more precisely, we have*

$$2 \cdot (1 + \sin(2\varphi(w))) = \max \mathrm{Spec}(R_w)\ .$$

Proof. Apply Theorem 2.49 with a canonical representation of w. □

2.8 Flat subspaces

As previously, we suppose that $(\mathbb{V}, \mathfrak{A})$ is an n-dimensional $\mathbb{C}$Q-space with $n \geq 2$; we denote its curvature tensor by R.

2.52 Definition. *An $\mathbb{R}$-linear subspace $\mathfrak{a} \subset \mathbb{V}$ is called* flat *if*

$$\forall\, u, v, w \in \mathfrak{a} \;:\; R(u,v)w = 0$$

*holds. A k-dimensional flat subspace is also called a k-*flat*.*

2.53 Example. Every real-1-dimensional linear subspace of $\mathbb{V}$ is flat.

2.54 Theorem. *If $\mathfrak{a} \subset \mathbb{V}$ is a flat subspace, then we have $\dim_{\mathbb{R}}(\mathfrak{a}) \leq 2$. The 2-flats of $\mathbb{V}$ are exactly the spaces*

$$\mathfrak{a} = \mathbb{R}x \oplus \mathbb{R}Jy$$

with $A \in \mathfrak{A}$, $x, y \in \mathbb{S}(V(A))$ and $x \perp y$.

Proof. First, let a space $\mathfrak{a} = \mathbb{R}x \oplus \mathbb{R}Jy$ with $A \in \mathfrak{A}$, $x, y \in \mathbb{S}(V(A))$ and $x \perp y$ be given. To prove that $\mathfrak{a}$ is flat, it suffices to show $\rho(x, Jy) = 0$ and $C(x, Jy) = 0$, where ρ and C are the functions of Proposition 2.47, and this is easily done.

Now, let a flat subspace $\mathfrak{a} \subset \mathfrak{m}$ with $\dim_{\mathbb{R}}(\mathfrak{a}) \geq 2$ be given. Because $\mathfrak{a}$ is flat, we have

$$\forall w \in \mathfrak{a} \;:\; \mathfrak{a} \subset \ker R_w \;, \tag{2.41}$$

where R_w denotes the Jacobi operator corresponding to w as in Section 2.7.

We first prove

$$\mathfrak{a} \text{ is not an isotropic subspace of } (\mathbb{W}, \mathfrak{A}) \tag{2.42}$$

by contradiction: Assume that $\mathfrak{a}$ is isotropic and fix $w \in \mathbb{S}(\mathfrak{a})$. Then any $A \in \mathfrak{A}$ is adapted to w, and therefore (2.41) and Corollary 2.50 show

$$\mathfrak{a} \subset \mathbb{R}w \oplus \mathbb{C}Aw \;. \tag{2.43}$$

Now, let $v \in \mathfrak{a}$ be given; because of (2.43), there exist $s \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ so that $v = s\cdot w + \lambda\cdot Aw$ holds. By assumption, $\mathfrak{a}$ is isotropic, therefore $v, w \in \mathfrak{a}$ implies via Proposition 2.20(a)

$$0 = \langle v, Aw\rangle_{\mathbb{C}} = \langle s\cdot w + \lambda \cdot Aw, Aw\rangle_{\mathbb{C}} = s \cdot \langle w, Aw\rangle_{\mathbb{C}} + \overline{\lambda} \cdot \langle Aw, Aw\rangle_{\mathbb{C}} = \overline{\lambda}$$

and hence $v \in \mathbb{R}w$. Thus, we have shown $\mathfrak{a} = \mathbb{R}w$, in contradiction to $\dim_{\mathbb{R}}(\mathfrak{a}) \geq 2$. This proves (2.42).

Because of (2.42), there exists $w \in \mathbb{S}(\mathfrak{a})$ with $\varphi(w) \neq \frac{\pi}{4}$. Let $A \in \mathfrak{A}$ be adapted to w and let $x, y \in \mathbb{S}(V(A))$ be such that $w = \cos(\varphi(w))\, x + \sin(\varphi(w))\, Jy$ is a canonical representation of w. In the case $0 < \varphi(w) < \frac{\pi}{4}$, we have $\ker R_w = \mathbb{R}x \oplus \mathbb{R}Jy$ by Corollary 2.50 and therefore (2.41) shows that $\mathfrak{a} = \mathbb{R}x \oplus \mathbb{R}Jy$ holds (remember $\dim_{\mathbb{R}}(\mathfrak{a}) \geq 2$).

In the case $\varphi(w) = 0$, we have $w = x$ and therefore (2.41) and Corollary 2.50 show that we have

$$\mathbb{R}x \subset \mathfrak{a} \subset \mathbb{R}x \ominus J((\mathbb{R}x)^{\perp,V(A)}) \; .$$

Therefore we have $\mathfrak{a} = \mathbb{R}x \oplus \mathfrak{a}'$ with some $\mathfrak{a}' \subset J((\mathbb{R}x)^{\perp,V(A)}) \subset JV(A)$. Because $R|(JV(A))^3$ is the curvature tensor of a space of constant curvature 2 (see Proposition 2.43(c)), we have $\dim_{\mathbb{R}}(\mathfrak{a}') = 1$, and therefore there exists $y' \in \mathbb{S}(V(A))$ with $y' \perp x$ and $\mathfrak{a}' = \mathbb{R}Jy'$; whence $\mathfrak{a} = \mathbb{R}x \oplus \mathbb{R}Jy'$ follows. □

2.55 Corollary. *Every vector of* $\mathbb{V}$ *is contained in a* 2*-dimensional flat subspace of* $\mathbb{V}$*.*

Proof. This follows from Theorem 2.54 because of the existence of canonical representations for all $v \in \mathbb{V}$ (Theorem 2.28(c)). □

Chapter 3

Isometries of the complex quadric

In this chapter we study the group of isometries of a complex quadric. In Section 3.1 we will see that any $\mathbb{C}$Q-automorphism of a $\mathbb{C}$Q-space $(\mathbb{V}, \mathfrak{A})$ gives rise to a holomorphic isometry of the corresponding quadric $Q := Q(\mathfrak{A})$. As a consequence, we see that there is "free mobility of $\mathbb{C}$Q-frames" in Q in the sense that any $\mathbb{C}$Q-isomorphism between tangent spaces of Q can be realized as the tangent map of a suitable holomorphic isometry of Q.

In Section 3.2, we study Q as a symmetric space. In particular, we explicitly describe the symmetric structure of Q (in the sense of the "Lie-theoretical approach" to symmetric spaces described in Appendix A.2) and the canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ it induces. Following the philosophy that the $\mathbb{C}$Q-structure together with the Riemannian metric and the complex structure of Q are the "fundamental geometric entities" of Q, we derive from the results of Sections 2.7 and 2.8 descriptions of the Cartan subalgebras, the roots and the corresponding root spaces of the symmetric space Q in terms of these fundamental entities.

As an application, we give in Section 3.3 a proof of the fact that (1) there are no holomorphic isometries on Q besides those that were already described in Section 3.1 and that (2) if the dimension of Q is $\neq 2$, then any isometry of Q is either holomorphic or anti-holomorphic. These facts are already found in [Rec95] (as Corollary 2 there). However, by making use of our terminology of complex quadrics with respect to an arbitrary conjugation, a much shorter proof than that from [Rec95] can be given for fact (1). Also the proof of fact (2) given here is different from that of [Rec95] (although it is based on a similar idea).

Finally, we consider 2-dimensional complex quadrics specifically, which play an exceptional role in several respects. For example, on a 2-dimensional complex quadric there exist isometries which are neither holomorphic nor anti-holomorphic. These exceptionalities can be traced to the fact that a 2-dimensional complex quadric is as a Hermitian symmetric space isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and is therefore, unlike the complex quadrics of other dimension, not irreducible. In Section 3.4 we explicitly describe an isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \to Q^2$ via the *Segre embedding*.

Throughout the chapter we make extensive use of the theory of symmetric spaces, including the theory of root systems. An exposition of the aspects of the theory which are of relevance here, and which also fixes the notations we shall use, is given in Appendix A.

3.1 Holomorphic and anti-holomorphic isometries of Q

We let a $\mathbb{C}$Q-space $(\mathbb{V},\mathfrak{A})$ of dimension $n \geq 3$ be given and consider the corresponding complex quadric $Q := Q(\mathfrak{A})$ of dimension $m := n-2$ along with the quadratic cone $\widehat{Q} := \widehat{Q}(\mathfrak{A})$ and the set $\widetilde{Q} := \widetilde{Q}(\mathfrak{A})$.

As before, we denote for any unitary or anti-unitary map $B : \mathbb{V} \to \mathbb{V}$ by $\underline{B} : \mathbb{P}(\mathbb{V}) \to \mathbb{P}(\mathbb{V}),\ [z] \mapsto [Bz]$ the corresponding holomorphic resp. anti-holomorphic map of $\mathbb{P}(\mathbb{V})$. We will also make extensive use of the Lie group $I(Q)$ of isometries of Q, of its subgroup $I_h(Q)$ of holomorphic isometries and of the coset $I_{ah}(Q)$ of anti-holomorphic isometries. We will use the Lie subgroups $\mathrm{Aut}(\mathfrak{A}), \mathrm{Aut}_s(\mathfrak{A}) \subset \mathrm{U}(\mathbb{V})$ of $\mathbb{C}$Q-automorphisms resp. strict $\mathbb{C}$Q-automorphisms of $(\mathbb{V},\mathfrak{A})$ and the coset $\overline{\mathrm{Aut}}(\mathfrak{A})$ of $\mathbb{C}$Q-anti-automorphisms (see Definition 2.10(a),(b),(c)). $\overline{\mathrm{Aut}}(\mathfrak{A})$ is contained in the set $\overline{\mathrm{U}}(\mathbb{V})$ of anti-unitary transformations of $\mathbb{V}$.

At first, we recapitulate well-known facts about isometries of $\mathbb{P}(\mathbb{V})$ already mentioned in Section 1.2.

3.1 Proposition. *(a) For every $B \in \mathrm{U}(\mathbb{V})$ we have $\underline{B} \in I_h(\mathbb{P}(\mathbb{V}))$. In particular, $T_p\underline{B} : T_p\mathbb{P}(\mathbb{V}) \to T_{\underline{B}(p)}\mathbb{P}(\mathbb{V})$ is a $\mathbb{C}$-linear isometry for any $p \in \mathbb{P}(\mathbb{V})$.*

(b) For every $B \in \overline{\mathrm{U}}(\mathbb{V})$ we have $\underline{B} \in I_{ah}(\mathbb{P}(\mathbb{V}))$. In particular, $T_p\underline{B} : T_p\mathbb{P}(\mathbb{V}) \to T_{\underline{B}(p)}\mathbb{P}(\mathbb{V})$ is an anti-linear isometry for any $p \in \mathbb{P}(\mathbb{V})$.

(c) If $B \in \mathrm{U}(\mathbb{V})$ satisfies $\underline{B} = \mathrm{id}_{\mathbb{P}(\mathbb{V})}$, then there exists $\lambda \in \mathbb{S}^1$ with $B = \lambda\,\mathrm{id}_{\mathbb{V}}$.

We now show that isometries of Q can be obtained in an analogous way:

3.2 Proposition. *(a) For every $B \in \mathrm{Aut}(\mathfrak{A})$ we have $\underline{B}|Q \in I_h(Q)$ and for any $p \in Q$, $T_p(\underline{B}|Q) : T_pQ \to T_{\underline{B}(p)}Q$ is a $\mathbb{C}$Q-isomorphism.*

(b) For every $B \in \overline{\mathrm{Aut}}(\mathfrak{A})$ we have $\underline{B}|Q \in I_{ah}(Q)$ and for any $p \in Q$, $T_p(\underline{B}|Q) : T_pQ \to T_{\underline{B}(p)}Q$ is a $\mathbb{C}$Q-anti-isomorphism.

3.3 Remark. Proposition 3.2(b) shows in particular that $\underline{A}|Q$ is an anti-holomorphic isometry on Q for $A \in \mathfrak{A}$. This isometry does not depend on the choice of $A \in \mathfrak{A}$, and is therefore "canonical" in the sense that it is derived exclusively from the geometric objects which define Q. As we will see in Section 5.4, the only point of Q of maximal distance from some $p \in Q$ is the point $\underline{A}(p)$. For this reason, we call $\underline{A}|Q$ the *antipode map* of Q.

Proof of Proposition 3.2. For (a). Let $B \in \mathrm{Aut}(\mathfrak{A})$ be given. As a $\mathbb{C}$Q-isomorphism of $(\mathbb{V},\mathfrak{A})$, B leaves $\widetilde{Q} = M_{\pi/4}$ invariant (see Proposition 2.34(a)), and therefore we have $\underline{B}(Q) = Q$. By Proposition 3.1(a) we have $\underline{B} \in I_h(\mathbb{P}(\mathbb{V}))$; because Q is a complex, regular submanifold of $\mathbb{P}(\mathbb{V})$, we conclude $\underline{B}|Q \in I_h(Q)$.

Now, let $p \in Q$ and $\zeta \in \perp^1_p(Q \hookrightarrow \mathbb{P}(\mathbb{V}))$ be given. $T_p(\underline{B}|Q) = T_p\underline{B}|T_pQ$ is a $\mathbb{C}$-linear isometry by Proposition 3.1(a). Moreover, Proposition 3.1(a) shows that $\underline{B}$ is an isometry of the ambient space $\mathbb{P}(\mathbb{V})$, and therefore $\underline{B}(Q) = Q$ implies $\underline{B}_*(\perp^1_p(Q \hookrightarrow \mathbb{P}(\mathbb{V}))) = \perp^1_{\underline{B}(p)}(Q \hookrightarrow \mathbb{P}(\mathbb{V}))$. Via the Weingarten equation, we conclude

$$A^Q_{\underline{B}_*\zeta} \circ (T_p\underline{B}|T_pQ) = (T_p\underline{B}|T_pQ) \circ A^Q_\zeta \ ,$$

where A^Q denotes the shape operator of $Q \hookrightarrow \mathbb{P}(\mathbb{V})$. This shows $T_p(\underline{B}|Q) = T_p\underline{B}|T_pQ$ to be a $\mathbb{C}$Q-isomorphism.

For (b). The proof is analogous to the proof of (a). □

3.4 Corollary. *The Lie group action* $\Psi : \mathrm{Aut}_s(\mathfrak{A})_0 \times Q \to Q,\ (B,p) \mapsto \underline{B}(p)$ *is transitive; in this way* Q *is a Riemannian homogeneous* $\mathrm{Aut}_s(\mathfrak{A})_0$*-space. In particular,* Q *is connected and complete.*

Proof. To prove the transitivity of the action Ψ let $p_1, p_2 \in Q$ be given. Then we have to show that there exists $B \in \mathrm{Aut}_s(\mathfrak{A})_0$ so that $\underline{B}(p_1) = p_2$ holds.

We choose $z_k \in \widetilde{Q} = M_{\pi/4}$ so that $p_k = [z_k]$ holds for $k \in \{1,2\}$. By Proposition 2.36 there exists $B \in \mathrm{Aut}(\mathfrak{A})$ so that $Bz_1 = z_2$ and therefore $\underline{B}(p_1) = p_2$ holds. Because we have $\underline{\lambda B} = \underline{B}$ for every $\lambda \in \mathbb{S}^1$, we may suppose without loss of generality that $B \in \mathrm{Aut}_s(\mathfrak{A})$ holds. In the case that we have $B \in \mathrm{Aut}_s(\mathfrak{A})_0$, we are finished.

Thus, we now consider the case $B \in \mathrm{Aut}_s(\mathfrak{A}) \setminus \mathrm{Aut}_s(\mathfrak{A})_0$. Let us fix $A \in \mathfrak{A}$. Then Proposition 2.17(a) shows that there exists $L \in \mathrm{O}(V(A))$ with $\det L = -1$ so that $B = L^{\mathbb{C}}$ holds. Because of $n \geq 3$ there exists $x \in \mathbb{S}(V(A))$ with $x \perp \mathrm{Re}_A\, z_1, \mathrm{Im}_A\, z_1$. The orthogonal map $S : V(A) \to V(A)$ with

$$Sx = -x \quad \text{and} \quad S|(\mathbb{R}x)^\perp = \mathrm{id}_{(\mathbb{R}x)^\perp}$$

satisfies $\det S = -1$. Thus we have $(L \circ S)^{\mathbb{C}} \in \mathrm{Aut}_s(\mathfrak{A})_0$, also $\underline{(L \circ S)^{\mathbb{C}}}(p_1) = \underline{B}(p_1) = p_2$ holds.

Proposition 3.2(a) shows that Ψ acts via isometries on Q and therefore, Q is a Riemannian homogeneous $\mathrm{Aut}_s(\mathfrak{A})_0$-space. Because $\mathrm{Aut}_s(\mathfrak{A})_0$ is connected, we see that Q is connected. As a Riemannian homogeneous space, Q is complete (see [O'N83], Remark 9.37, p. 257). □

3.5 Theorem. (Mobility in the quadric.) *Let* $p_1, p_2 \in Q$ *and a map* $L : T_{p_1}Q \to T_{p_2}Q$ *be given.*

(a) If L *is a* $\mathbb{C}$Q*-isomorphism, there exists one and only one* $f \in I_h(Q)$ *with*

$$f(p_1) = p_2 \quad \text{and} \quad T_{p_1}f = L \ ,$$

and we have $f = \underline{B}|Q$ *for some* $B \in \mathrm{Aut}_s(\mathfrak{A})$.

(b) If L *is a* $\mathbb{C}$Q*-anti-isomorphism, there exists one and only one* $f \in I_{ah}(Q)$ *with*

$$f(p_1) = p_2 \quad \text{and} \quad T_{p_1}f = L \ ,$$

and we have $f = \underline{B}|Q$ *for some* $B \in \overline{\mathrm{Aut}}(\mathfrak{A})$.

3.6 Remark. As we will see in Section 3.3, any holomorphic or anti-holomorphic isometry of Q is obtained in the way described in Theorem 3.5. Moreover, for $m \neq 2$, any isometry of Q is either holomorphic or anti-holomorphic.

Proof of Theorem 3.5. For (a). Because Q is connected (Corollary 3.4), the uniqueness of f follows from the rigidity of isometries.

For the existence proof, we fix $A \in \mathfrak{A}$ and denote by A^Q the shape operator of $Q \hookrightarrow \mathbb{P}(\mathbb{V})$, by $\pi : \mathbb{S}(\mathbb{V}) \to \mathbb{P}(\mathbb{V})$ the Hopf fibration, by ξ the unit normal field along $\pi|\widetilde{Q}$ introduced in Section 1.3 and by C the endomorphism field on the manifold $\mathbb{V}$ induced by A in the way also described in Section 1.3.

We fix $z_1 \in \pi^{-1}(\{p_1\})$, then $A^Q_{\xi(z_1)} \in \mathfrak{A}(Q,p_1)$ holds, and therefore we have $L \circ A^Q_{\xi(z_1)} \circ L^{-1} \in \mathfrak{A}(Q,p_2)$ because L is a $\mathbb{C}$Q-isomorphism. By Proposition 1.15 it follows that there exists $z_2 \in \pi^{-1}(\{p_2\})$ so that

$$L \circ A^Q_{\xi(z_1)} \circ L^{-1} = A^Q_{\xi(z_2)}$$

holds. Theorem 1.16 therefore shows that the $\mathbb{C}$-linear isometry $B_0 : \mathcal{H}_{z_1}Q \to \mathcal{H}_{z_2}Q$ determined by

$$(\pi_*|\mathcal{H}_{z_2}Q) \circ B_0 = L \circ (\pi_*|\mathcal{H}_{z_1}Q) \tag{3.1}$$

satisfies

$$B_0 \circ (C_{z_1}|\mathcal{H}_{z_1}Q) = (C_{z_2}|\mathcal{H}_{z_2}Q) \circ B_0 \; . \tag{3.2}$$

We now consider the $\mathbb{C}$-linear map $B : \mathbb{V} \to \mathbb{V}$ characterized by

$$Bz_1 = z_2, \quad B(Az_1) = Az_2 \quad \text{and} \quad \forall v \in \mathcal{H}_{z_1}Q \; : \; B(\overrightarrow{v}) = \overrightarrow{B_0 v} \; ; \tag{3.3}$$

B is well-defined and a $\mathbb{C}$-linear isometry because (z_k, Az_k) is a unitary basis of $(\overrightarrow{\mathcal{H}_{z_k}Q})^{\perp,\mathbb{V}}$ for $k \in \{1,2\}$. Equations (3.3) and (3.2) show that B is a strict $\mathbb{C}$Q-automorphism of $(\mathbb{V},\mathfrak{A})$. Therefore we have $f := \underline{B}|Q \in I_h(Q)$ by Proposition 3.2(a). We have $f(p_1) = \pi(Bz_1) = \pi(z_2) = p_2$. Moreover, if we abbreviate $\pi_Q := \pi|\widetilde{Q}$ and $B_Q := B|\widetilde{Q}$, we have $f \circ \pi_Q = \pi_Q \circ B_Q$ and therefore

$$\begin{aligned} T_{p_1}f \circ (\pi_*|\mathcal{H}_{z_1}Q) &= T_{z_1}(f \circ \pi_Q)|\mathcal{H}_{z_1}Q = T_{z_1}(\pi_Q \circ B_Q)|\mathcal{H}_{z_1}Q = (\pi_*|\mathcal{H}_{z_2}Q) \circ (T_{z_1}B_Q|\mathcal{H}_{z_1}Q) \\ &\overset{(3.3)}{=} (\pi_*|\mathcal{H}_{z_2}Q) \circ B_0 \overset{(3.1)}{=} L \circ (\pi_*|\mathcal{H}_{z_1}Q) \; , \end{aligned}$$

hence $T_{p_1}f = L$.

For (b). The uniqueness of f once again follows from the rigidity of isometries. For the existence proof, we fix $A \in \mathfrak{A}$ and consider the antipode map $\underline{A}|Q$, which is an anti-holomorphic isometry of Q as we already noted in Remark 3.3. Moreover, Proposition 3.2(b) shows that $T_{p_2}(\underline{A}|Q) : T_{p_2}Q \to T_{\underline{A}(p_2)}Q$ is a $\mathbb{C}$Q-anti-isomorphism. Hence

$$T_{p_2}(\underline{A}|Q) \circ L : T_{p_1}Q \to T_{\underline{A}(p_2)}Q$$

is a $\mathbb{C}$Q-isomorphism. It follows by (a) that there exists $\widetilde{B} \in \mathrm{Aut}_s(\mathfrak{A})$ so that $\widetilde{f} := \underline{\widetilde{B}}|Q \in I_h(Q)$ satisfies

$$\widetilde{f}(p_1) = \underline{A}(p_2) \quad \text{and} \quad T_{p_1}\widetilde{f} = T_{p_2}(\underline{A}|Q) \circ L \; .$$

Put $B := A \circ \widetilde{B} \in \overline{\mathrm{Aut}}(\mathfrak{A})$ and $f := \underline{B}|Q = (\underline{A}|Q) \circ \widetilde{f} \in I_{ah}(Q)$. $\underline{A}|Q$ is involutive along with A, and therefore we have

$$f(p_1) = \underline{A}(\widetilde{f}(p_1)) = \underline{A}(\underline{A}(p_2)) = p_2$$

and

$$T_{p_1} f = T_{\underline{A}(p_2)}(\underline{A}|Q) \circ T_{p_1}\widetilde{f} = T_{\underline{A}(p_2)}(\underline{A}|Q) \circ T_{p_2}(\underline{A}|Q) \circ L = T_{p_2}((\underline{A}|Q) \circ (\underline{A}|Q)) \circ L = L\,.$$

□

3.7 Remark. Another proof of the existence of isometries of Q corresponding to $\mathbb{C}$Q-(anti-)isomorphisms of tangent spaces of Q can be given via the Theorem of Cartan/Ambrose/Hicks (see [KN63], Theorem VI.7.4, p. 261f.), if one uses the fact that Q is a locally Riemannian symmetric space which is simply connected (Remark 1.24(a)) and complete (Corollary 3.4): If $L : T_{p_1}Q \to T_{p_2}Q$ is a $\mathbb{C}$Q-isomorphism or a $\mathbb{C}$Q-anti-isomorphism, then L is curvature-equivariant by Proposition 2.45, and therefore the Theorem of Cartan/Ambrose/Hicks shows that there is an affine diffeomorphism $f : Q \to Q$ with $f(p_1) = p_2$ and $T_{p_1} f = L$. Because the Riemannian metric and the complex structure of Q are parallel, the invariance of L with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ causes f to be an isometry, and the complex (anti-)linearity of L causes f to be (anti-)holomorphic.

3.8 Proposition. *The kernel of the Lie group homomorphism* $\Phi : \mathrm{Aut}(\mathfrak{A}) \to I_h(Q),\ B \mapsto \underline{B}|Q$ *is* $\{\,\lambda\,\mathrm{id}_{\mathbb{V}} \mid \lambda \in \mathbb{S}^1\,\}$.

Proof. It is clear that $\{\,\lambda\,\mathrm{id}_{\mathbb{V}} \mid \lambda \in \mathbb{S}^1\,\} \subset \ker(\Phi)$ holds. Conversely, let $B \in \mathrm{Aut}(\mathfrak{A})$ be given with $\underline{B}|Q = \mathrm{id}_Q$. Let us fix $p \in Q$, then we have

$$T_p\underline{B}|T_pQ = T_p(\underline{B}|Q) = \mathrm{id}_{T_pQ}\,. \tag{3.4}$$

Let us also fix $\zeta \in \perp^1_p(Q \hookrightarrow \mathbb{P}(\mathbb{V}))$ and denote by A^Q the shape operator of $Q \hookrightarrow \mathbb{P}(\mathbb{V})$. As in the proof of Proposition 3.2 we get

$$A^Q_{\underline{B}_*\zeta} \overset{(3.4)}{=} A^Q_{\underline{B}_*\zeta} \circ (T_p\underline{B}|T_pQ) = (T_p\underline{B}|T_pQ) \circ A^Q_\zeta \overset{(3.4)}{=} A^Q_\zeta\,. \tag{3.5}$$

The $\mathbb{C}$-linear map $\perp_p(Q \hookrightarrow \mathbb{P}(\mathbb{V})) \to \mathrm{End}(T_pQ),\ \eta \mapsto A^Q_\eta$ is injective, and therefore Equation (3.5) implies $\underline{B}_*\zeta = \zeta$. This fact together with Equation (3.4) shows that $T_p\underline{B} = \mathrm{id}_{T_p\mathbb{P}(\mathbb{V})}$ holds, whence $\underline{B} = \mathrm{id}_{\mathbb{P}(\mathbb{V})}$ follows by the rigidity of isometries. Thus Proposition 3.1(c) shows that $B = \lambda \cdot \mathrm{id}_{\mathbb{V}}$ holds for some $\lambda \in \mathbb{S}^1$. □

3.2 Q as a Hermitian symmetric $\mathrm{Aut}_s(\mathfrak{A})_0$-space

In the following proposition we explain in what way the complex quadric Q is a Hermitian symmetric $\mathrm{Aut}_s(\mathfrak{A})_0$-space (Q, Ψ, p_0, σ) (compare Appendix A.2, where the view of symmetric spaces as such a "datum" is described under the heading "Lie theoretical approach").

We denote by $\pi : \mathbb{S}(\mathbb{V}) \to \mathbb{P}(\mathbb{V})$ the Hopf fibration and abbreviate

$$\boxed{G := \mathrm{Aut}_s(\mathfrak{A})_0} \quad ;$$

then we have for any $A \in \mathfrak{A}$:

$$G = \{\, B \in \mathrm{U}(\mathbb{V}) \,|\, B \circ A = A \circ B,\ \det(B) = 1 \,\} \;.$$

Moreover, we denote by $\mathfrak{g} := \mathfrak{aut}_s(\mathfrak{A})$ the Lie algebra of G and consider its Killing form $\varkappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. $\varkappa$ has been described explicitly in Proposition 2.17(a), see Equation (2.6). G acts on Q via the Lie group action $\Psi : G \times Q \to Q$ described in Corollary 3.4.

3.9 Proposition. *Let $p_0 \in Q$ be given, say $p_0 = \pi(z_0)$ with $z_0 \in \widetilde{Q}$.*

(a) $W := \mathrm{span}_{\mathfrak{A}}\{z_0\}$ is a 2-dimensional $\mathbb{C}$Q-subspace of $\mathbb{V}$ which depends only on p_0, not on the choice of $z_0 \in \pi^{-1}(\{p_0\})$. We denote the induced $\mathbb{C}$Q-structure of the $\mathbb{C}$Q-subspaces W and $W^\perp$ by $\mathfrak{A}_W$ and $\mathfrak{A}_{W^\perp}$, respectively. The isotropy group K of Ψ at p_0 is then given by

$$K = \{\ B \in G \ | \ B|W \in \mathrm{Aut}_s(\mathfrak{A}_W)_0 \ \} \;. \tag{3.6}$$

Moreover, the map

$$F : K \to \mathrm{Aut}_s(\mathfrak{A}_W)_0 \times \mathrm{Aut}_s(\mathfrak{A}_{W^\perp})_0,\ B \mapsto (B|W,\ B|W^\perp)$$

is an isomorphism of Lie groups, consequently K is connected.

(b) The image of the isotropy representation

$$\Theta : K \to \mathrm{U}(T_{p_0}Q),\ B \mapsto T_{p_0}(\underline{B}|Q)$$

is $\mathrm{Aut}(\mathfrak{A}(Q,p_0))_0$. Θ is injective for m odd, whereas its kernel is $\{\pm\mathrm{id}_{\mathbb{V}}\}$ for m even. It follows that $\Theta : K \to \mathrm{Aut}(\mathfrak{A}(Q,p_0))_0$ is an isomorphism of Lie groups for m odd, a two-fold covering map of Lie groups for m even. Moreover, the action Ψ is almost effective.

(c) Let $S : \mathbb{V} \to \mathbb{V}$ be the linear involution characterized by

$$S|W = \mathrm{id}_W \quad \textit{and} \quad S|W^\perp = -\mathrm{id}_{W^\perp} \;.$$

Then we have $-S \in G$, and the involutive Lie group automorphism

$$\sigma : G \to G,\ B \mapsto S \circ B \circ S^{-1}$$

satisfies $\mathrm{Fix}(\sigma)_0 = K$.

Consequently (Q, Ψ, p_0, σ) is a Hermitian symmetric G-space in the sense of the "Lie theoretical approach" of Appendix A.2; its canonical covariant derivative is identical to the Levi-Civita covariant derivative of Q. This symmetric space is of compact type; it is irreducible for $m \neq 2$.

(d) *The canonical decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ *with respect to* σ *is given by*

$$\mathfrak{k} = \{ X \in \mathfrak{g} \,|\, X(W) \subset W,\ X(W^\perp) \subset W^\perp \}\,, \tag{3.7}$$

$$\mathfrak{m} = \{ X \in \mathfrak{g} \,|\, X(W) \subset W^\perp,\ X(W^\perp) \subset W \}\,. \tag{3.8}$$

Proof. For (a). Let us fix $A \in \mathfrak{A}$. Clearly, $W = \mathbb{C}z_0 \ominus \mathbb{C}Az_0$ is a 2-dimensional $\mathbb{C}$Q-subspace of $\mathbb{V}$ and $Y := V(A|W) = W \cap V(A)$ is a real-2-dimensional subspace of $V(A)$. Immediately we will show

$$K = \{\ B \in G \ |\ B|Y \in \mathrm{SO}(Y)\ \}\,; \tag{3.9}$$

Equation (3.6) follows therefrom by Proposition 2.17(a).

To prove Equation (3.9), we first note that with $x := \sqrt{2}\,\mathrm{Re}_A\, z_0$ and $y := \sqrt{2}\,\mathrm{Im}_A\, z_0$, (x, y) is an orthonormal basis of Y by Proposition 2.23(b). For given $B \in G$ we have:

$$\begin{aligned} B \in K &\iff \underline{B}(p_0) = p_0 \iff \exists\, \lambda \in \mathbb{S}^1\ :\ Bz_0 = \lambda z_0 \\ &\iff \exists\,(a+ib) \in \mathbb{S}^1\ :\ Bx + JBy = (a+ib)(x+Jy) = (ax-by) + J(bx+ay) \\ &\iff \exists\,(a+ib) \in \mathbb{S}^1\ :\ (Bx = ax - by \quad \text{and} \quad By = bx + ay)\,; \end{aligned}$$

this calculation shows that $B \in K$ holds if and only if $B(Y) = Y$ holds and there exist $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ so that $B|Y$ is represented by the matrix $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ with respect to the orthonormal basis (x, y) of Y. But this is the case if and only if $B|Y \in \mathrm{SO}(Y)$ holds. Thus we have shown Equation (3.9).

For any $B \in G$, we have $B|V(A) \in \mathrm{SO}(V(A))$ by Proposition 2.17(a). Therefore, we have for any $B \in K$ besides $B|Y \in \mathrm{SO}(Y)$ also $B|Y^{\perp,V(A)} \in \mathrm{SO}(Y^{\perp,V(A)})$. It follows that

$$K \to \mathrm{SO}(Y) \times \mathrm{SO}(Y^{\perp,V(A)}),\ B \mapsto (B|Y,\ B|Y^{\perp,V(A)})$$

is an isomorphism of Lie groups, whence the statement on F follows via Proposition 2.17(a).

For (b). It is clear that Θ is a homomorphism of Lie groups. Let $B \in K$ be given, then we have

$$B \in \ker(\Theta) \iff T_{p_0}(\underline{B}|Q) = \mathrm{id}_{T_{p_0}Q} \overset{(*)}{\iff} \underline{B}|Q = \mathrm{id}_Q \overset{(\dagger)}{\iff} B \in \{\,\lambda\,\mathrm{id}_{\mathbb{V}} \,|\, \lambda \in \mathbb{S}^1\,\} \cap G\,.$$

Here the equivalence marked $(*)$ is a consequence of the rigidity of isometries, and the equivalence marked $(\dagger)$ is justified by Proposition 3.8. We have $\{\,\lambda\,\mathrm{id}_{\mathbb{V}} \,|\, \lambda \in \mathbb{S}^1\,\} \cap \mathrm{Aut}_s(\mathfrak{A}) = \{\pm\mathrm{id}_{\mathbb{V}}\}$, and Proposition 2.17(a) shows that $-\mathrm{id}_{\mathbb{V}} \in G$ holds if and only if m is even. This shows that Θ is injective for m odd and has kernel $\{\pm\mathrm{id}_{\mathbb{V}}\}$ for m even.

Proposition 3.2(a) shows that $\Theta(K) \subset \mathrm{Aut}(\mathfrak{A}(Q, p_0))$ holds; because K is connected, we in fact have $\Theta(K) \subset \mathrm{Aut}(\mathfrak{A}(Q, p_0))_0$. Because the kernel of Θ is discrete and

$$\dim K = \dim(\mathrm{Aut}_s(\mathfrak{A}_W)_0 \times \mathrm{Aut}_s(\mathfrak{A}_{W^\perp})_0) = 1 + \tfrac{m(m-1)}{2} = \dim \mathrm{Aut}(\mathfrak{A}(Q, p_0))_0$$

holds (see (a) and Proposition 2.17), it follows that $\Theta(K) = \mathrm{Aut}(\mathfrak{A}(Q, p_0))_0$ holds. The statement about $\Theta : K \to \mathrm{Aut}(\mathfrak{A}(Q, p_0))_0$ being an isomorphism or a two-fold covering map of Lie group now follows immediately.

For $B \in G$ we have

$$\Psi_B = \mathrm{id}_Q \iff \big(B \in K \text{ and } \Theta(B) = \mathrm{id}_{T_pQ}\big) \iff B \in \{\pm\mathrm{id}_{\mathbb{V}}\} \cap G$$

because of the rigidity of isometries and the preceding result on $\ker(\Theta)$. This shows that Ψ is almost effective.

For (c). S is the complexification of the linear map $S' : V(A) \to V(A)$ characterized by $S'|Y = \mathrm{id}_Y$ and $S'|Y^{\perp,V(A)} = -\mathrm{id}_{Y^{\perp,V(A)}}$. We have $-S' \in \mathrm{SO}(V(A))$ and therefore $-S \in G$ by Proposition 2.17(a). Because we have $\sigma(B) = (-S) \circ B \circ (-S)^{-1}$ for every $B \in G$ and $-S$ is involutive, we see that $\sigma : G \to G$ is an involutive Lie group automorphism of G. We will now show

$$\mathrm{Fix}(\sigma) = \{\, B \in G \,|\, B|W \in \mathrm{Aut}_s(\mathfrak{A}_W)\,\} \,. \tag{3.10}$$

First, let $B \in \mathrm{Fix}(\sigma)$ be given. Then we have $S \circ B = B \circ S$ and therefore B leaves the eigenspace $\mathrm{Eig}(S,1) = W$ invariant. Because of $B \in \mathrm{Aut}_s(\mathfrak{A})$ it follows that $B|W \in \mathrm{Aut}_s(\mathfrak{A}_W)$ holds. Conversely, let $B \in G$ be given such that $B|W \in \mathrm{Aut}_s(\mathfrak{A}_W)$ holds. Then the unitary transformation B leaves the spaces $W = \mathrm{Eig}(S,1)$ and $W^\perp = \mathrm{Eig}(S,-1)$ invariant. Because we have $\mathbb{V} = \mathrm{Eig}(S,1) \oplus \mathrm{Eig}(S,-1)$, it follows that $B \circ S = S \circ B$ and therefore $B \in \mathrm{Fix}(\sigma)$ holds. This completes the proof of Equation (3.10).

It follows from Equation (3.10) that $\mathrm{Fix}(\sigma)_0 = \{\, B \in G \,|\, B|W \in \mathrm{Aut}_s(\mathfrak{A}_W)_0\,\} \overset{\text{(a)}}{=} K$ holds and thus we have

$$K \subset \mathrm{Fix}(\sigma) \quad \text{and} \quad \dim K = \dim \mathrm{Fix}(\sigma) \,.$$

Because G acts by Ψ transitively and via holomorphic isometries on Q, and Ψ is almost effective by (b), it follows that (Q,Ψ,p_0,σ) is a Hermitian symmetric G-space. The claim that the canonical covariant derivative of this symmetric space is identical to the Levi-Civita covariant derivative of Q is just a rephrasing for Q of the fact that any Riemannian symmetric space is naturally reductive with its canonical reductive structure, see Appendix A.3. It follows from Equation (2.6) in Proposition 2.17 that the Killing form $\varkappa$ of $\mathfrak{g}$ is negative definite, and therefore the Hermitian symmetric G-space Q is of compact type. Moreover, for $m \neq 2$, $\Theta(K) = \mathrm{Aut}(\mathfrak{A}(Q,p_0))_0$ acts irreducibly on $T_{p_0}Q$ by Proposition 2.39(b), and therefore Q is then irreducible.

For (d). The linearization of the Lie group automorphism σ is given by

$$\sigma_L : \mathfrak{g} \to \mathfrak{g},\ X \mapsto S \circ X \circ S^{-1} \,.$$

For $X \in \mathfrak{g}$ we therefore have $X \in \mathfrak{k} = \mathrm{Eig}(\sigma_L,1)$ if and only if X and S commute, which is the case if and only if X leaves the spaces $\mathrm{Eig}(S,1) = W$ and $\mathrm{Eig}(S,-1) = W^\perp$ invariant. Similarly, we have $X \in \mathfrak{m} = \mathrm{Eig}(\sigma_L,-1)$ if and only if $X(\mathrm{Eig}(S,\pm 1)) \subset \mathrm{Eig}(S,\mp 1)$ holds. $\square$

3.10 Remarks. (a) Proposition 3.9(a) shows that Q is as a homogeneous space isomorphic to the quotient space G/K. For fixed $A \in \mathfrak{A}$, G is (via $B \mapsto B|V(A)$) isomorphic to $\mathrm{SO}(V(A)) \cong \mathrm{SO}(m+2)$ (see Proposition 2.17(a)), and similarly, K is isomorphic to $\mathrm{SO}(2) \times \mathrm{SO}(m)$. Thus we obtain the conventional quotient representation $\mathrm{SO}(m+2)/(\mathrm{SO}(2) \times \mathrm{SO}(m))$ of Q.

(b) In the case $m = 2$, Q is indeed reducible. In fact, as we will see in Section 3.4, Q^2 is as a Hermitian symmetric space isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

3.11 Proposition. *Q is an extrinsically symmetric submanifold of $\mathbb{P}(\mathbb{V})$; this means that for every $p \in Q$ there exists an isometry $\widetilde{s}_p \in I(\mathbb{P}(\mathbb{V}))$ with $\widetilde{s}_p(p) = p$ and $\widetilde{s}_p(Q) = Q$ so that $T_p\widetilde{s}_p : T_p\mathbb{P}(\mathbb{V}) \to T_p\mathbb{P}(\mathbb{V})$ is the reflection in the normal space of Q at p (see for example [NT89], p. 157). Note that $\widetilde{s}_p|Q$ is the geodesic symmetry of Q at p (see the "geometric approach" in Appendix A.2).*

Proof. Let $p \in Q$ be given, fix $z \in \pi^{-1}(\{p\})$, put $W := \mathrm{span}_{\mathfrak{A}}\{z\}$ and consider the linear involution $S : \mathbb{V} \to \mathbb{V}$ characterized by

$$S|W = \mathrm{id}_W \quad \text{and} \quad S|W^\perp = -\mathrm{id}_{W^\perp}\ ;$$

note that we already used S for $p = p_0$ to describe the Lie group automorphism $\sigma : G \to G$ in Proposition 3.9. We have $S \in \mathrm{Aut}_s(\mathfrak{A})$, and therefore $\widetilde{s}_p := \underline{S} \in I_h(\mathbb{P}(\mathbb{V}))$ leaves Q invariant by Proposition 3.2(a); moreover we have $\widetilde{s}_p(p) = p$ and $T_p\widetilde{s}_p$ is the reflection in the normal space of Q at p. □

Let us fix $p_0 \in Q$ and consider the Hermitian symmetric G-space (Q, Ψ, p_0, σ) and the canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ as in Proposition 3.9. We have the canonical isomorphism

$$\tau : \mathfrak{m} \to T_{p_0}Q,\ X \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big(\underline{(\mathrm{Exp}(tX))}(p_0)\big)\ ,$$

where $\mathrm{Exp} : \mathfrak{g} \to G$ is the exponential map of G. Remember that for every $B \in K$ (where K is the isotropy group of the action Ψ at p_0), the diagram

$$\begin{tikzcd} \mathfrak{m} \arrow[r, "\mathrm{Ad}(B)|\mathfrak{m}"] \arrow[d, "\tau"'] & \mathfrak{m} \arrow[d, "\tau"] \\ T_{p_0}Q \arrow[r, "T_{p_0}\underline{B}"'] & T_{p_0}Q \end{tikzcd}$$

commutes, and that if we denote by R the curvature tensor of Q,

$$\forall X, Y, Z \in \mathfrak{m}\ :\ R(\tau(X), \tau(Y))\tau(Z) = -\tau([[X, Y], Z]) \tag{3.11}$$

holds (see Equations (A.11) and (A.12)).

In the sequel, we equip $\mathfrak{m}$ with the complex structure $J^{\mathfrak{m}}$, the complex inner product $\langle\cdot,\cdot\rangle^{\mathfrak{m}}_{\mathbb{C}}$ (which induces the real inner product $\langle\cdot,\cdot\rangle^{\mathfrak{m}}_{\mathbb{R}} = \mathrm{Re}(\langle\cdot,\cdot\rangle^{\mathfrak{m}}_{\mathbb{C}})$) and the $\mathbb{C}$Q-structure $\mathfrak{A}^{\mathfrak{m}}$ such that the linear isomorphism $\tau : \mathfrak{m} \to T_{p_0}Q$ becomes a $\mathbb{C}$Q-isomorphism.

It is the objective of the following proposition to provide explicit descriptions of τ and of the structures we equipped $\mathfrak{m}$ with. It shows in particular that the real inner product $\langle\cdot,\cdot\rangle^{\mathfrak{m}}_{\mathbb{R}}$ is a negative multiple of the Killing form $\varkappa$ of $\mathfrak{g}$; this fact enables us to apply the root theory as described in Appendix A.4 to the present situation.

We note that for any $z_0 \in \pi^{-1}(\{p_0\})$ and every $X \in \mathfrak{m}$, we have $X(z_0) \in W^\perp = \overrightarrow{\mathcal{H}_{z_0}Q}$ (where W is as in Proposition 3.9(a); note Proposition 3.9(d) and Theorem 2.26), therefore there exists one and only one map $\widehat{z_0} : \mathfrak{m} \to \mathcal{H}_{z_0}Q$ characterized by

$$\forall X \in \mathfrak{m} \;:\; \overrightarrow{\widehat{z_0}(X)} = X(z_0)\ . \tag{3.12}$$

It is clear that $\widehat{z_0}$ is $\mathbb{C}$-linear. Moreover, every given $X \in \mathfrak{m}$ commutes with $A \in \mathfrak{A}$, is skew-Hermitian and satisfies besides $X(W) \subset W^\perp$ also $X(W^\perp) \subset W$ (again see Proposition 3.9(d)); because of these properties X is already uniquely determined by $X(z_0) =: v$, namely via the equations

$$\forall w \in W \;:\; X(w) = \langle w, z_0\rangle_{\mathbb{C}}\, v + \langle w, Az_0\rangle_{\mathbb{C}}\, Av \tag{3.13}$$

and

$$\forall w' \in W^\perp \;:\; X(w') = -\big(\, \langle w', v\rangle_{\mathbb{C}}\, z_0 + \langle w', Av\rangle_{\mathbb{C}}\, Az_0 \,\big)\ . \tag{3.14}$$

Consequently, $\widehat{z_0}$ is an isomorphism of $\mathbb{C}$-linear spaces. As the following proposition shows, this isomorphism is closely related to the isomorphism τ we wish to describe.

We call in mind that $\mathcal{H}_{z_0}Q$ is a $\mathbb{C}$Q-subspace of $T_{z_0}\mathbb{V}$ by Theorem 2.26, and that the map $\mathcal{H}_{z_0}Q \to W^\perp$, $v \mapsto \overrightarrow{v}$ is a $\mathbb{C}$Q-isomorphism.

3.12 Proposition. *We fix $z_0 \in \pi^{-1}(\{p_0\})$ and put $W := \mathrm{span}_{\mathfrak{A}}\{z_0\}$ as in Proposition 3.9.*

(a) We have

$$\tau = \pi_* \circ \widehat{z_0}\ . \tag{3.15}$$

It follows by means of Theorem 2.26 that $\widehat{z_0} : \mathfrak{m} \to \mathcal{H}_{z_0}Q$ is an isomorphism of $\mathbb{C}$Q-spaces, and we obtain the following commutative diagram, where all arrows represent $\mathbb{C}$Q-isomorphisms:

$$\begin{tikzcd} \mathfrak{m} \arrow[r, "\widehat{z_0}"] \arrow[dr, "\tau"'] & \mathcal{H}_{z_0}Q \arrow[d, "\pi_*|\mathcal{H}_{z_0}Q"] \\ & T_{p_0}Q\ . \end{tikzcd} \tag{3.16}$$

(b) For every $X, Y \in \mathfrak{m}$, we have

$$\langle X, Y\rangle^{\mathfrak{m}}_{\mathbb{R}} = -\tfrac{1}{4m} \cdot \varkappa(X, Y)\ , \tag{3.17}$$

where $\varkappa$ is the Killing form of $\mathfrak{g}$. Moreover, we have for every $X \in \mathfrak{m}$

$$(J^{\mathfrak{m}}X)(z_0) = J(Xz_0) \tag{3.18}$$

and $\mathfrak{A}^{\mathfrak{m}} = \{\, A^{\mathfrak{m}} \,|\, A \in \mathfrak{A}\,\}$, where for every $A \in \mathfrak{A}$ the conjugation $A^{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}$ satisfies

$$(A^{\mathfrak{m}}X)(z_0) = A(Xz_0)\ . \tag{3.19}$$

Note that because of Equations (3.13) *and* (3.14), *Equations* (3.18) *and* (3.19) *characterize $J^{\mathfrak{m}}$ resp. $A^{\mathfrak{m}}$ uniquely.*

(c) *Let us denote by $R^{\mathfrak{m}}$ the curvature tensor of the* $\mathbb{C}$Q*-space* $\mathfrak{m}$ *in the sense of Section 2.7. Then we have*

$$\forall X, Y, Z \in \mathfrak{m} \; : \; R^{\mathfrak{m}}(X,Y)Z = -[[X,Y],Z] \; .$$

3.13 Remark. The information from Propositions 3.9(d) and 3.12(a) can be used to obtain an explicit description of the geodesics of Q: For any $v \in T_{p_0}Q$, the maximal geodesic $\gamma_v : \mathbb{R} \to Q$ of Q with $\gamma_v(0) = p_0$ and $\dot\gamma_v(0) = v$ is given by $\gamma_v(t) = \Psi(\mathrm{Exp}(tX), p_0)$, where $X := \tau^{-1}(v) \in \mathfrak{m}$ and $\mathrm{Exp} : \mathfrak{g} \to G$ is the exponential map of G.

In Section 5.4, we will instead describe the geodesics of Q via an explicit description of the maximal tori of the symmetric space Q.

Proof of Proposition 3.12. For (a). We let $X \in \mathfrak{m}$ be given. Again denoting by $\mathrm{Exp} : \mathfrak{g} \to G$ the exponential map of G, we then have

$$\overrightarrow{\widehat{z_0}(X)} = X(z_0) = \overrightarrow{\frac{\dot{\mathrm{d}}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Exp}(tX)z_0}$$

and hence

$$\widehat{z_0}(X) = \frac{\dot{\mathrm{d}}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Exp}(tX)z_0 \; .$$

Therefrom we obtain

$$\pi_* \circ \widehat{z_0}(X) = \pi_* \frac{\dot{\mathrm{d}}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Exp}(tX)z_0 = \frac{\dot{\mathrm{d}}}{\mathrm{d}t}\Big|_{t=0}\big(\pi \circ \mathrm{Exp}(tX)z_0\big) = \frac{\dot{\mathrm{d}}}{\mathrm{d}t}\Big|_{t=0}\big(\underline{\mathrm{Exp}(tX)}(p_0)\big) = \tau(X)$$

and therefore Equation (3.15).

Because $(\pi_*|\mathcal{H}_{z_0}Q) : \mathcal{H}_{z_0}Q \to T_{p_0}Q$ and $\tau : \mathfrak{m} \to T_{p_0}Q$ are $\mathbb{C}$Q-isomorphisms, it follows from Equation (3.15) that also $\widehat{z_0} : \mathfrak{m} \to \mathcal{H}_{z_0}Q$ is a $\mathbb{C}$Q-isomorphism.

For (b). $\widehat{z_0} : \mathfrak{m} \to \mathcal{H}_{z_0}Q$ is a $\mathbb{C}$Q-isomorphism by (a), and therefore $\overrightarrow{(\ldots)} \circ \widehat{z_0} : \mathfrak{m} \to \overrightarrow{\mathcal{H}_{z_0}Q},\; X \mapsto X(z_0)$ also is a $\mathbb{C}$Q-isomorphism. Equations (3.18) and (3.19) are obvious consequences of this fact.

For the proof of Equation (3.17), we fix $A \in \mathfrak{A}$. Any $X \in \mathfrak{m}$ is a skew-Hermitian map $X : \mathbb{V} \to \mathbb{V}$ which interchanges the spaces W and $W^\perp$ by Proposition 3.9(d), and therefore we have

$$(X|W^\perp) = -(X|W)^* : W^\perp \to W \; , \tag{3.20}$$

where we denote for any $\mathbb{C}$-linear map $Z : W \to W^\perp$ by $Z^* : W^\perp \to W$ the adjoint map of Z.

For unitary spaces V_1, V_2 we now denote the usual inner product on $L(V_1, V_2)$ by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{L(V_1,V_2)}$. It should be noted that with respect to an arbitrary basis $(b_1, \ldots, b_r)$ of V_1

$$\forall X, Y \in L(V_1, V_2) \; : \; \langle\!\langle X, Y \rangle\!\rangle_{L(V_1,V_2)} = \sum_{k=1}^{r} \langle X b_k, Y b_k \rangle_{\mathbb{C}} \tag{3.21}$$

holds, and that we have

$$\forall X, Y \in L(V_1, V_2) : \langle\!\langle X^*, Y^* \rangle\!\rangle_{L(V_2,V_1)} = \langle\!\langle X, Y \rangle\!\rangle_{L(V_1,V_2)} \; . \tag{3.22}$$

We now obtain for any $X, Y \in \mathfrak{m}$ via Equation (2.6) in Proposition 2.17(a):

$$\begin{aligned}
\varkappa(X,Y) &\overset{(2.6)}{=} (-m)\cdot \langle\!\langle X,Y\rangle\!\rangle_{L(\mathbb{V},\mathbb{V})} = (-m)\cdot(\langle\!\langle X|W, Y|W\rangle\!\rangle_{L(W,W^\perp)} + \langle\!\langle X|W^\perp, Y|W^\perp\rangle\!\rangle_{L(W^\perp,W)})\\
&\overset{(3.20)}{=} (-m)\cdot(\langle\!\langle X|W, Y|W\rangle\!\rangle_{L(W,W^\perp)} + \langle\!\langle -(X|W)^*, -(Y|W)^*\rangle\!\rangle_{L(W^\perp,W)})\\
&\overset{(3.22)}{=} (-2m)\cdot\langle\!\langle X|W, Y|W\rangle\!\rangle_{L(W,W^\perp)} \overset{(*)}{=} (-2m)\cdot(\langle Xz_0, Yz_0\rangle_{\mathbb{C}} + \langle XAz_0, YAz_0\rangle_{\mathbb{C}})\\
&= (-2m)\cdot(\langle Xz_0, Yz_0\rangle_{\mathbb{C}} + \langle AXz_0, AYz_0\rangle_{\mathbb{C}}) = (-2m)\cdot(\langle Xz_0, Yz_0\rangle_{\mathbb{C}} + \overline{\langle Xz_0, Yz_0\rangle_{\mathbb{C}}})\\
&= (-4m)\cdot\mathrm{Re}(\langle Xz_0, Yz_0\rangle_{\mathbb{C}}) = (-4m)\cdot\langle Xz_0, Yz_0\rangle_{\mathbb{R}} \overset{(\dagger)}{=} (-4m)\cdot\langle X,Y\rangle^{\mathfrak{m}}_{\mathbb{R}}\,,
\end{aligned}$$

where for $(*)$ we used Equation (3.21) and the fact that (z_0, Az_0) is a unitary basis of W, and $(\dagger)$ follows from the fact that $\mathfrak{m}\to W^\perp,\ X\mapsto X(z_0)$ is a linear isometry.

For (c). This follows immediately from Equation (3.11) and the fact that $\tau:\mathfrak{m}\to T_{p_0}Q$ is a $\mathbb{C}$Q-isomorphism. □

3.14 Remark. The curvature tensor of the $\mathbb{C}$Q-space $\mathfrak{m}$ (which is conjugate to the curvature tensor of Q under the $\mathbb{C}$Q-isomorphism $\tau:\mathfrak{m}\to T_{p_0}Q$) can be described, as Proposition 2.47 shows, by the functions ρ and C defined in that proposition. Because of Proposition 3.12(c) we therefore have a relationship between these functions and the "double Lie bracket" $\mathfrak{m}\times\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m},\ (X,Y,Z)\mapsto[[X,Y],Z]$. It is interesting, however, to note that $\rho=\rho^{\mathfrak{m}}$ and $C=C^{\mathfrak{m}}$ can already be expressed by a single Lie bracket, i.e. via the map $\mathfrak{m}\times\mathfrak{m}\to\mathfrak{k},\ (X,Y)\mapsto[X,Y]$, as it is now described.

We let $X, Y\in\mathfrak{m}$ be given. Then we have $Z:=[X,Y]\in\mathfrak{k}$, and this element is determined uniquely by $Z(z_0)$ and $Z|W^\perp$ (because $Z\in\mathfrak{aut}_s(\mathfrak{A})$ holds and we have $\mathbb{V}=\mathrm{span}_{\mathfrak{A}}\{z_0\}\oplus W^\perp$). Abbreviating $u:=X(z_0)\in W^\perp$ and $v:=Y(z_0)\in W^\perp$, we obtain

$$\begin{aligned}
Z(z_0) &= (XY-YX)z_0 = Xv - Yu\\
&\overset{(3.14)}{=} -\langle v,u\rangle_{\mathbb{C}}z_0 - \langle v,Au\rangle_{\mathbb{C}}Az_0 + \langle u,v\rangle_{\mathbb{C}}z_0 + \langle u,Av\rangle_{\mathbb{C}}Az_0\\
&= (\langle u,v\rangle_{\mathbb{C}} - \langle v,u\rangle_{\mathbb{C}})\,z_0 = 2i\cdot\mathrm{Im}(\langle u,v\rangle_{\mathbb{C}})\,z_0 = 2\langle u,Jv\rangle_{\mathbb{R}}\,Jz_0\\
&= 2\langle X, J^{\mathfrak{m}}Y\rangle^{\mathfrak{m}}_{\mathbb{R}}\cdot Jz_0 = \rho^{\mathfrak{m}}(X,Y)\cdot Jz_0\,,
\end{aligned}$$

and also for every $w\in W^\perp$ (where $\wedge$ has the same meaning as in Equation (2.36))

$$\begin{aligned}
Z(w) &= (XY-YX)w\\
&\overset{(3.14)}{=} X\big(-(\langle w,v\rangle_{\mathbb{C}}z_0 + \langle w,Av\rangle_{\mathbb{C}}Az_0)\big) - Y\big(-(\langle w,u\rangle_{\mathbb{C}}z_0 + \langle w,Au\rangle_{\mathbb{C}}Az_0)\big)\\
&= -\langle w,v\rangle_{\mathbb{C}}u - \langle w,Av\rangle_{\mathbb{C}}Au + \langle w,u\rangle_{\mathbb{C}}v + \langle w,Au\rangle_{\mathbb{C}}Av\\
&= -(u\wedge v + Au\wedge Av)w = -C^{W^\perp}(u,v)w\,,
\end{aligned}$$

where $C^{W^\perp}$ is the function defined in Proposition 2.47 for the $\mathbb{C}$Q-space $W^\perp$. By pull back with the $\mathbb{C}$Q-isomorphism $\psi:=\overrightarrow{(\ldots)}\circ\widehat{z_0}:\mathfrak{m}\to W^\perp,\ X\mapsto X(z_0)$, we obtain from the preceding equation

$$\psi^{-1}\circ Z\circ\psi = -C^{\mathfrak{m}}(X,Y)\,.$$

This provides the promised representations of $\rho^{\mathfrak{m}}$ and $C^{\mathfrak{m}}$ in terms of $Z=[X,Y]$.

We now study Q from the point of view of the root theory for symmetric spaces. In particular, we describe the Cartan subalgebras, the roots and the root spaces of $\mathfrak{m}$ explicitly in terms of the $\mathbb{C}$Q-space structure of $\mathfrak{m}$.

An exposition of the root theory is given in Appendix A.4, where the terms and notations involved in the following are also introduced. For the applicability of that theory, it is of importance that Q is a Riemannian symmetric space of compact type (see Proposition 3.9(c)) and that the inner product we consider on $\mathfrak{m}$ here is a negative multiple of the Killing form of $\mathfrak{g}$ (see Proposition 3.12(b)).

We now suppose $m \geq 2$ and continue to regard $\mathfrak{m}$ as a $\mathbb{C}$Q-space.

3.15 Theorem. *(a) The flat subspaces of $\mathfrak{m}$ in the usual sense (see Proposition A.6) coincide with the flat subspaces of the $\mathbb{C}$Q-space $\mathfrak{m}$ in the sense of Definition 2.52. Therefore it follows from Theorem 2.54:*

The Hermitian symmetric space Q is of rank 2, and the Cartan subalgebras (i.e. the 2-dimensional flat subspaces) $\mathfrak{a}$ of $\mathfrak{m}$ are exactly the spaces

$$\mathfrak{a} = \mathbb{R}X \oplus \mathbb{R}J^{\mathfrak{m}}Y$$

with $A \in \mathfrak{A}^{\mathfrak{m}}$, $X, Y \in \mathbb{S}(V(A))$, $\langle X, Y\rangle^{\mathfrak{m}}_{\mathbb{R}} = 0$.

(b) Let $\mathfrak{a} = \mathbb{R}X \oplus \mathbb{R}J^{\mathfrak{m}}Y$ be a Cartan subalgebra of $\mathfrak{m}$ as in (a). Then the following table gives besides $\lambda_0 := 0 \in \mathfrak{a}^*$ a system of positive roots λ_k of $\mathfrak{m}$ with respect to $\mathfrak{a}$ (via their Riesz vectors $\lambda_k^\sharp$), together with the corresponding root spaces $\mathfrak{m}_{\lambda_k}$ and their multiplicities n_{λ_k}:

k	$\lambda_k^\sharp \in \mathfrak{a}$	$\mathfrak{m}_{\lambda_k}$	n_{λ_k}
0	0	$\mathbb{R}X \oplus \mathbb{R}J^{\mathfrak{m}}Y$	2
1	$\sqrt{2}\cdot J^{\mathfrak{m}}Y$	$J^{\mathfrak{m}}((\mathbb{R}X \oplus \mathbb{R}Y)^\perp)$	$m-2$
2	$\sqrt{2}\cdot X$	$(\mathbb{R}X \oplus \mathbb{R}Y)^\perp$	$m-2$
3	$\sqrt{2}\cdot(X - J^{\mathfrak{m}}Y)$	$\mathbb{R}(J^{\mathfrak{m}}X + Y)$	1
4	$\sqrt{2}\cdot(X + J^{\mathfrak{m}}Y)$	$\mathbb{R}(J^{\mathfrak{m}}X - Y)$	1

Here $^\perp$ denotes the ortho-complement in $V(A)$. In the case $m = 2$ the roots λ_1 and λ_2 do not exist: their multiplicity is zero.

(c) Let $Z \in \mathfrak{m}$ be given. Then there exists a Cartan algebra $\mathfrak{a} \subset \mathfrak{m}$ with $Z \in \mathfrak{a}$.

3.16 Remark. As was already mentioned in Theorem A.8(b), assertion (c) of the above theorem is true for Riemannian symmetric spaces of compact type in general. But for the present specific situation, we can give an elementary proof by use of the $\mathbb{C}$Q-structure on $\mathfrak{m}$.

Proof of Theorem 3.15. We derive the theorem from the results of Sections 2.7 and 2.8 via Proposition 3.12(c). In particular we see from the latter proposition that the flat subspaces of $\mathfrak{m}$ in the usual sense (see Proposition A.6) coincide with the flat subspaces of the $\mathbb{C}$Q-space $\mathfrak{m}$

in the sense of Definition 2.52. Therefore (a) follows from Theorem 2.54, and (c) follows from Corollary 2.55.

It remains to prove (b). For this, we denote for any $Z \in \mathfrak{m}$ the Jacobi operator corresponding to Z by $R_Z^{\mathfrak{m}} := R^{\mathfrak{m}}(\,\cdot\,, Z)Z : \mathfrak{m} \to \mathfrak{m}$, and put for $t \in \mathbb{R}$

$$Z(t) := \cos(t)X + \sin(t)J^{\mathfrak{m}}Y \in \mathfrak{a}\ ;$$

note that $\mathbb{S}(\mathfrak{a}) = \{\, Z(t) \,|\, t \in \mathbb{R}\,\}$ holds.

To derive the data on the root system and the root spaces of $\mathfrak{m}$, we use Proposition A.11. As we did there, we define for any function $\mu : \mathfrak{a} \to \mathbb{R}$

$$E_\mu := \bigcap_{Z \in \mathfrak{a}} \mathrm{Eig}(R_Z, \mu(Z))$$

and

$$\Sigma := \{\, \mu : \mathfrak{a} \to \mathbb{R} \,|\, E_\mu \neq \{0\}\,\}\ .$$

Theorem 2.49 shows that in the present setting we have $\Sigma = \{\mu_0, \ldots, \mu_4\}$, where the functions $\mu_k : \mathfrak{a} \to \mathbb{R}$ are characterized by

$$\forall t, s \in \mathbb{R}\ :\ \mu_k(s\,Z(t)) = s^2 \cdot \varkappa_k(t) \tag{3.23}$$

via the eigenfunctions $\varkappa_k$ from Theorem 2.49; note that $\mu_0 = 0$ holds. We also have $E_{\mu_k} = E_k$, where the spaces E_k are also those of Theorem 2.49.

By Equation (A.29) in Proposition A.11(a) we now have

$$\Delta = \{\pm\lambda_1, \ldots, \pm\lambda_4\}$$

where the linear forms $\lambda_k \in \mathfrak{a}^* \setminus \{0\}$ are up to sign characterized by

$$\lambda_k^2 = \mu_k\ . \tag{3.24}$$

This equation and the information from Theorem 2.49 permits to calculate the Riesz vectors of the roots λ_k explicitly; for example one has for every $t \in \mathbb{R}$

$$\lambda_1(Z(t))^2 = \kappa_1(t) = 1 - \cos(2t) = 2\sin(t)^2 = (\langle Z(t), \sqrt{2}\,J^{\mathfrak{m}}Y\rangle_{\mathbb{R}}^{\mathfrak{m}})^2\ ;$$

therefrom $\lambda_1^\sharp = \pm\sqrt{2}\,J^{\mathfrak{m}}Y$ follows. By an appropriate choice of sign one sees that the vectors given as $\lambda_k^\sharp$ in the table indeed form a positive root system for $\mathfrak{m}$.

Finally we have for $k \in \{1, \ldots, 4\}$ by Equation (A.30)

$$\mathfrak{m}_{\lambda_k} = E_{\lambda_k^2} = E_{\mu_k} = E_k\ . \qquad \square$$

3.3 Curvature-equivariant maps and the classification of isometries of Q

We now prove that there do not exist holomorphic or anti-holomorphic isometries on a complex quadric besides those described in Theorem 3.5. Also, if the dimension of the quadric is $\neq 2$, then any of its isometries is either holomorphic or anti-holomorphic.

The crucial point in the proof is to show that any curvature-equivariant $\mathbb{C}$-linear isometry between $\mathbb{C}$Q-spaces already is a $\mathbb{C}$Q-isomorphism, and that any curvature-equivariant $\mathbb{R}$-linear isometry between $\mathbb{C}$Q-spaces of dimension $\neq 2$ is either $\mathbb{C}$-linear or anti-linear, and thus either a $\mathbb{C}$Q-isomorphism or a $\mathbb{C}$Q-anti-isomorphism (Theorem 3.18).

3.17 Proposition. *Suppose $m \geq 2$; let $(\mathbb{W}, \mathfrak{A})$ and $(\mathbb{W}', \mathfrak{A}')$ be two m-dimensional $\mathbb{C}$Q-spaces and $B : \mathbb{W} \to \mathbb{W}'$ be a curvature-equivariant $\mathbb{R}$-linear isometry. Here and in the following, we denote the objects derived from $\mathbb{W}'$ (for example, its curvature tensor) by appending a $'$ to the symbol for the corresponding object of $\mathbb{W}$.*

(a) If $\mathfrak{a}$ is a 2-flat of $\mathbb{W}$, then $B(\mathfrak{a})$ is a 2-flat of $\mathbb{W}'$.

(b) For any $w \in \mathbb{W}$, we have $B \circ R_w = R'_{Bw} \circ B$; in particular, we have $\mathrm{Spec}(R'_{Bw}) = \mathrm{Spec}(R_w)$ and for any $c \in \mathbb{R}$ we have $\mathrm{Eig}(R'_{Bw}, c) = B(\mathrm{Eig}(R_w, c))$.

(c) For any $w \in \mathbb{W} \setminus \{0\}$ we have $\varphi'(Bw) = \varphi(w)$, and therefore $M'_t = B(M_t)$ for every $t \in [0, \frac{\pi}{4}]$ and $\widehat{Q}(\mathfrak{A}') = B(\widehat{Q}(\mathfrak{A}))$.

Proof. (a) and (b) are obvious consequences of B being a curvature-equivariant $\mathbb{R}$-linear isomorphism. Because of Corollary 2.51, (b) also implies that $\varphi'(Bw) = \varphi(w)$ holds for any $w \in \mathbb{W} \setminus \{0\}$, and $M'_t = B(M_t)$ follows for any $t \in [0, \frac{\pi}{4}]$ because B is a linear isometry. Finally, we have $\widehat{Q}(\mathfrak{A}') = \mathbb{R}_+ \cdot M'_{\pi/4} = \mathbb{R}_+ \cdot B(M_{\pi/4}) = B(\widehat{Q}(\mathfrak{A}))$, see Example 2.37. □

3.18 Theorem. *Let $(\mathbb{W}, \mathfrak{A})$ and $(\mathbb{W}', \mathfrak{A}')$ be two m-dimensional $\mathbb{C}$Q-spaces and $B : \mathbb{W} \to \mathbb{W}'$ be a curvature-equivariant $\mathbb{R}$-linear isometry.*

(a) If B is $\mathbb{C}$-linear, then B is a $\mathbb{C}$Q-isomorphism.

(b) If B is anti-linear, then B is a $\mathbb{C}$Q-anti-isomorphism.

(c) If $m \neq 2$ holds, then B is either $\mathbb{C}$-linear or anti-linear.

Proof. For (a). In the case $m = 1$, any $\mathbb{C}$-linear isometry $B : \mathbb{W} \to \mathbb{W}'$ is a $\mathbb{C}$Q-isomorphism (such an isometry is then automatically curvature-equivariant), see Example 2.11. Thus, we may now suppose $m \geq 2$. Let $A \in \mathfrak{A}$ be given. Because B is a $\mathbb{C}$-linear isometry, the map $A' := B \circ A \circ B^{-1}$ is a conjugation on $\mathbb{W}'$ and we have by Proposition 1.11(a) and Proposition 3.17(c)

$$\widehat{Q}(A') = B(\widehat{Q}(A)) = \widehat{Q}(\mathfrak{A}') .$$

By Proposition 1.10, it follows that $A' \in \mathfrak{A}'$ holds and therefore B is a $\mathbb{C}$Q-isomorphism.

For (b). Let $A' \in \mathfrak{A}'$ be given. Because B is anti-linear, $A' \circ B$ is $\mathbb{C}$-linear, and $A' \circ B$ is a curvature-equivariant linear isometry along with A' and B (for the curvature-equivariance of A' see Proposition 2.45). By (a), we see that $A' \circ B$ is a $\mathbb{C}Q$-isomorphism, and hence B is a $\mathbb{C}Q$-anti-isomorphism.

We prepare the proof of (c) with a technical lemma:

3.19 Lemma. *Suppose $m \geq 2$, let $B : \mathbb{W} \to \mathbb{W}'$ be a curvature-equivariant $\mathbb{R}$-linear isometry, and let $A \in \mathfrak{A}$ and an orthonormal system (x, y) in $V(A)$ be given. Then there exists $A' \in \mathfrak{A}'$ and an orthonormal system (x', y') in $V(A')$ so that*

$$Bx = x' \quad \text{and} \quad B(Jy) = J'y' \tag{3.25}$$

holds. Moreover, there exists $\varepsilon \in \{\pm 1\}$ so that one of the following distinctive cases holds:

(1) $B(Jx) = \varepsilon \cdot J'x' \in J'V(A')$ and $By = \varepsilon \cdot y' \in V(A')$

(2) $B(Jx) = \varepsilon \cdot y' \in V(A')$ and $By = \varepsilon \cdot J'x' \in J'V(A')$

3.20 Remark. As we will see in the proof of Theorem 3.18(c) below, case (2) can in fact only occur for $m = 2$.

Proof of Lemma 3.19. Theorem 2.54 shows that $\mathfrak{a} := \mathbb{R}x \oplus \mathbb{R}Jy$ is a 2-flat of $\mathbb{W}$, $\mathfrak{a}' := B(\mathfrak{a})$ therefore is a 2-flat of $\mathbb{W}'$ by Proposition 3.17(a), and by a further application of Theorem 2.54 it follows that there exists $A' \in \mathfrak{A}'$ and an orthonormal system (x', y') in $V(A')$ with $\mathfrak{a}' = \mathbb{R}x' \oplus \mathbb{R}J'y'$. We have $\mathfrak{a} \cap M_0 = \{\pm x, \pm Jy\}$ and $\mathfrak{a}' \cap M_0' = \{\pm x', \pm J'y'\}$ and therefore Proposition 3.17(c) shows

$$\{\pm Bx, \pm B(Jy)\} = B(\mathfrak{a} \cap M_0) = \mathfrak{a}' \cap M_0' = \{\pm x', \pm J'y'\} . \tag{3.26}$$

In particular, we have $Bx \in \{\pm x', \pm J'y'\}$.

$\mathfrak{a}' = \mathbb{R}x'' \oplus \mathbb{R}J'y''$ with $x'' := J'y' \in J'V(A') = V(-A')$ and $y'' := J'x' \in J'V(A') = V(-A')$ is another representation of $\mathfrak{a}'$ of the kind of Theorem 2.54. Therefore we can ensure $Bx \in \{\pm x'\}$ by replacing (A', x', y') with $(-A', x'', y'')$ if necessary. Then Equation (3.26) shows that $B(Jy) \in \{\pm J'y'\}$ holds, because $B(Jy)$ is orthogonal to Bx. By adjusting the signs of x' and y' where necessary, we can therefore arrange

$$Bx = x' \quad \text{and} \quad B(Jy) = J'y' . \tag{3.27}$$

Now, put $t := \frac{\pi}{8}$ and $w := \cos(t)x + \sin(t)Jy$; by Equations (3.27), we have $Bw = \cos(t)x' + \sin(t)J'y'$. By combining Theorem 2.49 with Proposition 3.17(b), we see

$$\begin{aligned} \mathbb{R}(J'x' + y') &= \mathrm{Eig}(R'_{Bw}, \kappa_3(t)) = B(\mathrm{Eig}(R_w, \kappa_3(t))) = \mathbb{R}(B(Jx) + By) \quad \text{and} \\ \mathbb{R}(J'x' - y') &= \mathrm{Eig}(R'_{Bw}, \kappa_4(t)) = B(\mathrm{Eig}(R_w, \kappa_4(t))) = \mathbb{R}(B(Jx) - By) . \end{aligned}$$

Because B is an $\mathbb{R}$-linear isometry, it follows that there exist $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ so that

$$B(Jx) + By = \varepsilon_1 \cdot (J'x' + y') \quad \text{and} \quad B(Jx) - By = \varepsilon_2 \cdot (J'x' - y') \tag{3.28}$$

holds.

In the case $\varepsilon_1 = \varepsilon_2 =: \varepsilon$, Equations (3.28) show that we have

$$B(Jx) = \varepsilon\, J'x' \in J'V(A') \quad \text{and} \quad By = \varepsilon\, y' \in V(A')$$

and therefore case (1) of the lemma holds.

In the case $\varepsilon := \varepsilon_1 = -\varepsilon_2$, we analogously see that we have

$$B(Jx) = \varepsilon\, y' \in V(A') \quad \text{and} \quad By = \varepsilon\, J'x' \in J'V(A')$$

and hence, case (2) of the lemma holds. □

Proof of Theorem 3.18(c). If $m = 1$ holds, let us fix $x \in \mathbb{S}(\mathbb{W})$ and put $x' := Bx \in \mathbb{S}(\mathbb{W}')$. In this setting, we either have $B(Jx) = J'x'$, and then B is $\mathbb{C}$-linear, or else $B(Jx) = -J'x'$, and then B is anti-linear.

Therefore, we may now suppose $m \geq 3$. Let $A \in \mathfrak{A}$ and $x \in \mathbb{S}(V(A))$ be given. We extend x to an orthonormal system (x, y, z) in $V(A)$.

By applying Lemma 3.19 to the orthonormal system (x, y), we see that there exists $A' \in \mathfrak{A}'$ and an orthonormal system (x', y') in $V(A')$ with $Bx = x'$ and $B(Jy) = J'y'$.

We now show by contradiction that case (2) of the lemma cannot occur in the present situation. Assuming that case (2) holds, we have $B(Jx) \in V(A')$. We apply the lemma to the orthonormal system (x, z) of $V(A)$. Thus, there exists $\widetilde{A}' \in \mathfrak{A}'$ and an orthonormal system $(\widetilde{x}', \widetilde{z}')$ in $V(\widetilde{A}')$ with $Bx = \widetilde{x}'$ and $B(Jz) = J'\widetilde{z}'$. We have $\widetilde{x}' = Bx = x'$ and therefore $0 \neq Bx \in V(A') \cap V(\widetilde{A}')$, whence $\widetilde{A}' = A'$ follows. Because we have $B(Jx) \in V(A') = V(\widetilde{A}')$ by assumption, we see that case (2) of the lemma also holds with respect to (x, z).

Because case (2) of the lemma thus holds both with respect to (x, y) and to (x, z), there exist $\varepsilon, \widetilde{\varepsilon} \in \{\pm 1\}$ so that $\widetilde{\varepsilon} \cdot \widetilde{z}' = B(Jx) = \varepsilon \cdot y'$ and therefore $\widetilde{z}' = \varepsilon\widetilde{\varepsilon} \cdot y'$ holds. We have

$$\widetilde{z}' = \varepsilon\widetilde{\varepsilon} y' \implies J'\widetilde{z}' = \varepsilon\widetilde{\varepsilon}\, J'y' \overset{(3.25)}{\implies} B(Jz) = \varepsilon\widetilde{\varepsilon}\, B(Jy) \implies Jz = \varepsilon\widetilde{\varepsilon}\, Jy \implies z = \varepsilon\widetilde{\varepsilon}\, y\,,$$

which is a contradiction to y and z being orthogonal to each other.

Therefore, with regard to the orthonormal system (x, y) case (1) of the lemma holds. Hence, there exists $\varepsilon \in \{\pm 1\}$ so that

$$B(Jx) = \varepsilon \cdot J'x' = \varepsilon \cdot J'(Bx)$$

holds. Thus, we have shown

$$\forall A \in \mathfrak{A},\, x \in \mathbb{S}(V(A))\; \exists\, \varepsilon(x) \in \{\pm 1\}\; :\; B(Jx) = \varepsilon(x) \cdot J'(Bx)\,.$$

Because $M_0 = \dot{\bigcup}_{A\in\mathfrak{A}}\mathbb{S}(V(A))$ is connected, the continuous map $M_0 \to \{\pm 1\}$, $x \mapsto \varepsilon(x)$ is constant; in other words, there exists $\varepsilon \in \{\pm 1\}$ so that $(B\circ J)|M_0 = \varepsilon\cdot(J'\circ B)|M_0$ holds. Because $\mathrm{span}_{\mathbb{R}}(M_0) = \mathbb{W}$ holds, we conclude $B\circ J = \varepsilon\cdot J'\circ B$. Therefore, B is $\mathbb{C}$-linear for $\varepsilon = 1$ and anti-linear for $\varepsilon = -1$. □

3.21 Corollary. *Let $\mathbb{W}$ be a $2m$-dimensional euclidean space and R be a curvature-like tensor on $\mathbb{W}$.*

(a) If $\mathbb{W}$ is in fact an m-dimensional unitary space, then there exists at most one $\mathbb{C}$Q-structure $\mathfrak{A}$ on $\mathbb{W}$ so that R is the curvature tensor of the $\mathbb{C}$Q-space $(\mathbb{W},\mathfrak{A})$.

(b) For $m \neq 2$ there exist at most two orthogonal complex structures J and $-J$ on $\mathbb{W}$ and at most one $\mathbb{C}$Q-structure on $(\mathbb{W},\pm J)$ so that R is the curvature tensor of the $\mathbb{C}$Q-space $(\mathbb{W},\pm J,\mathfrak{A})$.

Proof. This is an immediate consequence of Theorem 3.18. □

3.22 Lemma. *Let M be a connected, affine, complex manifold so that the complex structure J of M is parallel. Moreover, let an affine map $f: M \to M$ be given.*

If there exists $p_0 \in M$ so that $T_{p_0}f : T_{p_0}M \to T_{f(p_0)}M$ is $\mathbb{C}$-linear or anti-linear, then f is holomorphic resp. anti-holomorphic.

Proof. We suppose that $T_{p_0}f$ is $\mathbb{C}$-linear; the anti-linear case is handled analogously. Let $p \in M$ be given; we have to show that $T_pf\circ J_p = J_{f(p)}\circ T_pf$ holds. Let $v \in T_pM$ be given. Because M is connected, there exists a curve $\gamma : [0,1] \to M$ with $\gamma(0) = p_0$ and $\gamma(1) = p$. Let us denote by $X \in \mathfrak{X}_\gamma(M)$ the parallel vector field along γ with $X_1 = v$. Because f is affine and J is a parallel endomorphism field, both $Tf\circ J\circ X$ and $J\circ Tf\circ X$ are parallel vector fields along $f\circ\gamma$; they coincide in 0 because $T_{p_0}f$ is $\mathbb{C}$-linear, and therefore they are equal. In particular, we have $T_pf\circ J_pv = J_{f(p)}\circ T_pf(v)$. □

3.23 Theorem. *Let $(\mathbb{V},\mathfrak{A})$ be an $(n = m+2)$-dimensional $\mathbb{C}$Q-space and $Q := Q(\mathfrak{A})$ the corresponding complex quadric.*

(a) $I_h(Q) = \{\,\underline{B}|Q \mid B \in \mathrm{Aut}_s(\mathfrak{A})\,\}$.

(b) $I_{ah}(Q) = \{\,\underline{B}|Q \mid B \in \overline{\mathrm{Aut}}(\mathfrak{A})\,\}$.

(c) If $m \neq 2$ holds, then we have $I(Q) = I_h(Q)\dot{\cup}I_{ah}(Q)$.

3.24 Remark. As we will see in Section 3.4, Q^2 is holomorphically isometric to $\mathbb{P}^1\times\mathbb{P}^1$. This fact shows that there are indeed isometries on Q^2 which are neither holomorphic nor anti-holomorphic.

Proof of Theorem 3.23. For (a). We already proved $\{\,\underline{B}|Q \mid B \in \mathrm{Aut}_s(\mathfrak{A})\,\} \subset I_h(Q)$ in Proposition 3.2(a). Conversely, let $f \in I_h(Q)$ be given and fix $p \in Q$. Then, $L := T_p f : T_p Q \to T_{f(p)} Q$ is a curvature-equivariant $\mathbb{C}$-linear isometry and consequently a $\mathbb{C}Q$-isomorphism by Theorem 3.18(a). By Theorem 3.5(a), it follows that there exists $B \in \mathrm{Aut}_s(\mathfrak{A})$ so that $\underline{B}|Q \in I_h(Q)$ satisfies

$$\underline{B}(p) = f(p) \quad \text{and} \quad T_p(\underline{B}|Q) = L\,.$$

By the rigidity of isometries, it follows that $f = \underline{B}|Q$ holds.

For (b). The proof is analogous to (a).

For (c). Let $f \in I(Q)$ be given. Once again, we fix $p \in Q$, then $L := T_p f : T_p Q \to T_{f(p)} Q$ is a curvature-equivariant $\mathbb{R}$-linear isometry. By Theorem 3.18(c) we thus see that L is either $\mathbb{C}$-linear or anti-linear. By Lemma 3.22, it follows that f is holomorphic resp. anti-holomorphic. □

3.25 Corollary. *Any (anti-)holomorphic isometry of Q can be extended to an (anti-)holomorphic isometry of $\mathbb{P}(\mathbb{V})$.*

Proof. Let an (anti-)holomorphic isometry f of Q be given. By Theorem 3.23(a),(b) there exists $B \in \mathrm{Aut}_s(\mathfrak{A})$ resp. $B \in \overline{\mathrm{Aut}}(\mathfrak{A})$ with $f = \underline{B}|Q$. Proposition 3.1(a),(b) then shows that $\underline{B}$ is an (anti-)holomorphic isometry of $\mathbb{P}(\mathbb{V})$. □

3.4 Q^2 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and therefore reducible

The two series Q^m and $\mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2-1}$ of Hermitian symmetric spaces intersect at one point, namely Q^2 is as Hermitian symmetric space isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. As has already been noted, it is a consequence of this fact that the symmetric space Q^2 is not irreducible (unlike complex quadrics of every other dimension, see Proposition 3.9(c)).

In the present section, we will construct the isomorphism $Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ explicitly using the Segre embedding, which is a holomorphic isometric embedding $\mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2-1} \to \mathbb{P}^{n_1 n_2 - 1}$. It will turn out that in the case $n_1 = n_2 = 2$, its image is a 2-dimensional complex quadric in $\mathbb{P}^3$. It should be mentioned that the coordinate-free description of the Segre embedding given here is based on a discussion with Prof. H. Reckziegel.

The complex quadrics of dimension 1, 3, 4 and 6 are also isomorphic to members of other series of Hermitian symmetric spaces. We will construct the corresponding isomorphisms explicitly in Chapter 8.

At first, we let W_1 and W_2 be unitary spaces of arbitrary complex dimension n_1 resp. n_2. We further suppose that W_1 is equipped with a conjugation $W_1 \to W_1,\ w \mapsto \overline{w}$.

We regard the complex projective spaces $\mathbb{P}(W_1)$ and $\mathbb{P}(W_2)$ as Hermitian manifolds via the Fubini/Study metric as usual. Then the Hermitian manifold $\mathbb{P}(W_1) \times \mathbb{P}(W_2)$ becomes a Hermitian homogeneous $(\mathrm{SU}(W_1) \times \mathrm{SU}(W_2))$-space via the Lie group action

$$\begin{array}{ccccc} (\mathrm{SU}(W_1) \times \mathrm{SU}(W_2)) & \times & (\mathbb{P}(W_1) \times \mathbb{P}(W_2)) & \longrightarrow & \mathbb{P}(W_1) \times \mathbb{P}(W_2) \, , \\ (\,(B_1, B_2) & , & (p_1, p_2)\,) & \longmapsto & (\underline{B_1}\, p_1, \underline{B_2}\, p_2) \, . \end{array}$$

Moreover, if we fix $(p_1, p_2) \in \mathbb{P}(W_1) \times \mathbb{P}(W_2)$ and denote for $k \in \{1, 2\}$ by $S_k : W_k \to W_k$ the $\mathbb{C}$-linear transformation described by $S_k | p_k = \mathrm{id}_{p_k}$ and $S_k | p_k^\perp = -\mathrm{id}_{p_k^\perp}$, then the involutive Lie group automorphism

$$\sigma : \mathrm{SU}(W_1) \times \mathrm{SU}(W_2) \to \mathrm{SU}(W_1) \times \mathrm{SU}(W_2),\ (B_1, B_2) \mapsto (\,S_1\, B_1\, S_1^{-1}\,,\ S_2\, B_2\, S_2^{-1}\,)$$

describes a Hermitian symmetric structure on $\mathbb{P}(W_1) \times \mathbb{P}(W_2)$; in the sequel we will regard $\mathbb{P}(W_1) \times \mathbb{P}(W_2)$ as a Hermitian symmetric $(\mathrm{SU}(W_1) \times \mathrm{SU}(W_2))$-space in this way.

We now consider the $\mathbb{C}$-linear space $\mathbb{V} := L(W_1, W_2)$ of $\mathbb{C}$-linear maps $W_1 \to W_2$ as a unitary space with its canonical inner product, which we also denote by $\langle \cdot, \cdot \rangle$ and which can via an arbitrary unitary basis $(a_1, \dots, a_{n_1})$ of W_1 be characterized by

$$\forall\, T, S \in L(W_1, W_2) \ : \ \langle T, S \rangle = \sum_{k=1}^{n_1} \langle T a_k, S a_k \rangle \, . \tag{3.29}$$

Note that $\mathbb{V}$ has complex dimension $n_1\, n_2$, and therefore the complex projective space $\mathbb{P}(\mathbb{V})$, which we regard as a Hermitian symmetric $\mathrm{SU}(\mathbb{V})$-space as usual, has complex dimension $n_1\, n_2 - 1$.

In the sequel, we denote by

$$\pi_1 : \mathbb{S}(W_1) \to \mathbb{P}(W_1)\, , \quad \pi_2 : \mathbb{S}(W_2) \to \mathbb{P}(W_2) \quad \text{and} \quad \pi : \mathbb{S}(\mathbb{V}) \to \mathbb{P}(\mathbb{V})$$

the Hopf fibrations of the respective unitary spaces.

3.26 Proposition. *The map*

$$b : W_1 \times W_2 \to \mathbb{V},\ (w_1, w_2) \mapsto \langle\, \cdot \,, \overline{w_1} \rangle\, w_2 \tag{3.30}$$

is $\mathbb{C}$-bilinear and satisfies

$$\forall\, w_1 \in W_1,\ w_2 \in W_2 \ : \ \|b(w_1, w_2)\| = \|w_1\| \cdot \|w_2\| \, , \tag{3.31}$$

in particular $b(\mathbb{S}(W_1) \times \mathbb{S}(W_2)) \subset \mathbb{S}(\mathbb{V})$.

Therefore, there exists one and only one map $f : \mathbb{P}(W_1) \times \mathbb{P}(W_2) \to \mathbb{P}(\mathbb{V})$ so that the following diagram commutes:

$$\begin{tikzcd} \mathbb{S}(W_1) \times \mathbb{S}(W_2) \arrow[r, "b"] \arrow[d, "\pi_1 \times \pi_2"'] & \mathbb{S}(\mathbb{V}) \arrow[d, "\pi"] \\ \mathbb{P}(W_1) \times \mathbb{P}(W_2) \arrow[r, "f"'] & \mathbb{P}(\mathbb{V}) \, . \end{tikzcd}$$

f is a holomorphic isometric embedding. f is called the Segre embedding*.*

Proof. It is clear that b is $\mathbb{C}$-bilinear. For the verification of Equation (3.31), let $w_1 \in W_1$ and $w_2 \in W_2$ be given, and let $(a_1, \dots, a_{n_1})$ be a unitary basis of W_1. Then we have

$$\begin{aligned} \|b(w_1, w_2)\|^2 = \langle b(w_1, w_2), b(w_1, w_2)\rangle &= \sum_{k=1}^{n_1} \langle\, \langle a_k, \overline{w_1}\rangle\, w_2 \ ,\ \langle a_k, \overline{w_1}\rangle\, w_2 \,\rangle \\ &= \sum_{k=1}^{n_1} |\langle a_k, \overline{w_1}\rangle|^2 \cdot \|w_2\|^2 = \|w_1\|^2 \cdot \|w_2\|^2 \ , \end{aligned}$$

whence Equation (3.31) follows. The statement on the existence and uniqueness of f is an immediate consequence.

Next, we show that f is injective. For this purpose, we let $(w_1, w_2), (w_1', w_2') \in \mathbb{S}(W_1) \times \mathbb{S}(W_2)$ be given so that

$$f(\pi_1(w_1), \pi_2(w_2)) = f(\pi_1(w_1'), \pi_2(w_2'))$$

holds. Then there exists $\lambda \in \mathbb{S}^1$ so that

$$b(w_1, w_2) = \lambda \cdot b(w_1', w_2')$$

and therefore

$$\forall v \in W_1 \ :\ \langle v, \overline{w_1}\rangle\, w_2 = \lambda \cdot \langle v, \overline{w_1'}\rangle\, w_2'$$

holds. If we plug $v := \overline{w_1} \in W_1$ into the latter equation, we obtain

$$\underbrace{\langle \overline{w_1}, \overline{w_1}\rangle}_{=1}\, w_2 = \underbrace{\lambda \cdot \langle \overline{w_1}, \overline{w_1'}\rangle}_{=:\mu}\, w_2' \ . \tag{3.32}$$

Because both w_2 and w_2' are of unit length, we have $|\mu| = 1$; because we also have $|\lambda| = 1$, it follows from the definition of μ that

$$1 = |\langle \overline{w_1}, \overline{w_1'}\rangle| = |\overline{\langle w_1, w_1'\rangle}| = |\langle w_1, w_1'\rangle|$$

holds, see Proposition 2.3(d). Because we also have $\|w_1\| = \|w_1'\| = 1$, we see that with respect to w_1 and w_1', equality holds in the complex Cauchy/Schwarz inequality. Hence there exists $\nu \in \mathbb{C}$ with

$$w_1 = \nu \cdot w_1' \ ; \tag{3.33}$$

because both w_1 and w_1' are of unit length, we have $|\nu| = 1$. By Equations (3.33) and (3.32), we have $(w_1, w_2) = (\nu\, w_1', \mu\, w_2')$ and thus $(\pi_1(w_1), \pi_2(w_2)) = (\pi_1(w_1'), \pi_2(w_2'))$. This completes the proof of the injectivity of f.

Immediately, we will show that f is a holomorphic isometric immersion. Because its domain of definition is compact, it then also follows that f is an embedding.

For the proof that f is a holomorphic isometric immersion, we let $(w_1, w_2) \in \mathbb{S}(W_1) \times \mathbb{S}(W_2)$ be given. We denote by $\mathcal{H}_{w_k} \subset T_{w_k}\mathbb{S}(W_k)$ $(k \in \{1, 2\})$ resp. $\mathcal{H}_{b(w_1,w_2)} \subset T_{b(w_1,w_2)}\mathbb{S}(\mathbb{V})$ the horizontal space of the Riemannian submersion π_k resp. π at the point w_k resp. $b(w_1, w_2)$. By Equation (1.6) we have

$$\overrightarrow{\mathcal{H}_{w_k}} = (\mathbb{C}\, w_k)^{\perp, W_k} \quad \text{and} \quad \overrightarrow{\mathcal{H}_{b(w_1,w_2)}} = (\mathbb{C}\, b(w_1, w_2))^{\perp, \mathbb{V}} \ . \tag{3.34}$$

Also, if we identify the tangent space $T_{(w_1,w_2)}(\mathbb{S}(W_1)\times\mathbb{S}(W_2))$ with $T_{w_1}\mathbb{S}(W_1)\oplus T_{w_2}\mathbb{S}(W_2)$, we have

$$\forall\,(\xi_1,\xi_2)\in T_{w_1}\mathbb{S}(W_1)\oplus T_{w_2}\mathbb{S}(W_2)\ :\ \overrightarrow{(T_{(w_1,w_2)}b)(\xi_1,\xi_2)} = b(\overrightarrow{\xi_1},w_2)+b(w_1,\overrightarrow{\xi_2})\,. \tag{3.35}$$

To prove that f is isometrically immersive at the point (w_1,w_2), it suffices to show that

$$b_*(\mathcal{H}_{w_1}\oplus\mathcal{H}_{w_2})\subset\mathcal{H}_{b(w_1,w_2)} \tag{3.36}$$

and

$$\forall\,(\xi_1,\xi_2)\in\mathcal{H}_{w_1}\oplus\mathcal{H}_{w_2}\ :\ \langle b_*(\xi_1,\xi_2),b_*(\xi_1,\xi_2)\rangle = \langle(\xi_1,\xi_2),(\xi_1,\xi_2)\rangle \tag{3.37}$$

holds. From the fact that $T_{(w_1,w_2)}b$ is $\mathbb{C}$-linear (as can be seen from Equation (3.35)), together with (3.36), it then also follows that f is holomorphic at (w_1,w_2).

For the proof of (3.36), let $(\xi_1,\xi_2)\in\mathcal{H}_{w_1}\oplus\mathcal{H}_{w_2}$ be given. Then we have $\langle\overrightarrow{\xi_1},w_1\rangle=\langle\overrightarrow{\xi_2},w_2\rangle=0$ by Equations (3.34). Letting $(a_1,\dots,a_{n_1})$ be a unitary basis of W_1, we therefore have by Equation (3.35)

$$\begin{aligned}
\langle\overrightarrow{(T_{(w_1,w_2)}b)(\xi_1,\xi_2)},b(w_1,w_2)\rangle &= \langle\,b(\overrightarrow{\xi_1},w_2)+b(w_1,\overrightarrow{\xi_2})\,,b(w_1,w_2)\rangle\\
&=\sum_k\Big(\big\langle\,\langle a_k,\overline{\overrightarrow{\xi_1}}\rangle w_2+\langle a_k,\overline{w_1}\rangle\overrightarrow{\xi_2}\,,\,\langle a_k,\overline{w_1}\rangle w_2\,\big\rangle\Big)\\
&=\sum_k\Big(\langle a_k,\overline{\overrightarrow{\xi_1}}\rangle\,\underbrace{\overline{\langle a_k,\overline{w_1}\rangle}}_{\overset{(*)}{=}\langle\overline{w_1},a_k\rangle}\,\underbrace{\langle w_2,w_2\rangle}_{=1}+\langle a_k,\overline{w_1}\rangle\,\overline{\langle a_k,\overline{w_1}\rangle}\,\underbrace{\langle\overrightarrow{\xi_2},w_2\rangle}_{=0}\Big)\\
&=\Big\langle\,\sum_k\langle\overline{w_1},a_k\rangle\,a_k\,,\,\overline{\overrightarrow{\xi_1}}\,\Big\rangle=\langle\overline{w_1},\overline{\overrightarrow{\xi_1}}\rangle\overset{(*)}{=}\overline{\langle w_1,\overrightarrow{\xi_1}\rangle}=0
\end{aligned}$$

(for the equals signs marked $(*)$, see Proposition 2.3(d)), whence $(T_{(w_1,w_2)}b)(\xi_1,\xi_2)\in\mathcal{H}_{b(w_1,w_2)}$ follows by Equations (3.34).

For the proof of Equation (3.37), we let $(\xi_1,\xi_2)\in\mathcal{H}_{w_1}\oplus\mathcal{H}_{w_2}$ be given. We first note that we have by Equation (3.31):

$$\begin{aligned}
&\langle b(\overrightarrow{\xi_1},w_2),b(\overrightarrow{\xi_1},w_2)\rangle=\|\overrightarrow{\xi_1}\|^2\cdot\|w_2\|^2=\|\xi_1\|^2\\
\text{and}\quad&\langle b(w_1,\overrightarrow{\xi_2}),b(w_1,\overrightarrow{\xi_2})\rangle=\|w_1\|^2\cdot\|\overrightarrow{\xi_2}\|^2=\|\xi_2\|^2\,;
\end{aligned}\tag{3.38}$$

we also have

$$\langle b(\overrightarrow{\xi_1},w_2),b(w_1,\overrightarrow{\xi_2})\rangle=\sum_k\big\langle\,\langle a_k,\overline{\overrightarrow{\xi_1}}\rangle\,w_2\,,\,\langle a_k,\overline{w_1}\rangle\,\overrightarrow{\xi_2}\,\big\rangle=\sum_k\langle a_k,\overline{\overrightarrow{\xi_1}}\rangle\,\overline{\langle a_k,\overline{w_1}\rangle}\,\underbrace{\langle w_2,\overrightarrow{\xi_2}\rangle}_{=0}=0\,. \tag{3.39}$$

Thus we obtain:

$$\begin{aligned}
&\langle\,\overrightarrow{(T_{(w_1,w_2)}b)(\xi_1,\xi_2)}\,,\,\overrightarrow{(T_{(w_1,w_2)}b)(\xi_1,\xi_2)}\,\rangle\\
\overset{(3.35)}{=}&\langle\,b(\overrightarrow{\xi_1},w_2)+b(w_1,\overrightarrow{\xi_2})\,,\,b(\overrightarrow{\xi_1},w_2)+b(w_1,\overrightarrow{\xi_2})\,\rangle\\
=\ &\underbrace{\langle b(\overrightarrow{\xi_1},w_2),b(\overrightarrow{\xi_1},w_2)\rangle}_{\overset{(3.38)}{=}\|\xi_1\|^2}+\underbrace{\langle b(\overrightarrow{\xi_1},w_2),b(w_1,\overrightarrow{\xi_2})\rangle}_{\overset{(3.39)}{=}0}+\underbrace{\langle b(w_1,\overrightarrow{\xi_2}),b(\overrightarrow{\xi_1},w_2)\rangle}_{\overset{(3.39)}{=}0}+\underbrace{\langle b(w_1,\overrightarrow{\xi_2}),b(w_1,\overrightarrow{\xi_2})\rangle}_{\overset{(3.38)}{=}\|\xi_2\|^2}\\
=\ &\|\xi_1\|^2+\|\xi_2\|^2=\langle(\xi_1,\xi_2),(\xi_1,\xi_2)\rangle\,.
\end{aligned}$$

□

From now on, we suppose that also W_2 is equipped with a conjugation, which we also denote by $w \mapsto \overline{w}$. We will use the following notations: For any $T \in L(W_1, W_2)$, we denote by $T^* \in L(W_2, W_1)$ the adjoint endomorphism of T; we also consider the endomorphism $\overline{T} \in L(W_1, W_2)$ characterized by

$$\forall\, w \in W_1 \;:\; \overline{T}(w) = \overline{T\,\overline{w}}\,. \tag{3.40}$$

It should be noted that $\overline{T}^* = \overline{T^*}$ holds; with respect to unitary bases of W_1 and W_2 which are adapted to the respective conjugations on these spaces (see Definition 2.7(b)), this endomorphism is represented by the transpose of the matrix representing T. Also, if W_3 is another unitary space equipped with a conjugation, we have

$$\forall\, T \in L(W_1, W_2),\; S \in L(W_2, W_3) \;:\; \big(\, (S \circ T)^* = T^* \circ S^* \quad \text{and} \quad \overline{S \circ T} = \overline{S} \circ \overline{T}\,\big)\,, \tag{3.41}$$

where the notations S^* and $\overline{S}$ are used analogously for $S \in L(W_2, W_3)$.

Applying the notations T^* and $\overline{T}$ also to endomorphisms of W_k ($k \in \{1,2\}$), we moreover note that we have

$$\forall\, B \in \mathrm{SU}(W_k) \;:\; B^*, \overline{B} \in \mathrm{SU}(W_k)\,. \tag{3.42}$$

Proof of (3.42). Let $B \in \mathrm{SU}(W_k)$ be given. Then we obviously have $B^* = B^{-1} \in \mathrm{SU}(W_k)$. Moreover, if we fix a unitary basis $(a_1, \dots, a_{n_k})$ of W_k which is adapted to the conjugation on this space, then $\overline{B}$ transforms this basis into another unitary basis of W_k, and therefore $\overline{B} \in \mathrm{U}(W_k)$ holds. Again using the basis (a_k) and the Leibniz formula for the determinant, one easily sees

$$\forall\, T \in \mathrm{End}(W_k) \;:\; \det(\overline{T}) = \overline{\det(T)}\,,$$

whence $\overline{B} \in \mathrm{SU}(W_k)$ follows. □

3.27 Proposition. *For every $B_1 \in \mathrm{SU}(W_1)$ and $B_2 \in \mathrm{SU}(W_2)$ the map*

$$F(B_1, B_2) : \mathbb{V} \to \mathbb{V},\; T \mapsto B_2 \circ T \circ \overline{B_1}^*$$

is an element of $\mathrm{SU}(\mathbb{V})$, the map $F : \mathrm{SU}(W_1) \times \mathrm{SU}(W_2) \to \mathrm{SU}(\mathbb{V}),\; (B_1, B_2) \mapsto F(B_1, B_2)$ is a homomorphism of Lie groups, and (f, F) is a homomorphism of homogeneous spaces (see Appendix A.1) from the $(\mathrm{SU}(W_1) \times \mathrm{SU}(W_2))$-space $\mathbb{P}(W_1) \times \mathbb{P}(W_2)$ into the $\mathrm{SU}(\mathbb{V})$-space $\mathbb{P}(\mathbb{V})$.

Proof. Let $B_1 \in \mathrm{SU}(W_1)$ and $B_2 \in \mathrm{SU}(W_2)$ be given and fix a unitary basis $(a_1, \dots, a_{n_1})$ of W_1 which is adapted to the conjugation of this space. Then we have for every $T, S \in \mathbb{V}$

$$\begin{aligned}\langle\, F(B_1, B_2)T\,,\, F(B_1, B_2)S\,\rangle = \langle B_2 \circ T \circ \overline{B_1}^*, B_2 \circ S \circ \overline{B_1}^* \rangle &= \sum_{k=1}^{n_1} \langle B_2\, T\, \overline{B_1}^*\, a_k, B_2\, S\, \overline{B_1}^*\, a_k \rangle \\ &= \sum_{k=1}^{n_1} \langle T\, \overline{B_1}^*\, a_k, S\, \overline{B_1}^*\, a_k \rangle = \langle T, S \rangle\,,\end{aligned}$$

where the last equals sign is justified by the fact that $(\overline{B_1}^*\, a_k)_k$ is another unitary basis of W_1 (note that we have $\overline{B_1}^* \in \mathrm{SU}(W_1)$ by (3.42)). This shows that $F(B_1, B_2) \in \mathrm{U}(\mathbb{V})$ holds.

To show that F in fact maps into $\mathrm{SU}(\mathbb{V})$, we consider the Lie group homomorphism

$$g := \det \circ F : \mathrm{SU}(W_1) \times \mathrm{SU}(W_2) \to \mathbb{S}^1 .$$

Then we have to show $g \equiv 1$. Because of the connectedness of $\mathrm{SU}(W_1) \times \mathrm{SU}(W_2)$ this is already implied by

$$\forall X_1 \in \mathfrak{su}(W_1),\ X_2 \in \mathfrak{su}(W_2) \ :\ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g(\gamma_{X_1}(t), \gamma_{X_2}(t)) = 0 , \tag{3.43}$$

where $\gamma_{X_k} : \mathbb{R} \to \mathrm{SU}(W_k)$ denotes the 1-parameter subgroup of $\mathrm{SU}(W_k)$ induced by X_k.

For the proof of Equation (3.43), we let $X_k \in \mathfrak{su}(W_k)$ ($k \in \{1,2\}$) be given; then

$$X_k \in \mathrm{End}_-(W_k) \quad \text{and} \quad \mathrm{tr}(X_k) = 0 \tag{3.44}$$

holds, and we have

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g(\gamma_{X_1}(t), \gamma_{X_2}(t)) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\, t \mapsto \det(\, T \mapsto \gamma_{X_2}(t) \circ T \circ \overline{\gamma_{X_1}(t)}^{*} \,)\,) \\ &= \mathrm{tr}(T \mapsto X_2 \circ T + T \circ \overline{X_1}^{*}) \\ &= \mathrm{tr}(T \mapsto X_2 \circ T) + \mathrm{tr}(T \mapsto T \circ \overline{X_1}^{*}) .\end{aligned} \tag{3.45}$$

To calculate these traces, we fix besides the basis (a_k) also a unitary basis $(b_1, \dots, b_{n_2})$ of W_2 adapted to the conjugation of this space and consider for $j \in \{1, \dots, n_1\}$, $k \in \{1, \dots, n_2\}$ the linear maps

$$T_{jk} := b(a_j, b_k) : W_1 \to W_2,\ w \mapsto \langle w, a_j \rangle\, b_k . \tag{3.46}$$

Then $(T_{jk})_{j,k}$ is a unitary basis of $\mathbb{V}$, and therefore we have

$$\begin{aligned}\mathrm{tr}(T \mapsto X_2 \circ T) &= \sum_{j,k} \langle X_2 \circ T_{jk}, T_{jk} \rangle = \sum_{j,k} \sum_{\ell} \langle X_2\, T_{jk}\, a_\ell, T_{jk}\, a_\ell \rangle \overset{(3.46)}{=} \sum_{j,k} \langle X_2\, b_k, b_k \rangle \\ &= n_1 \cdot \mathrm{tr}(X_2) \overset{(3.44)}{=} 0\end{aligned} \tag{3.47}$$

and

$$\begin{aligned}\mathrm{tr}(T \mapsto T \circ \overline{X_1}^{*}) &= \sum_{j,k} \langle T_{jk} \circ \overline{X_1}^{*}, T_{jk} \rangle = \sum_{j,k} \sum_{\ell} \langle T_{jk}\, \overline{X_1}^{*}\, a_\ell, T_{jk}\, a_\ell \rangle \overset{(3.46)}{=} \sum_{j,k} \langle\, \langle \overline{X_1}^{*}\, a_j, a_j \rangle b_k\, , b_k \rangle \\ &= \sum_{j,k} \langle \overline{X_1}^{*}\, a_j, a_j \rangle = \sum_{j,k} \langle a_j, \overline{X_1}\, a_j \rangle = \sum_{j,k} \langle \overline{a_j}, \overline{X_1\, a_j} \rangle = \sum_{j,k} \langle X_1\, a_j, a_j \rangle \\ &= n_2 \cdot \mathrm{tr}(X_1) \overset{(3.44)}{=} 0 .\end{aligned} \tag{3.48}$$

By plugging Equations (3.47) and (3.48) into Equation (3.45), we obtain the desired result (3.43).

It is now clear that F is a Lie group homomorphism. To prove that (f, F) is a homomorphism of homogeneous spaces, it suffices to show

$$\begin{aligned}&\forall\, (B_1, B_2) \in \mathrm{SU}(W_1) \times \mathrm{SU}(W_2),\ (w_1, w_2) \in W_1 \times W_2 \ : \\ &\qquad\qquad b(B_1\, w_1, B_2\, w_2) = F(B_1, B_2)(\, b(w_1, w_2)\,) ,\end{aligned} \tag{3.49}$$

because the Hopf fibrations π_k and π are $\mathrm{SU}(W_k)$-equivariant resp. $\mathrm{SU}(\mathbb{V})$-equivariant.

Let $(B_1, B_2) \in \mathrm{SU}(W_1) \times \mathrm{SU}(W_2)$ and $(w_1, w_2) \in W_1 \times W_2$ be given. Then we have for every $w \in W_1$

$$\begin{aligned} b(B_1\, w_1, B_2\, w_2)w &= \langle w, \overline{B_1\, w_1}\rangle\, B_2\, w_2 = \langle w, \overline{B_1}\,\overline{w_1}\rangle\, B_2\, w_2 = B_2\big(\, \langle \overline{B_1}^{\,*} w, \overline{w_1}\rangle\, w_2\,\big) \\ &= (B_2 \circ b(w_1, w_2) \circ \overline{B_1}^{\,*})w = F(B_1, B_2)(\, b(w_1, w_2)\,)w\ . \end{aligned}$$

This proves Equation (3.49). □

We now specialize to the situation where $W_1 = W_2 =: W$ is a unitary space of even complex dimension $2m$ equipped with a conjugation $W \to W,\ w \mapsto \overline{w}$; we put $W_{\mathbb{R}} := V(\overline{\Box}) = \{\, w \in W \,|\, w = \overline{w}\,\}$. As before we regard the space $\mathrm{End}(W) = L(W, W) =: \mathbb{V}$ as a unitary space; it is easily verified that the map $\mathbb{V} \to \mathbb{V},\ T \mapsto \overline{T}$ (see Equation (3.40)) is anti-linear, involutive and orthogonal with respect to $\mathrm{Re}(\langle\cdot,\cdot\rangle)$, and hence a conjugation on $\mathbb{V}$.

We will now construct a $\mathbb{C}$Q-structure $\mathfrak{A}$ on $\mathbb{V}$ (not induced by the conjugation $T \mapsto \overline{T}$) so that the Segre embedding $f : \mathbb{P}(W) \times \mathbb{P}(W) \to \mathbb{P}(\mathbb{V})$ (as described in Proposition 3.26) maps into the complex quadric $Q(\mathfrak{A})$. For this purpose, we fix an orthogonal complex structure $\tau : W_{\mathbb{R}} \to W_{\mathbb{R}}$ on the euclidean space $W_{\mathbb{R}}$ and denote the complexification of τ again by $\tau : W \to W$. It should be noted that

$$\tau^2 = -\mathrm{id}_W \quad \text{and} \quad \overline{\tau} = \tau \tag{3.50}$$

holds.

3.28 Proposition. *(a) The map*

$$A : \mathbb{V} \to \mathbb{V},\ T \mapsto \tau \circ \overline{T} \circ \tau^{-1}$$

is a conjugation on $\mathbb{V}$ with the "eigenspace" $V(A) = \{\, T \in \mathbb{V} \,|\, \tau \circ \overline{T} = T \circ \tau\,\}$, and we have

$$\forall T, S \in V(A)\ :\ T \circ S \in V(A)\ . \tag{3.51}$$

In the sequel, we regard $\mathbb{V}$ as a $\mathbb{C}$Q-space via the $\mathbb{C}$Q-structure $\mathfrak{A} := \mathbb{S}^1 \cdot A$.

(b) The Segre embedding $f : \mathbb{P}(W) \times \mathbb{P}(W) \to \mathbb{P}(\mathbb{V})$ maps into the complex quadric $Q(\mathfrak{A})$.

Proof. For (a). Using Equations (3.50), it is easily verified that A is anti-linear and involutive. Moreover, if we denote by $\langle\cdot,\cdot\rangle_{\mathbb{R}} := \mathrm{Re}(\langle\cdot,\cdot\rangle)$ the real inner product on $\mathbb{V}$ induced by its complex inner product, and fix a unitary basis $(a_1, \dots, a_{2m})$ of W, then we have for any $T, S \in \mathbb{V}$

$$\langle A(T), A(S)\rangle_{\mathbb{R}} = \sum_{k=1}^{2m} \langle \tau\, \overline{T}\, \tau^{-1}\, a_k, \tau\, \overline{S}\, \tau^{-1}\, a_k\rangle_{\mathbb{R}} = \sum_{k=1}^{2m} \langle \overline{T}\, \tau^{-1}\, a_k, \overline{S}\, \tau^{-1}\, a_k\rangle_{\mathbb{R}} \overset{(*)}{=} \langle \overline{T}, \overline{S}\rangle_{\mathbb{R}} = \langle T, S\rangle_{\mathbb{R}}\ ,$$

where the equals sign marked $(*)$ is justified by the fact that $(\tau^{-1}(a_k))_{k=1,\dots,2m}$ is another unitary basis of $\mathbb{V}$. Thus we see that A is orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$. By Proposition 2.3(h) we conclude that A is a conjugation on $\mathbb{V}$. (3.51) follows from the equation

$$\forall T, S \in \mathbb{V}\ :\ A(T \circ S) = A(T) \circ A(S)\ ,$$

which is easily verified using Equations (3.41) and (3.50).

For (b). It is sufficient to show that $b(w_1, w_2)$ is $\mathfrak{A}$-isotropic for every $w_1, w_2 \in W$, where $b : W \times W \to \mathbb{V}$ is the bilinear map from Equation (3.30). For this, we fix an orthonormal basis $(a_1, \ldots, a_{2m})$ of $W_{\mathbb{R}}$ so that

$$\forall k \in \{1, \ldots, m\} \;:\; \big(\, \tau(a_k) = a_{m+k} \quad \text{and therefore also} \quad \tau(a_{m+k}) = -a_k \,\big) \tag{3.52}$$

holds. Then $(a_1, \ldots, a_{2m})$ also is a unitary basis of W. If we now let $w_1, w_2 \in W$ be given and abbreviate $T := b(w_1, w_2)$, we have

$$A(T)a_k = (\tau \circ \overline{T} \circ \tau^{-1})a_k = \tau(\overline{T\,\overline{\tau^{-1}\,a_k}}) = \tau(\overline{T\,\tau^{-1}\,a_k}) \tag{3.53}$$

(for the last equals sign, note that we have $\tau^{-1}\,a_k \in W_{\mathbb{R}}$ because of (3.52)) and therefore

$$\begin{aligned}
\langle T, A(T)\rangle &= \sum_{k=1}^{2m} \langle T\,a_k, A(T)\,a_k\rangle \overset{(3.53)}{=} \sum_{k=1}^{2m} \langle T\,a_k, \tau(\overline{T\,\tau^{-1}\,a_k})\rangle \\
&= \sum_{k=1}^{2m} \big\langle\, \langle a_k, \overline{w_1}\rangle w_2 \,,\; \tau(\, \overline{\langle \tau^{-1}\,a_k, \overline{w_1}\rangle w_2}\,)\,\big\rangle \\
&= \sum_{k=1}^{2m} \langle a_k, \overline{w_1}\rangle \cdot \langle \tau^{-1}\,a_k, \overline{w_1}\rangle \cdot \langle w_2, \tau\,\overline{w_2}\rangle \\
&= \langle w_2, \tau\,\overline{w_2}\rangle \cdot \sum_{k=1}^{m} \big(\, \langle a_k, \overline{w_1}\rangle \cdot \langle \tau^{-1}\,a_k, \overline{w_1}\rangle + \langle a_{m+k}, \overline{w_1}\rangle \cdot \langle \tau^{-1}\,a_{m+k}, \overline{w_1}\rangle \,\big) \\
&\overset{(3.52)}{=} \langle w_2, \tau\,\overline{w_2}\rangle \cdot \sum_{k=1}^{m} \big(\, -\langle a_k, \overline{w_1}\rangle \cdot \langle a_{m+k}, \overline{w_1}\rangle + \langle a_{m+k}, \overline{w_1}\rangle \cdot \langle a_k, \overline{w_1}\rangle \,\big) = 0\,,
\end{aligned}$$

showing that $b(w_1, w_2)$ is $\mathfrak{A}$-isotropic. □

3.29 Theorem. *We now suppose that $m = 1$ holds. Then W is a 2-dimensional unitary space, $\mathbb{P}(W)$ is a 1-dimensional complex projective space, and $Q(\mathfrak{A})$ is a 2-dimensional complex quadric. We consider the Segre embedding $f : \mathbb{P}(W) \times \mathbb{P}(W) \to \mathbb{P}(\mathbb{V})$ (see Proposition 3.26) and the Lie group homomorphism $F : \mathrm{SU}(W) \times \mathrm{SU}(W) \to \mathrm{SU}(\mathbb{V})$ from Proposition 3.27. In this situation, we have:*

(a) $f : \mathbb{P}(W) \times \mathbb{P}(W) \to Q(\mathfrak{A})$ is a holomorphic isometry.

(b) We have $F(\mathrm{SU}(W) \times \mathrm{SU}(W)) = \mathrm{Aut}_s(\mathfrak{A})_0$ and $F : \mathrm{SU}(W) \times \mathrm{SU}(W) \to \mathrm{Aut}_s(\mathfrak{A})_0$ is a two-fold covering map of Lie groups with kernel $\{\pm(\mathrm{id}_W, \mathrm{id}_W)\}$. Herein we recognize the well-known isomorphy of Lie groups

$$\boxed{\mathrm{Spin}(4) \cong \mathrm{SU}(2)\times\mathrm{SU}(2)}\;.$$

(c) (f, F) is an almost-isomorphism of Hermitian symmetric spaces from the $(\mathrm{SU}(W) \times \mathrm{SU}(W))$-space $\mathbb{P}(W) \times \mathbb{P}(W)$ onto the $\mathrm{Aut}_s(\mathfrak{A})_0$-space $Q(\mathfrak{A})$.

Thus, we have shown the following isomorphy in the category of Hermitian symmetric spaces:

$$\boxed{Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1} .$$

Proof. For (a). By Proposition 3.26 and Proposition 3.28(b), $f : \mathbb{P}(W) \times \mathbb{P}(W) \to Q(\mathfrak{A})$ is a holomorphic isometric embedding. From the facts that $\dim_{\mathbb{C}}(\mathbb{P}(W) \times \mathbb{P}(W)) = 2 = \dim_{\mathbb{C}} Q(\mathfrak{A})$ holds, $\mathbb{P}(W) \times \mathbb{P}(W)$ is compact and $Q(\mathfrak{A})$ is connected, it follows that we have $f(\mathbb{P}(W) \times \mathbb{P}(W)) = Q(\mathfrak{A})$, and thus $f : \mathbb{P}(W) \times \mathbb{P}(W) \to Q(\mathfrak{A})$ is in fact a holomorphic isometry.

For (b). We first show that $\ker(F) = \{\pm(\mathrm{id}_W, \mathrm{id}_W)\}$ holds. The inclusion "$\supset$" is obvious. For the converse inclusion, let $B_1, B_2 \in \mathrm{SU}(W)$ be given with $F(B_1, B_2) = \mathrm{id}_{\mathbb{V}}$. We thus have

$$\forall T \in \mathbb{V} \ : \ B_2 \circ T \circ \overline{B_1}^{*} = T .$$

We have $\overline{B_1} \in \mathrm{SU}(W)$ by (3.42), and therefore the preceding equation implies

$$\forall T \in \mathbb{V} \ : \ B_2 \circ T = T \circ \overline{B_1} . \tag{3.54}$$

By specializing $T = \mathrm{id}_W \in \mathbb{V}$ in this equation, we obtain $B_2 = \overline{B_1} =: B \in \mathrm{SU}(W)$. Now Equation (3.54) shows that B lies in the center of $\mathbb{V} = \mathrm{End}(W)$, whence it follows that $B = \lambda \cdot \mathrm{id}_W$ holds for some $\lambda \in \mathbb{C}$. Because of $B \in \mathrm{SU}(W)$ we have (remember, $\dim W = 2$)

$$1 = \det(B) = \det(\lambda\, \mathrm{id}_W) = \lambda^2$$

and thus $\lambda \in \{\pm 1\}$. Thus we have shown $(B_1, B_2) \in \{\pm(\mathrm{id}_W, \mathrm{id}_W)\}$. It follows that F is a two-fold covering map of Lie groups onto its image.

Below, we will show

$$\mathrm{SU}(W) \subset V(A) . \tag{3.55}$$

Combining Equation (3.51) with (3.55) and (3.42), we then find

$$\forall (B_1, B_2) \in \mathrm{SU}(W) \times \mathrm{SU}(W) \ : \ F(B_1, B_2)(V(A)) \subset V(A) ;$$

because we have $F(\mathrm{SU}(W) \times \mathrm{SU}(W)) \subset \mathrm{SU}(\mathbb{V})$ by Proposition 3.27, we therefrom conclude (see also Proposition 2.17(a))

$$F(\mathrm{SU}(W) \times \mathrm{SU}(W)) \ \subset \ \mathrm{Aut}_s(\mathfrak{A})_0 .$$

Because F is a covering map of Lie groups over its image and we have $\dim(\mathrm{SU}(W) \times \mathrm{SU}(W)) = 6 = \dim(\mathrm{Aut}_s(\mathfrak{A})_0)$, we in fact have

$$F(\mathrm{SU}(W) \times \mathrm{SU}(W)) = \mathrm{Aut}_s(\mathfrak{A})_0 .$$

It only remains to show (3.55). For this, we let $B \in \mathrm{SU}(W)$ be given. We fix $a_1 \in \mathbb{S}(W_{\mathbb{R}})$ and put $a_2 := \tau(a_1)$. Then (a_1, a_2) is an orthogonal basis of $W_{\mathbb{R}}$ and a unitary basis of W, and there exist $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$, so that B is with respect to this unitary basis represented by the matrix

$$M_B := \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} .$$

The endomorphisms $\overline{B}$ and τ are represented with respect to the same basis by the matrices

$$M_{\overline{B}} := \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix} \quad \text{resp.} \quad M_\tau := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Now, one easily calculates

$$M_\tau \cdot M_{\overline{B}} = \begin{pmatrix} -\overline{\beta} & -\alpha \\ \overline{\alpha} & -\beta \end{pmatrix} = M_B \cdot M_\tau ,$$

whence $\tau \circ \overline{B} = B \circ \tau$, hence $A(B) = \tau \circ \overline{B} \circ \tau^{-1} = B$ and therefore $B \in V(A)$ follows.

For (c). It follows from (a), (b) and Proposition 3.27 that (f, F) is an almost-isomorphism of homogeneous spaces. By Proposition A.5, (f, F) is an almost-isomorphism of affine symmetric spaces; in fact it is an almost-isomorphism of Hermitian symmetric spaces because f is a holomorphic isometry by (a). □

Chapter 4

The classification of curvature-invariant subspaces

One of the central results of this dissertation is the classification of the totally geodesic submanifolds of the complex quadric Q. By a well-known theorem, the connected, complete, totally geodesic submanifolds of the Riemannian symmetric space Q passing through a point $p \in Q$ are in one-to-one correspondence with the curvature-invariant subspaces of T_pQ (see for example [KN69], Theorem XI.4.3, p. 237). Therefore, the task of classifying the (connected, complete) totally geodesic submanifolds of Q splits into the following two problems:

(a) Classify the curvature-invariant subspaces of T_pQ, or equivalently, of a $\mathbb{C}$Q-space $(\mathbb{V}, \mathfrak{A})$.

(b) For each of the curvature-invariant subspaces U of T_pQ found in the solution of problem (a), construct a totally geodesic, connected, complete submanifold M_U of Q with $p \in M_U$ and $T_pM_U = U$.

In the present chapter, we will solve problem (a). In Section 4.1 we state the classification result (Theorem 4.2) and prove some facts about the various types of curvature-invariant subspaces.

For the proof of the classification we proceed as follows: We suppose $Q = Q(\mathfrak{A}_Q)$, then Q is a Hermitian symmetric $(G := \mathrm{Aut}_s(\mathfrak{A}_Q)_0)$-space as we saw in Section 3.2; moreover with respect to a fixed $p_0 \in Q$ we have the canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and the canonical isomorphism $\tau : \mathfrak{m} \to T_{p_0}Q$. A subspace $U \subset T_{p_0}Q$ is curvature-invariant if and only if $\tau^{-1}(U) \subset \mathfrak{m}$ is a Lie triple system in $\mathfrak{m}$ (see Equation (3.11)). Therefore it is sufficient to classify the Lie triple systems $\mathfrak{m}' \subset \mathfrak{m}$. In doing so, we will use the canonical $\mathbb{C}$Q-space structure on $\mathfrak{m}$ described in Section 3.2. But it will also be of importance that $\mathfrak{m}$ carries further structure beyond the $\mathbb{C}$Q-space structure because of its embedding into the Lie algebra $\mathfrak{g}$. In particular, because of this further structure we are able to apply the theory of roots and root spaces (see Appendix A.4) in this situation; this theory is the central tool in showing that the classification given in Theorem 4.2 is complete.

In Section 4.2 we prepare the completeness proof by studying a more general situation: We let M be a general Riemannian symmetric G-space of compact type with canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, we also suppose that a Lie triple system $\mathfrak{m}' \subset \mathfrak{m}$ is given. Then we develop a root theory for $\mathfrak{m}'$; in particular we describe relations between the roots and root spaces of $\mathfrak{m}'$ and the roots resp. root spaces of the ambient symmetric space M.

In Sections 4.3 and 4.4 we return to the specific situation of a complex quadric Q to attain the classification of Lie triple systems $\mathfrak{m}' \subset \mathfrak{m}$. For this purpose we combine the relations between the roots and root spaces of $\mathfrak{m}'$ and Q derived in Section 4.2 with the explicit description of the roots and root spaces of Q via the $\mathbb{C}$Q-structure of $\mathfrak{m}$ given in Theorem 3.15.

It should be mentioned that it is also possible to prove the classification of curvature-invariant subspaces in a $\mathbb{C}$Q-space $\mathbb{V}$ without use of the root theory by directly investigating the eigenvalues and eigenspaces of the Jacobi operator corresponding to the curvature tensor of $\mathbb{V}$ (see Theorem 2.49). However, the use of the root theory permits to give a more systematic proof; moreover it seems probable that the results of Section 4.2 would be of use also for the classification of totally geodesic submanifolds in other symmetric spaces than complex quadrics.

4.1 The classification theorem

4.1 Definition. *Let $(\mathbb{V}, \mathfrak{A})$ be a $\mathbb{C}$Q-space with curvature tensor R.*

(a) A (real) linear subspace $U \subset \mathbb{V}$ is called curvature-invariant*, if*

$$\forall u, v, w \in U \;:\; R(u,v)w \in U$$

holds.

(b) A curvature-invariant subspace $U \neq \mathbb{V}$ of $\mathbb{V}$ is called maximal*, if there exists no curvature-invariant subspace U' of $\mathbb{V}$ with $U \subsetneq U' \subsetneq \mathbb{V}$.*

(c) Let $U \subset \mathbb{V}$ be a curvature-invariant subspace. We call the maximal dimension of an R-flat subspace of $\mathbb{V}$ which is contained in U the rank *of U and denote this number by* $\mathrm{rk}(U)$.[6]

The aim of the present chapter is to prove the following theorem:

4.2 Theorem. *Let $(\mathbb{V}, \mathfrak{A})$ be an m-dimensional $\mathbb{C}$Q-space with $m \geq 2$. Then, a real linear subspace $\{0\} \neq U \subsetneq \mathbb{V}$ is curvature-invariant if and only if it is of one of the types described in the following list:*

(Geo, t) *$U = \mathbb{R}v$ holds for some $v \in \mathbb{S}(\mathbb{V})$ with $\varphi(v) = t$; here we have $t \in [0, \frac{\pi}{4}]$.*

(G1, k) *U is a k-dimensional $\mathbb{C}$Q-subspace of $\mathbb{V}$ (see Proposition 2.13); here we have $2 \leq k \leq m-1$.*

[6] See Definition 2.52. Because of Theorem 2.54, we necessarily have $\mathrm{rk}(U) \in \{1,2\}$.

($G2, k_1, k_2$) *There exist* $A \in \mathfrak{A}$ *and linear subspaces* $W_1, W_2 \subset V(A)$ *of real dimension* k_1 *resp.* k_2 *so that* $W_1 \perp W_2$ *and* $U = W_1 \oplus JW_2$ *holds; here we have* $k_1, k_2 \geq 1$ *and* $k_1 + k_2 \leq m$.

(G3) *There exists* $A \in \mathfrak{A}$ *and an orthonormal system* (x, y) *in* $V(A)$ *so that* $U = \mathbb{C}(x - Jy) \oplus \mathbb{R}(x + Jy)$ *holds.*

($P1, k$) *There exists* $A \in \mathfrak{A}$ *so that* U *is a* k*-dimensional* $\mathbb{R}$*-linear subspace of* $V(A)$ *; here we have* $1 \leq k \leq m$.

(P2) *There exists* $A \in \mathfrak{A}$ *and* $x \in \mathbb{S}(V(A))$ *so that* $U = \mathbb{C}x$ *holds.*

(A) *There exists* $A \in \mathfrak{A}$ *and an orthonormal system* (x, y, z) *in* $V(A)$ *so that*

$$U = \mathbb{R}(2x + Jy) \oplus \mathbb{R}(y + Jx + \sqrt{3}Jz)$$

holds; this type exists only for $m \geq 3$.

($I1, k$) U *is a complex* k*-dimensional isotropic subspace of* $\mathbb{V}$ *(see Propositions 2.20(e),(f) and Proposition 2.21); here we have* $1 \leq k \leq \frac{m}{2}$.

($I2, k$) U *is a totally real, real-*k*-dimensional isotropic subspace of* $\mathbb{V}$ *; here we have* $1 \leq k \leq \frac{m}{2}$.

By the type of a curvature-invariant subspace*, we mean the full specification* (Geo, t) , $(\mathrm{G1}, k)$ *etc., including the numbers* t , k *etc. where relevant. We identify the types* $(\mathrm{G2}, k_1, k_2)$ *with* $(\mathrm{G2}, k_2, k_1)$, *the type* $(\mathrm{P1}, 1)$ *with* $(\mathrm{Geo}, 0)$, *and the type* $(\mathrm{I2}, 1)$ *with* $(\mathrm{Geo}, \frac{\pi}{4})$. *Then no curvature-invariant subspace is of more than one type.*

If U *and* U' *are curvature-invariant subspace of* $\mathbb{V}$, U *can be transformed into* U' *by an element of* $\mathrm{Aut}(\mathfrak{A})$ *if and only if* U *and* U' *are of the same type.*

Moreover, the various types of curvature-invariant spaces have the following properties:

type of U	$\dim_{\mathbb{R}} U$	*U complex or totally real?*	$\mathrm{rk}(U)$	$\varphi(\mathbb{S}(U))$	*U maximal?*
(Geo, t)	1	*totally real*	1	$\{t\}$	*no*
$(\mathrm{G1}, k)$	$2k$	*complex*	2	$[0, \frac{\pi}{4}]$	*for* $k = m - 1 \geq 2$
$(\mathrm{G2}, k_1, k_2)$	$k_1 + k_2$	*totally real*	2	$[0, \frac{\pi}{4}]$	*for* $k_1 + k_2 = m \geq 3$
$(\mathrm{G3})$	3	*neither*	2	$[0, \frac{\pi}{4}]$	*only for* $m = 2$
$(\mathrm{P1}, k)$	k	*totally real*	1	$\{0\}$	*for* $k = m$
$(\mathrm{P2})$	2	*complex*	1	$\{0\}$	*only for* $m = 2$
(A)	2	*neither*	1	$\{\arctan(\frac{1}{2})\}$	*only for* $m = 3$
$(\mathrm{I1}, k)$	$2k$	*complex*	1	$\{\frac{\pi}{4}\}$	*for* $2k = m \geq 4$
$(\mathrm{I2}, k)$	k	*totally real*	1	$\{\frac{\pi}{4}\}$	*no*

4.3 Remarks. (a) If $\mathbb{V}$ is a 1-dimensional $\mathbb{C}$Q-space, then every $\mathbb{R}$-linear subspace of $\mathbb{V}$ is curvature-invariant.

(b) The letters "G", "P", "A" and "I" in the type specifications for curvature-invariant subspaces stand for the words "generic", "principal", "arctan($\frac{1}{2}$)" and "isotropic", respectively. Indeed, as the table in Theorem 4.2 shows, the curvature-invariant subspaces of type $(\mathrm{P1}, k)$ and of type $(\mathrm{P2})$ consist of principal vectors only (see Definition 2.7(a)), the spaces of type (A) consist of vectors with $\mathfrak{A}$-angle $\arctan(\frac{1}{2})$ only, and the spaces of type $(\mathrm{I1}, k)$ and $(\mathrm{I2}, k)$ are isotropic (see Definition 2.19). The spaces of type $(\mathrm{G1}, k)$, $(\mathrm{G2}, k_1, k_2)$ and $(\mathrm{G3})$ are "generic" in the sense that they contain vectors of every $\mathfrak{A}$-angle $t \in [0, \frac{\pi}{4}]$. In the type specification (Geo, t), the abbreviation "Geo" obviously stands for "geodesic", as the totally geodesic submanifolds of Q corresponding to curvature-invariant subspaces of this type are the traces of geodesics in Q.

(c) The 2-flats of $\mathbb{V}$ are exactly the curvature-invariant subspaces of type $(\mathrm{G2}, 1, 1)$ (compare Theorem 2.54).

(d) The curvature-invariant subspaces of type $(\mathrm{P2})$ are exactly the 1-dimensional $\mathbb{C}Q$-subspaces of $\mathbb{V}$. These spaces consist of principal vectors only (unlike the spaces of type $(\mathrm{G1}, k)$ with $k \geq 2$), and therefore we do not call this type $(\mathrm{G1}, 1)$ in order to remain consistent with the meaning of the letters P and G described in Remark (b).

Proof of Theorem 4.2. First we verify that the spaces of the types given in the theorem are in fact curvature-invariant: For (Geo, t) this is obvious, for $(\mathrm{G1}, k)$, $(\mathrm{G2}, k_1, k_2)$ and $(\mathrm{P2})$ it follows easily by inspection of the representation of the curvature tensor of $(\mathbb{V}, \mathfrak{A})$ given in Proposition 2.43(b) (note that spaces of these types are invariant with respect to at least one $A \in \mathfrak{A}$), for $(\mathrm{G3})$ it is checked by an straightforward explicit calculation, for $(\mathrm{P1}, k)$ it follows from Proposition 2.43(c), and for $(\mathrm{I1}, k)$ and $(\mathrm{I2}, k)$ it follows from Proposition 2.43(d). For (A): Let U be of type (A); then there exists $A \in \mathfrak{A}$ and an orthonormal system (x, y, z) in $V(A)$ so that with $a := \frac{1}{\sqrt{5}}(2x + Jy)$ and $b := \frac{1}{\sqrt{5}}(y + Jx + \sqrt{3}Jz)$, (a, b) is an orthonormal basis of U. Via Proposition 2.43(b), one now calculates

$$R(a, b)a = -\tfrac{2}{5}\, b \quad \text{and} \quad R(a, b)b = \tfrac{2}{5}\, a\,. \tag{4.1}$$

It follows that U is curvature-invariant.

It is easily seen that the information in the table on the dimension of the curvature-invariant spaces and on them being complex or totally real subspaces is correct.

For the data on the rank of U: In any case, we have $\mathrm{rk}(U) \in \{1, 2\}$ because of Theorem 2.54. If U is of any of the types $(\mathrm{G1}, k)$, $(\mathrm{G2}, k_1, k_2)$ or $(\mathrm{G3})$, it is easily seen that U contains a 2-flat of $\mathbb{V}$ (again, see Theorem 2.54), and therefore the rank of U then has to be 2. It is clear that the spaces of type (Geo, t) are of rank 1. Proposition 2.43(c) shows that if U of type $(\mathrm{P1}, k)$, the restriction of R to U is the curvature tensor of a sphere of radius $1/\sqrt{2}$, and therefore U is then of rank 1. If U is of type $(\mathrm{P2})$, then one easily calculates that the restriction of R to U is the curvature tensor of a 2-sphere of radius $1/\sqrt{2}$, and if U is of type (A), then Equations (4.1) show that the restriction of R to U is the curvature tensor of a 2-sphere of radius $\sqrt{10}/2$. Therefore, also in these cases, U is of rank 1. Finally, if U is of type $(\mathrm{I1}, k)$ or $(\mathrm{I2}, k)$, then Proposition 2.43(d) shows that the restriction of R to U is the curvature tensor of a complex projective space of constant holomorphic sectional curvature 4 resp. a sphere of radius 1, and thus also in these cases U is of rank 1.

For the data on $\varphi(\mathbb{S}(U))$: If U is of type (Geo, t), it is clear that $\varphi(\mathbb{S}(U)) = \{t\}$ holds. If U is of any of the types $(\mathrm{G1}, k)$, $(\mathrm{G2}, k_1, k_2)$ or $(\mathrm{G3})$, U contains a 2-flat $\mathfrak{a} = \mathbb{R}x \oplus \mathbb{R}Jy$, where (x, y) is a suitable orthonormal system in $V(A)$. Thus, we have $v_t := \cos(t)x + \sin(t)Jy \in U$ for every $t \in [0, \frac{\pi}{4}]$; because of $\varphi(v_t) = t$, $\varphi(\mathbb{S}(U)) = [0, \frac{\pi}{4}]$ follows. It is clear that the spaces of type $(\mathrm{P1}, k)$ and $(\mathrm{P2})$ contain principal vectors only and therefore satisfy $\varphi(\mathbb{S}(U)) = \{0\}$, an explicit calculation via Theorem 2.28(a) shows that spaces of type (A) contain only vectors of $\mathfrak{A}$-angle $\arctan(\frac{1}{2})$, and because the spaces of type $(\mathrm{I1}, k)$ and $(\mathrm{I2}, k)$ are isotropic, they satisfy $\varphi(\mathbb{S}(U)) = \{\frac{\pi}{4}\}$ by Proposition 2.29(b).

To prove the statements on the maximality of curvature-invariant subspaces, we presume that the list of curvature-invariant subspaces given in the theorem is complete; this fact will be proved in the remainder of the chapter. We now consider the various types individually:

(Geo, t) If U is of type (Geo, t), then U is contained in a 2-flat, i.e. in a space of type $(\mathrm{G2}, 1, 1)$ (see Corollary 2.55) and therefore cannot be maximal.

$(\mathrm{G1}, k)$ This type exists only for $m \geq 3$. If U is of type $(\mathrm{G1}, k)$ with $k \leq m-2$, then U is contained in a space of type $(\mathrm{G1}, m-1)$ and therefore cannot be maximal. On the other hand, the spaces of type $(\mathrm{G1}, m-1)$ are of real codimension 2 in $\mathbb{V}$. There exist no curvature-invariant subspaces of $\mathbb{V}$ of real codimension 1 because of $m \geq 3$, and therefore the spaces of type $(\mathrm{G1}, m-1)$ are then maximal.

$(\mathrm{G2}, k_1, k_2)$ If U is of type $(\mathrm{G2}, k_1, k_2)$ with $k_1 + k_2 < m$, then U is contained in a space of type $(\mathrm{G2}, k_1, m-k_1)$ and is therefore not maximal. Moreover, any space U of type $(\mathrm{G2}, 1, 1)$ is contained in a space of type $(\mathrm{G3})$ and is therefore not maximal in the case $m = 2$. On the other hand, if U is of type $(\mathrm{G2}, k_1, k_2)$ with $k_1 + k_2 = m \geq 3$, then U is maximal: Assume to the contrary that there exists a curvature-invariant subspace U' of $\mathbb{V}$ with $U \subsetneq U' \subsetneq \mathbb{V}$. Then we have $\dim_{\mathbb{R}} U' > \dim_{\mathbb{R}} U = m$, and therefore U' is of type $(\mathrm{G1}, k)$ for some k (see the table in the theorem) and hence complex. Thus we have $U' \supset U \oplus JU = \mathbb{V}$, which is a contradiction.

$(\mathrm{G3})$ For $m = 2$, the spaces of type $(\mathrm{G3})$ have real codimension 1 in $\mathbb{V}$ and are therefore maximal. On the other hand, for $m \geq 3$, the space U of type $(\mathrm{G3})$ described in the theorem is contained in the space $\mathbb{C}(x - Jy) \oplus \mathbb{C}(x + Jy) = \mathbb{C}x \oplus \mathbb{C}y$ of type $(\mathrm{G1}, 2)$, and therefore cannot be maximal.

$(\mathrm{P1}, k)$ If U is of type $(\mathrm{P1}, k)$ with $k < m$, then U is contained in a space of type $(\mathrm{P1}, m)$ and therefore cannot be maximal. On the other hand, if U is of type $(\mathrm{P1}, m)$, then we have $U = V(A)$ for some $A \in \mathfrak{A}$. An inspection of the table in the theorem shows that there exists no curvature-invariant subspace U' of $\mathbb{V}$ with $V(A) \subsetneq U' \subsetneq \mathbb{V}$.

$(\mathrm{P2})$ Let U be a curvature-invariant subspace of type $(\mathrm{P2})$. In the case $m = 2$, U is maximal: Assume to the contrary that there exists a curvature-invariant subspace U' of $\mathbb{V}$ with $U \subsetneq U' \subsetneq \mathbb{V}$. Then U' is of real dimension 3 and therefore of type $(\mathrm{G3})$, so that there exists an orthonormal system (x, y) in some $V(A)$, $A \in \mathfrak{A}$ with $U' = \mathbb{C}(x - Jy) \oplus \mathbb{R}(x + Jy)$. U is complex, and therefore we have $U = U \cap JU \subset U' \cap JU' = \mathbb{C}(x - Jy)$, which is a contradiction because all elements of U are principal, whereas $\mathbb{C}(x - Jy)$ is an isotropic subspace of $\mathbb{V}$.

On the other hand, in the case $m \geq 3$, U is contained in a space of type $(\mathrm{G1}, 2)$ and therefore cannot be maximal.

(A) Let U be a curvature-invariant subspace of type (A); then we necessarily have $m \geq 3$. Using the notation in the definition of this type in the theorem, we see that the $\mathbb{C}$Q-span (see Definition 2.10(e)) of U is given by $\widehat{U} := \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}z$; this space is of complex dimension 3. Thus, in the case $m \geq 4$, U is contained in the space $\widehat{U}$ of type $(\mathrm{G1}, 3)$ and therefore cannot be maximal.

In the case $m = 3$, we again show the maximality of U by contradiction: Assume that U' is a curvature-invariant subspace of $\mathbb{V}$ with $U \subsetneq U' \subsetneq \mathbb{V}$. Then we have $\arctan(\frac{1}{2}) \in \varphi(\mathbb{S}(U'))$ and therefore U' is of one of the types $(\mathrm{G1}, k)$, $(\mathrm{G2}, k_1, k_2)$ and (G3). If U' is of type $(\mathrm{G1}, k)$, then U' is a $\mathbb{C}$Q-subspace of $\mathbb{V}$ and therefore contains $\widehat{U}$; because we have $\dim_{\mathbb{C}} \widehat{U} = 3 = \dim_{\mathbb{C}} \mathbb{V}$, $U' = \mathbb{V}$ follows, a contradiction. If U' is of type $(\mathrm{G2}, k_1, k_2)$, then U' is totally real in $\mathbb{V}$, and hence U is totally real, also a contradiction. Finally, if U' is of type (G3), then the $\mathbb{C}$Q-span of U' is complex-2-dimensional, in contradiction to $\dim_{\mathbb{C}} \widehat{U} = 3$.

$(\mathrm{I1}, k)$ Let U be a curvature-invariant subspace of type $(\mathrm{I1}, k)$. Proposition 2.20(e),(f) shows that the $\mathbb{C}$Q-span $\widehat{U}$ of U is of complex dimension $2k$. In the case $2k < m$ $\widehat{U}$ is therefore a curvature-invariant subspace of $\mathbb{V}$ of type $(\mathrm{G1}, 2k)$; because we have $\widehat{U} \supsetneq U$ it follows that U is not maximal. In the case $2k = m = 2$, U is contained in a space of type (G3) and therefore not maximal either. In the case $2k = m \geq 4$, we once again prove the maximality of U by contradiction: Assume that U' is a curvature-invariant subspace of $\mathbb{V}$ with $U \subsetneq U' \subsetneq \mathbb{V}$. Then we have $\dim_{\mathbb{R}} U' > \dim_{\mathbb{R}} U = 2k = m$, and therefore U' is of type $(\mathrm{G1}, k')$ for some k', and hence a $\mathbb{C}$Q-space. Thus we have $\widehat{U} \subset U'$; because of $\dim_{\mathbb{C}}(\widehat{U}) = 2k = m$, we have $\widehat{U} = \mathbb{V}$ and therefore $U' = \mathbb{V}$ follows, a contradiction.

$(\mathrm{I2}, k)$ If U is of type $(\mathrm{I2}, k)$, then U is contained in the space $U \ominus JU$ of type $(\mathrm{I1}, k)$ and therefore cannot be maximal.

To prove that no curvature-invariant subspace of $\mathbb{V}$ is of more than one type (observing the identifications of types given in the theorem) and the statement on the action of $\mathrm{Aut}(\mathfrak{A})$ on the set of curvature-invariant subspaces of $\mathbb{V}$, we give for each type of curvature-invariant subspaces a set of properties which characterizes the curvature-invariant subspaces of that type among all curvature-invariant subspaces of $\mathbb{V}$:

type	characterizing properties of the spaces U of that type
(Geo, t)	$\varphi(\mathbb{S}(U)) = \{t\}$, $\dim_{\mathbb{R}} U = 1$
$(\mathrm{G1}, k)$	U is a $\mathbb{C}$Q-subspace, $\dim_{\mathbb{C}} U = k$
$(\mathrm{G2}, k_1, k_2)$	There exist $A \in \mathfrak{A}$ and linear subspaces $W_1, W_2 \subset V(A)$ of real dimension k_1 resp. k_2 so that $W_1 \perp W_2$ and $U = W_1 \ominus JW_2$ holds.
(G3)	U is neither complex nor totally real, $\dim_{\mathbb{R}} U = 3$

type	characterizing properties of the spaces U of that type
$(\mathrm{P1},k)$	U is totally real, $\varphi(\mathbb{S}(U))=\{0\}$, $\dim_{\mathbb{R}} U = k$
$(\mathrm{P2})$	U is complex, $\varphi(\mathbb{S}(U))=\{0\}$
(A)	U is neither complex nor totally real, $\dim_{\mathbb{R}} U = 2$
$(\mathrm{I1},k')$	U is a complex, isotropic subspace, $\dim_{\mathbb{C}} U = k'$
$(\mathrm{I2},k')$	U is a totally real, isotropic subspace, $\dim_{\mathbb{R}} U = k'$

From this table, we draw the following conclusions:

(a) The properties given for different types (again, note the identifications given in the theorem) are mutually exclusive, therefore no curvature-invariant subspace can be of more than one type.

(b) The properties in the table are all invariant under replacement of U by $B(U)$ (where $B\in\mathrm{Aut}(\mathfrak{A})$), therefore for any curvature-invariant subspace U, U and $B(U)$ are of the same type.

Next, we prove that two curvature-invariant subspaces U and U' can be transformed into each other by an element of $\mathrm{Aut}(\mathfrak{A})$ if and only if they are of the same type. One implication has already been shown as (b) above. For the other implication, we let curvature-invariant subspaces U, U' of $\mathbb{V}$ of the same type be given. If they are of type (Geo,t), then Proposition 2.36(a) shows that there exists $B\in\mathrm{Aut}(\mathfrak{A})$ with $U'=B(U)$. If they are of another type, then it is easy to construct a $\mathbb{C}$Q-automorphism $B\in\mathrm{Aut}(\mathfrak{A})$ which transports the data described in the definition of the respective type for U into the data for U' and therefore U into U'. As an example, we describe the construction of B more explicitly for the type $(\mathrm{I1},k)$:

Suppose that U,U' are of type $(\mathrm{I1},k)$, therefore U,U' are complex k-dimensional $\mathfrak{A}$-isotropic subspaces of $\mathbb{V}$. We fix $A\in\mathfrak{A}$. By Proposition 2.20(e),(f) there exist $2k$-dimensional linear subspaces $Y,Y'\subset V(A)$ and orthogonal complex structures $\tau: Y\to Y$ and $\tau': Y'\to Y'$ so that

$$U=\{\,x+J\tau x\,|\,x\in Y\,\}\quad\text{and}\quad U'=\{\,x+J\tau' x\,|\,x\in Y'\,\}$$

holds. Let $L_0:(Y,\tau)\to(Y',\tau')$ be a $\mathbb{C}$-linear isometry between the complex-k-dimensional unitary spaces (Y,τ) and (Y',τ'); that means $L_0: Y\to Y'$ is an $\mathbb{R}$-linear isometry and $\tau'\circ L_0 = L_0\circ\tau$ holds. L_0 can be extended to an orthogonal transformation $L: V(A)\to V(A)$, and its complexification $B:=L^{\mathbb{C}}$ is a (strict) $\mathbb{C}$Q-automorphism by Proposition 2.15. It is now easily seen that $B(U)=U'$ holds.

It remains to prove that every curvature-invariant subspace of $\mathbb{V}$ is of one of the types given in the theorem, and this is the objective of the remainder of the present chapter.

4.2 The root space decomposition of a Lie triple system

As was explained in the introduction of the present chapter, the (connected, complete) totally geodesic submanifolds of a complex quadric Q passing through the "origin point" $p_0 \in Q$ are in one-to-one correspondence with the Lie triple systems $\mathfrak{m}'$ contained in the space $\mathfrak{m}$ of the canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ corresponding to the symmetric space Q as described in Propositions 3.9(d) and 3.12. The classification of these Lie triple systems $\mathfrak{m}'$ given in Sections 4.3 and 4.4 makes fundamental use of a root space decomposition for $\mathfrak{m}'$ analogous to the one described for $\mathfrak{m}$ in Appendix A.4.

In the present section, we describe such a root space decomposition for Lie triple systems in a general setting: We let $(M, \varphi, p_0, \sigma)$ be a symmetric G-space of compact type and consider the linearization σ_L of the involutive Lie group automorphism $\sigma : G \to G$ and the canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G it induces. The Killing form $\varkappa$ of $\mathfrak{g}$ is negative definite; as in Section A.4 we regard $\mathfrak{g}$ as an euclidean space via the inner product $\langle \cdot, \cdot \rangle := -c \cdot \varkappa$ with some $c \in \mathbb{R}_+$.

4.4 Definition. *A linear subspace* $\mathfrak{m}' \subset \mathfrak{m}$ *is called a* Lie triple system *if* $[\,[\mathfrak{m}', \mathfrak{m}']\,, \mathfrak{m}'\,] \subset \mathfrak{m}'$ *holds.*

Let $\mathfrak{m}' \subset \mathfrak{m}$ be a Lie triple system; we wish to derive a root space decomposition for $\mathfrak{m}'$. As is well-known, there exists a Riemannian symmetric subspace M' of M with $p_0 \in M'$ and $\tau^{-1}(T_{p_0}M') = \mathfrak{m}'$ ([KN69], Theorem XI.4.3, p. 237; $\tau : \mathfrak{m} \to T_{p_0}M$ is the canonical isomorphism). However, we cannot apply the root theory of Appendix A.4 to M' directly, as M' need not be of compact type. Rather, we derive a root space decomposition of $\mathfrak{m}'$ from the root space decomposition of $\mathfrak{m}$ described in Appendix A.4.

For any given Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$, we consider the corresponding root system of $\mathfrak{m}$, which we now denote by $\Delta(\mathfrak{m}, \mathfrak{a}) \subset \mathfrak{a}^*$; also we put for any $\lambda \in \mathfrak{a}^*$

$$\mathfrak{m}_\lambda := \{\, X \in \mathfrak{m} \,|\, \forall Z \in \mathfrak{a} : \mathrm{ad}(Z)^2 X = -\lambda(Z)^2\, X \,\} .$$

Then we have the root space decomposition of $\mathfrak{m}$ as in Proposition A.10(c).

4.5 Definition. *Let* $\mathfrak{m}' \subset \mathfrak{m}$ *be a Lie triple system.*

(a) *We call the maximal dimension of a flat subspace (see Proposition A.6) of* $\mathfrak{m}$ *lying in* $\mathfrak{m}'$ *the* rank of $\mathfrak{m}'$, *denoted by* $\mathrm{rk}(\mathfrak{m}')$. *Obviously* $\mathrm{rk}(\mathfrak{m}') \le \mathrm{rk}(M)$ *holds.*

(b) *We call any flat subspace* $\mathfrak{a}'$ *of* $\mathfrak{m}'$ *with* $\dim(\mathfrak{a}') = \mathrm{rk}(\mathfrak{m}')$ *a* Cartan subalgebra *of* $\mathfrak{m}'$.

(c) *Suppose that* $\mathfrak{a}$ *is a Cartan subalgebra of* $\mathfrak{m}$ *so that* $\mathfrak{a}' := \mathfrak{a} \cap \mathfrak{m}'$ *is a Cartan subalgebra of* $\mathfrak{m}'$.[7]

[7] Such an $\mathfrak{a}$ does not necessarily exist for every configuration of $(\mathfrak{m}, \mathfrak{m}')$. However, its existence is guaranteed for $\mathrm{rk}(\mathfrak{m}') = \mathrm{rk}(\mathfrak{m})$ (then $\mathfrak{a}$ can be chosen as a Cartan subalgebra of $\mathfrak{m}'$), and for $\mathrm{rk}(\mathfrak{m}') = 1$ (by [Hel78], Theorem V.6.2(ii), p. 246).

In this situation we define for any $\alpha \in (\mathfrak{a}')^*$

$$\mathfrak{m}'_\alpha := \{ X \in \mathfrak{m}' \,|\, \forall Z \in \mathfrak{a}' : \mathrm{ad}(Z)^2 X = -\alpha(Z)^2 X \} \tag{4.2}$$

and

$$\Delta(\mathfrak{m}', \mathfrak{a}') := \{ \alpha \in (\mathfrak{a}')^* \setminus \{0\} \,|\, \mathfrak{m}'_\alpha \neq \{0\} \} . \tag{4.3}$$

We call $\Delta(\mathfrak{m}', \mathfrak{a}')$ *the* root system *of* $\mathfrak{m}'$ *with respect to* $\mathfrak{a}'$, *and the space* $\mathfrak{m}'_\alpha$ *the* root space *of* $\mathfrak{m}'$ *corresponding to the root* $\alpha \in \Delta(\mathfrak{m}', \mathfrak{a}')$. *Like in Proposition A.10(b) we call a subset* $\Delta'_+ \subset \Delta(\mathfrak{m}', \mathfrak{a}')$ *a* system of positive roots *if*

$$\Delta'_+ \cup (-\Delta'_+) = \Delta(\mathfrak{m}', \mathfrak{a}') \quad \textit{and} \quad \Delta'_+ \cap (-\Delta'_+) = \varnothing$$

holds.

For $\alpha \in \Delta(\mathfrak{m}', \mathfrak{a}')$ *we denote by* $R'_\alpha : \mathfrak{a}' \to \mathfrak{a}'$ *the orthogonal reflection in the hyperplane* $\alpha^{-1}(\{0\})$. *Then we call the group of orthogonal transformations of* $\mathfrak{a}'$ *generated by* $\{ R'_\alpha \,|\, \alpha \in \Delta(\mathfrak{m}', \mathfrak{a}') \}$ *the* Weyl group $W(\mathfrak{m}', \mathfrak{a}')$ *of* $\mathfrak{m}'$ *(with respect to* $\mathfrak{a}'$*).* $W(\mathfrak{m}', \mathfrak{a}')$ *also acts on* $(\mathfrak{a}')^*$ *via the action* $(g, \alpha) \mapsto \alpha \circ g^{-1}$.

4.6 Proposition. *Let* $\mathfrak{m}' \subset \mathfrak{m}$ *be a Lie triple system and suppose that* $\mathfrak{a}$ *is a Cartan subalgebra of* $\mathfrak{m}$ *so that* $\mathfrak{a}' := \mathfrak{a} \cap \mathfrak{m}'$ *is a Cartan subalgebra of* $\mathfrak{m}'$.

(a) We have for any system of positive roots $\Delta'_+ \subset \Delta(\mathfrak{m}', \mathfrak{a}')$

$$\mathfrak{m}' = \mathfrak{a}' \oplus \bigoplus_{\alpha \in \Delta'_+} \mathfrak{m}'_\alpha ; \tag{4.4}$$

moreover:

$$\mathfrak{m}'_0 = \mathfrak{a}' , \tag{4.5}$$

$$\Delta(\mathfrak{m}', \mathfrak{a}') \subset \{ \lambda|\mathfrak{a}' \mid \lambda \in \Delta(\mathfrak{m}, \mathfrak{a}), \lambda|\mathfrak{a}' \neq 0 \} , \tag{4.6}$$

$$\forall \alpha \in \Delta(\mathfrak{m}', \mathfrak{a}') \; : \; \mathfrak{m}'_\alpha = \left(\bigoplus\nolimits_{\substack{\lambda \in \Delta(\mathfrak{m}, \mathfrak{a}) \\ \lambda|\mathfrak{a}' = \alpha}} \mathfrak{m}_\lambda \right) \cap \mathfrak{m}' . \tag{4.7}$$

(b) We have $\mathrm{rk}(\mathfrak{m}') = \mathrm{rk}(\mathfrak{m})$ *if and only if* $\mathfrak{a}' = \mathfrak{a}$ *holds. If this is the case, then we have*

$$\Delta(\mathfrak{m}', \mathfrak{a}') \subset \Delta(\mathfrak{m}, \mathfrak{a}) , \quad \forall \alpha \in \Delta(\mathfrak{m}', \mathfrak{a}') : \mathfrak{m}'_\alpha = \mathfrak{m}_\alpha \cap \mathfrak{m}' \quad \textit{and} \quad W(\mathfrak{m}', \mathfrak{a}') \subset W(\mathfrak{m}, \mathfrak{a}) . \tag{4.8}$$

Moreover, the Weyl group $W(\mathfrak{m}', \mathfrak{a}')$ *then leaves* $\Delta(\mathfrak{m}', \mathfrak{a}')$ *invariant.*[8]

Proof. Let us abbreviate $\Delta := \Delta(\mathfrak{m}, \mathfrak{a})$ and $\Delta' := \Delta(\mathfrak{m}', \mathfrak{a}')$.

Because M is a Riemannian symmetric G-space of compact type, the Killing form $\varkappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is negative definite, and therefore $\langle \cdot, \cdot \rangle := -\varkappa$ is a positive definite inner product on $\mathfrak{g}$. We regard $\mathfrak{g}$ and especially the subspace $\mathfrak{m}$ as euclidean spaces in this way.

[8]If the symmetric subspace M' of M which corresponds to $\mathfrak{m}'$ is of compact type, then $W(\mathfrak{m}', \mathfrak{a}')$ leaves $\Delta(\mathfrak{m}', \mathfrak{a}')$ invariant by Proposition A.15(b) without regard to $\mathrm{rk}(\mathfrak{m}')$.

For (a). We first prove Equation (4.5). We have $[\mathfrak{a}', \mathfrak{a}'] = \{0\}$ by Proposition A.6(b) and therefore $\mathfrak{a}' \subset \mathfrak{m}_0'$. Conversely, let $X \in \mathfrak{m}_0'$ be given. Then we have for every $Z \in \mathfrak{a}'$: $\mathrm{ad}(Z)^2 X = 0$ and therefore

$$0 = \langle \mathrm{ad}(Z)^2 X, X \rangle = -\langle \mathrm{ad}(Z)X, \mathrm{ad}(Z)X \rangle \;,$$

whence $\mathrm{ad}(Z)X = 0$ follows by the positive definity of $\langle \cdot, \cdot \rangle$. From this fact and $[\mathfrak{a}', \mathfrak{a}'] = \{0\}$, we see that $[\mathfrak{a}' + \mathbb{R}X, \mathfrak{a}' + \mathbb{R}X] = \{0\}$ holds, and therefore $\mathfrak{a}' + \mathbb{R}X$ is flat by a further application of Proposition A.6. Because of the maximality of $\mathfrak{a}'$, we conclude $X \in \mathfrak{a}'$.

We now consider the endomorphisms $R_Z : \mathfrak{m} \to \mathfrak{m},\ X \mapsto -\mathrm{ad}(Z)^2 X$ with $Z \in \mathfrak{a}$. (R_Z is equivalent to a Jacobi operator, see Equation (A.13)). As was shown in Proposition A.11, there exists a finite set Σ of functions $\mathfrak{a} \to \mathbb{R}$ so that

$$\mathfrak{m} = \bigoplus_{\mu \in \Sigma} E_\mu \quad \text{and} \quad \forall \mu \in \Sigma \;:\; E_\mu \neq \{0\} \tag{4.9}$$

holds, where we define for every function $\mu : \mathfrak{a} \to \mathbb{R}$:[9]

$$E_\mu := \bigcap_{Z \in \mathfrak{a}} \mathrm{Eig}(R_Z, \mu(Z)) \;. \tag{4.10}$$

We have

$$\Sigma = \{\, \mu : \mathfrak{a} \to \mathbb{R} \mid E_\mu \neq \{0\} \,\} \;. \tag{4.11}$$

For every $Z \in \mathfrak{a}'$ the endomorphism R_Z leaves $\mathfrak{m}'$ invariant because $\mathfrak{m}'$ is a Lie triple system. The endomorphisms $R_Z|\mathfrak{m}' : \mathfrak{m}' \to \mathfrak{m}'$ (with $Z \in \mathfrak{a}'$) are self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle$, and any two such endomorphisms commute with each other. Therefore the family of endomorphisms $(R_Z|\mathfrak{m}')_{Z \in \mathfrak{a}'}$ is jointly orthogonally diagonalizable; via this fact we obtain a decomposition for $\mathfrak{m}'$ analogous to the decomposition of $\mathfrak{m}$ from Equation (4.9): There exists a finite set Σ' of functions $\mathfrak{a}' \to \mathbb{R}$ so that

$$\mathfrak{m}' = \bigoplus_{\nu \in \Sigma'} F'_\nu \quad \text{and} \quad \forall \nu \in \Sigma' \;:\; F'_\nu \neq \{0\} \tag{4.12}$$

holds, where we define for every function $\nu : \mathfrak{a}' \to \mathbb{R}$

$$F'_\nu := \bigcap_{Z \in \mathfrak{a}'} \mathrm{Eig}(R_Z|\mathfrak{m}', \nu(Z)) \;. \tag{4.13}$$

We have

$$\Sigma' = \{\, \nu : \mathfrak{a}' \to \mathbb{R} \mid F'_\nu \neq \{0\} \,\} \;. \tag{4.14}$$

To study the relationship between Σ' and Σ resp. between F'_ν and E_μ, we put for every function $\nu : \mathfrak{a}' \to \mathbb{R}$

$$\Sigma(\nu) := \{\ \mu \in \Sigma \mid \mu|\mathfrak{a}' = \nu\ \} \quad \text{and} \quad F_\nu := \bigcap_{Z \in \mathfrak{a}'} \mathrm{Eig}(R_Z, \nu(Z)) \;. \tag{4.15}$$

[9] For Equation (4.10) remember that we use the notation $\mathrm{Eig}(B, \lambda) := \ker(B - \lambda\,\mathrm{id})$ even when λ is not an eigenvalue of B, compare Section 0.2.

Then we have by Equations (4.13) and (4.15)

$$F'_\nu = F_\nu \cap \mathfrak{m}' \,. \tag{4.16}$$

We now prove the following equation, which is of central importance in the present consideration:

$$F_\nu = \bigoplus_{\mu \in \Sigma(\nu)} E_\mu \,. \tag{4.17}$$

Indeed, in this equation the inclusion "$\supset$" follows immediately from Equations (4.10) and (4.15). For the converse inclusion, we let $X \in F_\nu$ be given. In particular $X \in \mathfrak{m}$ holds; by Equation (4.9) it follows that we have $X = \sum_{\mu\in\Sigma} X_\mu$ with suitable $X_\mu \in E_\mu$. For every $Z \in \mathfrak{a}'$ we now have

$$\sum_{\mu\in\Sigma} \nu(Z)\, X_\mu = \nu(Z)\, X \overset{(4.15)}{=} R_Z(X) = \sum_{\mu\in\Sigma} R_Z(X_\mu) \overset{(4.10)}{=} \sum_{\mu\in\Sigma} \mu(Z)\, X_\mu \,.$$

This calculation shows that for any $\mu \in \Sigma$ with $X_\mu \neq 0$, we have $\nu(Z) = \mu(Z)$ for every $Z \in \mathfrak{a}'$, and therefore $\mu \in \Sigma(\nu)$. Thus we see that $X = \sum_{\mu\in\Sigma(\nu)} X_\mu$ is a member of the right-hand side of Equation (4.17).

By combining Equation (4.17) with Equation (4.16) we obtain

$$\forall \nu \in \Sigma' \;:\; F'_\nu = \left(\bigoplus_{\mu\in\Sigma(\nu)} E_\mu \right) \cap \mathfrak{m}' \tag{4.18}$$

and therefore also

$$\forall \nu \in \Sigma' \;:\; \Sigma(\nu) \neq \varnothing \,. \tag{4.19}$$

To obtain the desired results we now describe relations between the objects involved in the diagonalizations we studied and the roots and root spaces of $\mathfrak{m}$ resp. $\mathfrak{m}'$:

$$\forall \lambda \in \mathfrak{a}^* \;:\; \mathfrak{m}_\lambda = E_{\lambda^2} \quad \text{and} \quad \forall \alpha \in (\mathfrak{a}')^* \;:\; \mathfrak{m}'_\alpha = F'_{\alpha^2} \,; \tag{4.20}$$

$$\Sigma = \{\, \lambda^2 \,|\, \lambda \in \Delta \,\} \dot\cup \{0\} \quad \text{and} \quad \Sigma' = \{\, \alpha^2 \,|\, \alpha \in \Delta' \,\} \dot\cup \{0\} \,. \tag{4.21}$$

The first equation in (4.20) resp. (4.21) is just Equation (A.30) resp. Equation (A.28) from Proposition A.11(a). The second equation in (4.20) follows from the fact that we have for any $\alpha \in (\mathfrak{a}')^*$

$$\mathfrak{m}'_\alpha \overset{(4.2)}{=} \bigcap_{Z\in\mathfrak{a}'} \mathrm{Eig}(R_Z|\mathfrak{m}', \alpha(Z)^2)$$

and Equation (4.13).

For the proof of the second equation in (4.21): Let $\nu \in \Sigma'$ with $\nu \neq 0$ be given. By (4.19) there exists some $\mu \in \Sigma$ so that $\mu|\mathfrak{a}' = \nu$ holds. By the first equation in (4.21) there exists $\lambda \in \Delta$ with $\mu = \lambda^2$. We have $\alpha := \lambda|\mathfrak{a}' \in (\mathfrak{a}')^* \setminus \{0\}$ and $\nu = \alpha^2$, hence $\mathfrak{m}'_\alpha \overset{(4.20)}{=} F'_{\alpha^2} = F'_\nu \neq \{0\}$. Therefrom $\alpha \in \Delta'$ follows by Equation (4.3), and therefore ν is a member of the right-hand side of the equation to be shown. For the converse inclusion: We have $F'_0 \overset{(4.20)}{=} \mathfrak{m}'_0 \overset{(4.5)}{=} \mathfrak{a} \neq \{0\}$

and therefore $0 \in \Sigma'$ by (4.14); also we have for any $\alpha \in \Delta'$: $F'_{\alpha^2} \overset{(4.20)}{=} \mathfrak{m}'_\alpha \neq \{0\}$ and therefore $\alpha^2 \in \Sigma'$ again by (4.14).

We now show Equations (4.6) and (4.7). Let $\alpha \in \Delta'$ be given. Then we have

$$\mathfrak{m}'_\alpha \overset{(4.20)}{=} F'_{\alpha^2} \overset{(4.18)}{=} \left(\bigoplus_{\mu \in \Sigma(\alpha^2)} E_\mu \right) \cap \mathfrak{m}' \overset{(4.21)}{=} \left(\bigoplus_{\substack{\lambda \in \Delta_+ \\ \lambda^2 \in \Sigma(\alpha^2)}} E_{\lambda^2} \right) \cap \mathfrak{m}' \overset{(4.20)}{=} \left(\bigoplus_{\substack{\lambda \in \Delta \\ \lambda|\mathfrak{a}' = \alpha}} \mathfrak{m}_\lambda \right) \cap \mathfrak{m}'$$

(where $\Delta_+ \subset \Delta$ is a positive root system); for the last equals sign notice that $(\lambda|\mathfrak{a}')^2 = \alpha^2$ implies $\lambda|\mathfrak{a}' = \pm\alpha$. This shows Equation (4.7). Because of $\mathfrak{m}'_\alpha \neq \{0\}$ it follows in particular that there exists $\lambda \in \Delta$ with $\lambda|\mathfrak{a}' = \alpha \neq 0$, whence (4.6) follows.

Finally, Equation (4.4) is derived in the following way:

$$\mathfrak{m}' \overset{(4.12)}{=} \bigoplus_{\nu \in \Sigma'} F'_\nu \overset{(4.21)}{=} F'_0 \oplus \bigoplus_{\alpha \in \Delta'_+} F'_{\alpha^2} \overset{(4.20)}{=} \mathfrak{m}'_0 \oplus \bigoplus_{\alpha \in \Delta'_+} \mathfrak{m}'_\alpha \overset{(4.5)}{=} \mathfrak{a}' \oplus \bigoplus_{\alpha \in \Delta'_+} \mathfrak{m}'_\alpha \,.$$

For (b). Because $\mathfrak{a}'$ and $\mathfrak{a}$ are Cartan subalgebras of $\mathfrak{a}'$ and $\mathfrak{a}$ respectively, we have $\mathrm{rk}(\mathfrak{m}') = \dim \mathfrak{a}'$ and $\mathrm{rk}(\mathfrak{m}) = \dim \mathfrak{a}$. From these facts and $\mathfrak{a}' \subset \mathfrak{a}$ it follows that $\mathrm{rk}(\mathfrak{m}') = \mathrm{rk}(\mathfrak{m})$ is equivalent to $\mathfrak{a}' = \mathfrak{a}$.

We now suppose that $\mathfrak{a}' = \mathfrak{a}$ holds. Then the first two parts of (4.8) follow from Equations (4.6) and (4.7); from $\mathfrak{a}' = \mathfrak{a}$ and $\Delta' \subset \Delta$ it also follows that the Weyl group $W(\mathfrak{m}', \mathfrak{a}')$ is a subgroup of $W(\mathfrak{m}, \mathfrak{a})$.

It remains to show that $W(\mathfrak{m}', \mathfrak{a}')$ leaves Δ' invariant.

For this, we let $\lambda \in \Delta'$ be given and fix $X \in \mathbb{S}(\mathfrak{m}'_\lambda)$. By (4.8), we have $X \in \mathfrak{m}_\lambda$, and we let $\widehat{X} \in \mathfrak{k}_\lambda \setminus \{0\}$ be the element related to X (see Definition A.12 and Proposition A.13(a)). Because of $\lambda \neq 0$ there exists some $Z_0 \in \mathfrak{a}$ with $\lambda(Z_0) = 1$, and then Definition A.12 shows that we have

$$\widehat{X} = [Z_0, X] \,. \tag{4.22}$$

Furthermore we put

$$\widehat{g} := \mathrm{Exp}(t_0\,\widehat{X}\,) \in K \quad \text{with} \quad t_0 := \tfrac{\pi}{\|\lambda^\sharp\|} \,,$$

where K is the isotropy group of the G-action on M at the "origin point" p_0 and $\mathrm{Exp} : \mathfrak{k} \to K$ is the exponential map of K. Then we have by Proposition A.15(a)

$$\mathrm{Ad}(\widehat{g})|\mathfrak{a} = R_\lambda \,. \tag{4.23}$$

Below, we will show

$$\mathrm{Ad}(\widehat{g})\mathfrak{m}' = \mathfrak{m}' \,. \tag{4.24}$$

We then obtain via Proposition A.15(b) for every $\mu \in \Delta'$

$$\mathfrak{m}'_{\mu\circ(R_\lambda)^{-1}} \overset{(4.8)}{=} \mathfrak{m}_{\mu\circ(R_\lambda)^{-1}} \cap \mathfrak{m}' \overset{(A.37)}{=} \mathrm{Ad}(\widehat{g})\mathfrak{m}_\mu \cap \mathfrak{m}' \overset{(4.24)}{=} \mathrm{Ad}(\widehat{g})(\mathfrak{m}_\mu \cap \mathfrak{m}') \overset{(4.8)}{=} \mathrm{Ad}(\widehat{g})\mathfrak{m}'_\mu \neq \{0\}$$

and therefore $\mu\circ(R_\lambda)^{-1} \in \Delta'$. This shows that Δ' is invariant under the Weyl group $W(\mathfrak{m}',\mathfrak{a}')$.

For the proof of Equation (4.24), we let $Y \in \mathfrak{m}'$ be given and consider the function

$$f : \mathbb{R} \to \mathfrak{m},\ t \mapsto \mathrm{Ad}(\mathrm{Exp}(t\,\widehat{X}))Y = \exp(t\ \mathrm{ad}(\widehat{X}))Y\ ,$$

where $\exp : \mathrm{End}(\mathfrak{g}) \to \mathrm{GL}(\mathfrak{g})$ is the usual exponential map of endomorphisms. f solves the differential equation

$$y' = \mathrm{ad}(\widehat{X})y\ . \tag{4.25}$$

Because $\mathfrak{m}'$ is a Lie triple system, it follows from Equation (4.22) that the endomorphism $\mathrm{ad}(\widehat{X})$ leaves $\mathfrak{m}'$ invariant. Because we also have $f(0) = Y \in \mathfrak{m}'$, the solution f of the differential equation (4.25) runs entirely in $\mathfrak{m}'$. In particular we have $\mathrm{Ad}(\widehat{g})Y = f(t_0) \in \mathfrak{m}'$. Thus we have shown $\mathrm{Ad}(\widehat{g})\mathfrak{m}' \subset \mathfrak{m}'$; because $\mathrm{Ad}(\widehat{g})$ is a linear isomorphism, we conclude (4.24). □

4.7 Definition. *Let $\mathfrak{m}' \subset \mathfrak{m}$ be a Lie triple system and $\mathfrak{a}$ a Cartan subalgebra of $\mathfrak{m}$ so that $\mathfrak{a}' := \mathfrak{a}\cap\mathfrak{m}'$ is a Cartan subalgebra of $\mathfrak{m}'$. Let $\alpha \in \Delta(\mathfrak{m}',\mathfrak{a}')$ be given. Remember that by Proposition 4.6(a) there exists at least one root $\lambda \in \Delta(\mathfrak{m},\mathfrak{a})$ with $\lambda|\mathfrak{a}' = \alpha$. We call α*

(a) elementary*, if there is only one root $\lambda \in \Delta(\mathfrak{m},\mathfrak{a})$ with $\lambda|\mathfrak{a}' = \alpha$;*

(b) composite*, if there are at least two different roots $\lambda,\mu \in \Delta(\mathfrak{m},\mathfrak{a})$ with $\lambda|\mathfrak{a}' = \mu|\mathfrak{a}' = \alpha$.*

In the situation described in Definition 4.7, elementary roots play a special role: If $\alpha \in \Delta(\mathfrak{m}',\mathfrak{a}')$ is elementary, then the root space $\mathfrak{m}'_\alpha$ is contained in the root space $\mathfrak{m}_\lambda$, where $\lambda \in \Delta(\mathfrak{m},\mathfrak{a})$ is the unique root with $\lambda|\mathfrak{a}' = \alpha$ (see Proposition 4.6(a)). As we will see in Proposition 4.9 and its corollary below, this property causes restrictions for the possible positions (in relation to $\mathfrak{a}'$) of λ.

It should be mentioned that in the case $\mathrm{rk}(\mathfrak{m}') = \mathrm{rk}(M)$ we have $\mathfrak{a}' = \mathfrak{a}$, and therefore in that case

$$\text{every } \alpha \in \Delta(\mathfrak{m}',\mathfrak{a}') \text{ is elementary}$$

(see Proposition 4.6(b)).

4.8 Lemma. *Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{m}$. Then we have*

$$\forall \lambda \in \Delta(\mathfrak{m},\mathfrak{a}),\ X \in \mathfrak{m}_\lambda,\ Z \in \mathfrak{a}\ :\ \mathrm{ad}(X)^2 Z = -\|X\|^2 \cdot \lambda(Z) \cdot \lambda^\sharp\ .$$

Proof. Let $\lambda \in \Delta(\mathfrak{m},\mathfrak{a})$ and $X \in \mathfrak{m}_\lambda$ be given. By Proposition A.13(a), there is exactly one $\widehat{X} \in \mathfrak{k}_\lambda$ which is related to $X \in \mathfrak{m}_\lambda$ in the sense of Definition A.12, meaning in particular that we have for given $Z \in \mathfrak{a}$

$$[Z,X] = \lambda(Z) \cdot \widehat{X}\ . \tag{4.26}$$

By Proposition A.13(b) we also have

$$[X,\widehat{X}] = \|X\|^2 \cdot \lambda^\sharp\ . \tag{4.27}$$

Using these equations, we calculate:

$$\mathrm{ad}(X)^2 Z = [X,[X,Z]] = -[X,[Z,X]] \overset{(4.26)}{=} -\lambda(Z)\cdot[X,\widehat{X}] \overset{(4.27)}{=} -\lambda(Z)\cdot\|X\|^2\cdot\lambda^\sharp\,. \qquad \square$$

4.9 Proposition. *Let $\mathfrak{m}' \subset \mathfrak{m}$ be a Lie triple system, and $\mathfrak{a}$ a Cartan subalgebra of $\mathfrak{m}$ so that $\mathfrak{a}' := \mathfrak{a}\cap\mathfrak{m}'$ is a Cartan subalgebra of $\mathfrak{m}'$. If $\alpha \in \Delta(\mathfrak{m}',\mathfrak{a}')$ is an elementary root and $\lambda \in \Delta(\mathfrak{m},\mathfrak{a})$ is the unique root with $\lambda|\mathfrak{a}' = \alpha$, then we have*

$$\lambda^\sharp \in \mathfrak{a}'\,.$$

If, on the other hand, $\lambda \in \Delta(\mathfrak{m},\mathfrak{a})$ satisfies $\lambda|\mathfrak{a}' = 0$, then we obviously have

$$\lambda^\sharp \perp \mathfrak{a}'\,.$$

Proof. Let $\alpha \in \Delta(\mathfrak{m}',\mathfrak{a}')$ be an elementary root and $\lambda \in \Delta(\mathfrak{m},\mathfrak{a})$ be the root with $\lambda|\mathfrak{a}' = \alpha$. Then we fix $Z \in \mathfrak{a}'$ so that $\lambda(Z) = \alpha(Z) = -1$ holds and $X \in \mathbb{S}(\mathfrak{m}'_\alpha)$ arbitrarily. We have $X \in \mathfrak{m}'_\alpha \subset \mathfrak{m}_\lambda$ by Proposition 4.6(a) and the fact that α is elementary, and therefore by Lemma 4.8

$$\mathfrak{m}' \overset{(*)}{\ni} \mathrm{ad}(X)^2 Z = -\|X\|^2\cdot\lambda(Z)\cdot\lambda^\sharp = \lambda^\sharp\,,$$

where $(*)$ follows from the fact that $\mathfrak{m}'$ is a Lie triple system. Therefore we have $\lambda^\sharp \in \mathfrak{m}'\cap\mathfrak{a} = \mathfrak{a}'$.

The statement on the case $\lambda \in \Delta(\mathfrak{m},\mathfrak{a})$ with $\lambda|\mathfrak{a}' = 0$ is obvious. $\square$

4.10 Corollary. *Let $\mathfrak{m}' \subset \mathfrak{m}$ be a Lie triple system with $\mathrm{rk}(\mathfrak{m}') = 1$, and let $X \in \mathfrak{m}'\setminus\{0\}$ be given. Then $\mathfrak{a}' := \mathbb{R}X$ is a Cartan subalgebra of $\mathfrak{m}'$ and there exists a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ so that $\mathfrak{a}' = \mathfrak{a}\cap\mathfrak{m}'$ holds.*

If $\alpha \in \Delta(\mathfrak{m}',\mathfrak{a}')$ is an elementary root of $\mathfrak{m}'$ and $\lambda \in \Delta(\mathfrak{m},\mathfrak{a})$ the unique root with $\lambda|\mathfrak{a}' = \alpha$, then $\lambda^\sharp$ is parallel to X.

Proof. The existence of $\mathfrak{a}$ follows from Theorem A.8(b) and the remainder is an immediate consequence of Proposition 4.9. $\square$

4.11 Remark. Investigating root systems of Lie algebras, ESCHENBURG used similar concepts as our elementary/composite roots, see [Esc84], Abschnitt 91, p. 131ff.. That situation is different from ours, because in contrary to symmetric spaces, the root spaces of Lie algebras are always 1-dimensional.

4.3 The classification of the rank 2 Lie triple systems

We now start with the proof that the list of curvature-invariant subspaces of the $\mathbb{C}$Q-space $(\mathbb{V},\mathfrak{A})$ given in Theorem 4.2 is in fact complete. We put $m := \dim_{\mathbb{C}}(\mathbb{V})$, let $(\widetilde{\mathbb{V}},\widetilde{\mathfrak{A}})$ be an arbitrary $(m+2)$-dimensional $\mathbb{C}$Q-space, let $Q := Q(\widetilde{\mathfrak{A}})$ be the m-dimensional complex quadric induced thereby and put $G := \mathrm{Aut}_s(\widetilde{\mathfrak{A}})_0$. We regard Q as a Hermitian symmetric G-space (Q, Ψ, p_0, σ) as in Section 3.2 and consider the canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G with respect to σ. Then $\mathfrak{m}$ is an m-dimensional $\mathbb{C}$Q-space in the way described in Proposition 3.12. As $\mathbb{C}$Q-space, it is isomorphic to $(\mathbb{V}, \mathfrak{A})$ by Corollary 2.16, and thus we may suppose without loss of generality that $(\mathbb{V}, \mathfrak{A})$ is equal to the $\mathbb{C}$Q-space $\mathfrak{m}$. In the sequel, we denote the complex inner product given on $\mathfrak{m}$ by $\langle \cdot, \cdot \rangle_{\mathbb{C}}$, the real inner product by $\langle \cdot, \cdot \rangle = \mathrm{Re}(\langle \cdot, \cdot \rangle_{\mathbb{C}})$, the complex structure of $\mathfrak{m}$ by $J : \mathfrak{m} \to \mathfrak{m},\ X \mapsto i\,X$ and the $\mathbb{C}$Q-structure of $\mathfrak{m}$ by $\mathfrak{A}$.

We let a curvature-invariant subspace $\mathfrak{m}' \neq \{0\}$ of the $\mathbb{C}$Q-space $\mathfrak{m}$ be given. Then Proposition 3.12(c) shows that $\mathfrak{m}'$ is a Lie triple system in $\mathfrak{m}$. We have $\mathrm{rk}(\mathfrak{m}') \leq \mathrm{rk}(Q) = 2$ and therefore $\mathrm{rk}(\mathfrak{m}') \in \{1, 2\}$. The two resulting cases $\mathrm{rk}(\mathfrak{m}') = 2$ and $\mathrm{rk}(\mathfrak{m}') = 1$ divide the proof of the classification theorem into two main parts. We treat the case $\mathrm{rk}(\mathfrak{m}') = 2$ in the present section, and the case $\mathrm{rk}(\mathfrak{m}') = 1$ in the next section.

Thus we now suppose $\mathrm{rk}(\mathfrak{m}') = 2$. We fix a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}'$; because of $\mathrm{rk}(\mathfrak{m}') = 2 = \mathrm{rk}(Q)$, $\mathfrak{a}$ also is a Cartan subalgebra of $\mathfrak{m}$. In the sequel, we denote by $\Delta := \Delta(\mathfrak{m}, \mathfrak{a})$ and $\Delta' := \Delta(\mathfrak{m}', \mathfrak{a})$ the root systems of $\mathfrak{m}$ resp. of $\mathfrak{m}'$ with respect to $\mathfrak{a}$. In this relation, we use the notations introduced in Section 4.2. Then we have by Proposition 4.6(b)

$$\Delta' \subset \Delta \quad \text{and} \quad \forall \alpha \in \Delta' \ : \ \mathfrak{m}'_\alpha = \mathfrak{m}_\alpha \cap \mathfrak{m}' \subset \mathfrak{m}_\alpha \ . \tag{4.28}$$

Therefore $\Delta'_+ := \Delta_+ \cap \Delta'$ is a system of positive roots of Δ', where $\Delta_+ := \{\lambda_1, \dots, \lambda_4\}$ is the system of positive roots of Δ described in Theorem 3.15(b). Further, we have by Proposition 4.6(a)

$$\mathfrak{m}' = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta'_+} \mathfrak{m}'_\alpha \ . \tag{4.29}$$

Moreover, the root system Δ' is invariant under the Weyl group $W(\mathfrak{m}', \mathfrak{a}')$ by Proposition 4.6(b), and this fact imposes restrictions on the subsets of $\Delta_+ = \{\lambda_1, \dots, \lambda_4\}$ which can occur as Δ'_+. For example $\Delta'_+ = \{\lambda_1, \lambda_4\}$ is impossible, because then $\Delta' = \Delta'_+ \dot\cup (-\Delta'_+)$ would not be invariant under the reflection in the line orthogonal to λ_1. (For the calculation of the action of the Weyl group on the λ_k, note the relationship between its action on λ_k and on $\lambda_k^\sharp$ given by Equation (A.36) and the explicit description of the $\lambda_k^\sharp$ in Theorem 3.15(b).)

By this consideration we see that Δ'_+ must be one of the following eight sets:

$$\varnothing\ , \quad \{\lambda_1\}\ , \quad \{\lambda_2\}\ , \quad \{\lambda_3\}\ , \quad \{\lambda_4\}\ , \quad \{\lambda_1, \lambda_2\}\ , \quad \{\lambda_3, \lambda_4\}\ , \quad \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}\ .$$

We now inspect the eight cases of possible Δ'_+ individually to verify that the corresponding Lie triple systems $\mathfrak{m}'$ are all of of one of the types $(\mathrm{G1}, k)$, $(\mathrm{G2}, k_1, k_2)$ and $(\mathrm{G3})$ as they are described in Theorem 4.2.

For this purpose, we note that by Theorem 3.15(a), there exist $A \in \mathfrak{A}$ and an orthonormal system (X, Y) in $V(A)$ so that $\mathfrak{a} = \mathbb{R}X \oplus \mathbb{R}JY$ holds. Also, we put $n_\alpha := \dim(\mathfrak{m}'_\alpha)$ for $\alpha \in \Delta'$, and continually use the data on the root system $\Delta_+ = \{\lambda_1, \dots, \lambda_4\}$ and the root spaces $\mathfrak{m}_{\lambda_k}$ given in Theorem 3.15(b).

The case $\Delta'_+ = \varnothing$. By Equation (4.29) we have $\mathfrak{m}' = \mathfrak{a} = \mathbb{R}X \oplus \mathbb{R}JY$, and therefore, $\mathfrak{m}'$ is of type $(\mathrm{G2},1,1)$ with $W_1 := \mathbb{R}X$, $W_2 := \mathbb{R}Y$.

The case $\Delta'_+ = \{\lambda_1\}$. By Equation (4.29) we have $\mathfrak{m}' = \mathfrak{a} \oplus \mathfrak{m}'_{\lambda_1}$; by (4.28) and Theorem 3.15(b) we have $\mathfrak{m}'_{\lambda_1} \subset \mathfrak{m}_{\lambda_1} = J((\mathbb{R}X \oplus \mathbb{R}Y)^{\perp,V(A)})$. It follows that $\mathfrak{m}'$ is of type $(\mathrm{G2},1,1+n'_{\lambda_1})$ with $W_1 := \mathbb{R}X$ and $W_2 := \mathbb{R}Y \oplus J\mathfrak{m}'_{\lambda_1}$.

The case $\Delta'_+ = \{\lambda_2\}$. Analogously as in the case $\Delta'_+ = \{\lambda_1\}$ we see that $\mathfrak{m}'$ is of type $(\mathrm{G2},1+n'_{\lambda_2},1)$ with $W_1 := \mathbb{R}X \oplus \mathfrak{m}'_{\lambda_2}$ and $W_2 := \mathbb{R}Y$.

The case $\Delta'_+ = \{\lambda_3\}$. By Equation (4.29) we have $\mathfrak{m}' = \mathfrak{a} \oplus \mathfrak{m}'_{\lambda_3}$. We have $\{0\} \neq \mathfrak{m}'_{\lambda_3} \subset \mathfrak{m}_{\lambda_3}$; because $\mathfrak{m}_{\lambda_3}$ is 1-dimensional, therefrom already $\mathfrak{m}'_{\lambda_3} = \mathfrak{m}_{\lambda_3} = \mathbb{R}(JX+Y)$ follows. Thus we have

$$\begin{aligned}\mathfrak{m}' &= \mathfrak{a} \oplus \mathfrak{m}_{\lambda_3} = \mathbb{R}X \oplus \mathbb{R}JY \oplus \mathbb{R}(JX+Y)\\ &= \mathbb{R}(X+JY) \oplus \mathbb{R}(X-JY) \oplus \mathbb{R}(JX+Y) = \mathbb{R}(X+JY) \oplus \mathbb{C}(X-JY)\,,\end{aligned}$$

and therefore $\mathfrak{m}'$ is of type $(\mathrm{G3})$.

The case $\Delta'_+ = \{\lambda_4\}$. Analogously as in the case $\Delta'_+ = \{\lambda_3\}$ we obtain $\mathfrak{m}' = \mathbb{R}(X-JY) \oplus \mathbb{C}(X+JY)$. By replacing Y with $-Y$, we see that also in this case $\mathfrak{m}'$ is of type $(\mathrm{G3})$.

The case $\Delta'_+ = \{\lambda_1, \lambda_2\}$. By Equation (4.29) we have

$$\mathfrak{m}' = \mathfrak{a} \oplus \mathfrak{m}'_{\lambda_1} \oplus \mathfrak{m}'_{\lambda_2} = W_1 \oplus J(W_2) \tag{4.30}$$

with $W_1 := \mathbb{R}X \oplus \mathfrak{m}'_{\lambda_2}$ and $W_2 := \mathbb{R}Y \oplus J(\mathfrak{m}'_{\lambda_1})$. Together with Equation (4.28), the table in Theorem 3.15(b) shows that $\mathfrak{m}'_{\lambda_1}, \mathfrak{m}'_{\lambda_2} \subset (\mathbb{R}X \oplus \mathbb{R}Y)^{\perp,V(A)} \subset V(A)$ holds, and therefore we have $W_1, W_2 \subset V(A)$.

We now show $W_1 \perp W_2$: Let $u \in W_2$ and $v \in W_1$ be given, and assume that $\langle u, v\rangle \neq 0$ holds. We have $Ju, v \in \mathfrak{m}'$ by Equation (4.30), and therefore Corollary 2.48 shows that $\mathfrak{m}'$ is a complex-linear subspace of $\mathfrak{m}$. Because we have $X + JY \in \mathfrak{a} \subset \mathfrak{m}'$, it follows that we also have $-Y + JX = J(X+JY) \in \mathfrak{m}'$. Hence we have $\mathfrak{m}_{\lambda_4} = \mathbb{R}(JX - Y) \subset \mathfrak{m}'$ (see Theorem 3.15(b)) and therefore $\mathfrak{m}'_{\lambda_4} = \mathfrak{m}_{\lambda_4} \cap \mathfrak{m}' = \mathfrak{m}_{\lambda_4}$ (see Proposition 4.6(b)), whence $\lambda_4 \in \Delta'_+$ follows. But this is a contradiction to the hypothesis $\Delta'_+ = \{\lambda_1, \lambda_2\}$ defining the present case.

Therefore $\mathfrak{m}'$ is of type $(\mathrm{G2}, 1+n'_{\lambda_2}, 1+n'_{\lambda_1})$ with the present choice of W_1 and W_2.

The case $\Delta'_+ = \{\lambda_3, \lambda_4\}$. For $k \in \{3,4\}$ we have $\dim \mathfrak{m}_{\lambda_k} = 1$, and therefore the same argument as in the treatment of the case $\Delta'_+ = \{\lambda_3\}$ shows that $\mathfrak{m}'_{\lambda_k} = \mathfrak{m}_{\lambda_k}$ holds. Thus we have by Equation (4.29)

$$\begin{aligned}\mathfrak{m}' &= \mathfrak{a} \oplus \mathfrak{m}'_{\lambda_3} \oplus \mathfrak{m}'_{\lambda_4} = (\mathbb{R}X \oplus \mathbb{R}JY) \oplus \mathbb{R}(JX+Y) \oplus \mathbb{R}(JX-Y)\\ &= \mathbb{R}X \oplus \mathbb{R}JY \oplus \mathbb{R}JX \oplus \mathbb{R}Y = \mathbb{C}X \oplus \mathbb{C}Y\,.\end{aligned}$$

Thus we have $\mathfrak{m}' = W \oplus JW$ with $W := \mathbb{R}X \oplus \mathbb{R}Y \subset V(A)$. Therefore $\mathfrak{m}'$ is a 2-dimensional $\mathbb{C}$Q-subspace and hence of type $(\mathrm{G1},2)$.

The case $\Delta'_+ = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. By Equation (4.29) we have

$$\mathfrak{m}' = \mathfrak{a} \oplus \mathfrak{m}'_{\lambda_1} \oplus \mathfrak{m}'_{\lambda_2} \oplus \mathfrak{m}'_{\lambda_3} \oplus \mathfrak{m}'_{\lambda_4} \,, \tag{4.31}$$

and by an analogous argument as for the case $\Delta'_+ = \{\lambda_3, \lambda_4\}$, we see that

$$\mathfrak{m}' \overset{(4.31)}{\supset} \mathfrak{a} \oplus \mathfrak{m}'_{\lambda_3} \oplus \mathfrak{m}'_{\lambda_4} = \mathbb{C}X \oplus \mathbb{C}Y \tag{4.32}$$

holds. In particular we have $X, JX \in \mathfrak{m}'$, whence it follows by Corollary 2.48 that $\mathfrak{m}'$ is a complex-linear subspace of $\mathfrak{m}$. Therefrom $\mathfrak{m}'_{\lambda_1} = J(\mathfrak{m}'_{\lambda_2})$ follows, and thus we obtain from Equations (4.31) and (4.32):

$$\mathfrak{m}' = \mathbb{C}X \oplus \mathbb{C}Y \oplus J(\mathfrak{m}'_{\lambda_2}) \oplus \mathfrak{m}'_{\lambda_2} = W \oplus JW$$

with $W := \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathfrak{m}'_{\lambda_2} \subset V(A)$. Therefore $\mathfrak{m}'$ is a $(2 + n'_{\lambda_2})$-dimensional $\mathbb{C}$Q-subspace and hence of type $(\mathrm{G1}, 2 + n'_{\lambda_2})$.

This completes the classification of the rank 2 Lie triple systems in $\mathfrak{m}$.

4.4 The classification of the rank 1 Lie triple systems

We continue to use the general notations of the previous section, but now suppose that $\{0\} \neq \mathfrak{m}'$ is a Lie triple system of $\mathfrak{m}$ of rank 1. For $H \in \mathfrak{m} \setminus \{0\}$ we denote by $\varphi(H)$ the $\mathfrak{A}$-angle of H as in Section 2.5.

4.12 **Lemma.** *If* $\dim \mathfrak{m}' \geq 2$ *holds, then all* $Z \in \mathfrak{m}' \setminus \{0\}$ *have one and the same* $\mathfrak{A}$*-angle* $\varphi_0 \in \{0, \arctan(\frac{1}{2}), \frac{\pi}{4}\}$. *In the case* $\varphi_0 = \arctan(\frac{1}{2})$, $\mathfrak{m}'$ *has no elementary roots (see Definition 4.7).*

Proof. The crucial point here is to show

$$\forall Z \in \mathfrak{m}' \setminus \{0\} \;:\; \varphi(Z) \in \{0, \arctan(\tfrac{1}{2}), \tfrac{\pi}{4}\} \,. \tag{4.33}$$

For this, we let $Z \in \mathfrak{m}' \setminus \{0\}$ be given; without loss of generality we may suppose $\|Z\| = 1$. Then we have the canonical decomposition

$$Z = \cos(\varphi(Z)) \cdot X + \sin(\varphi(Z)) \cdot JY \tag{4.34}$$

with suitable $A \in \mathfrak{A}$ and $X, Y \in \mathbb{S}(V(A))$.

Because $\mathfrak{m}'$ is of rank 1, $\mathfrak{a}' := \mathbb{R}Z$ is a Cartan subalgebra of $\mathfrak{m}'$; also $\mathfrak{a} := \mathbb{R}X \oplus \mathbb{R}JY$ is a Cartan subalgebra of $\mathfrak{m}$ such that $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{m}'$ holds. Because of $\dim(\mathfrak{m}') > \mathrm{rk}(\mathfrak{m}')$, we have $\Delta(\mathfrak{m}', \mathfrak{a}') \neq \varnothing$. Now let some $\alpha \in \Delta(\mathfrak{m}', \mathfrak{a}')$ be given.

Let us first suppose that α is elementary. Then there exists one and only one $\lambda \in \Delta(\mathfrak{m}, \mathfrak{a})$ with $\lambda|\mathfrak{a}' = \alpha$, and Corollary 4.10 shows that Z is parallel to $\lambda^\sharp$, hence we have $\varphi(Z) = \varphi(\lambda^\sharp)$.

From the explicit representation of the root vectors $\lambda_k^\sharp$ in Theorem 3.15(b) and Theorem 2.28(a) one easily calculates

$$\varphi(\pm\lambda_1^\sharp) = \varphi(\pm\lambda_2^\sharp) = 0 \quad \text{and} \quad \varphi(\pm\lambda_3^\sharp) = \varphi(\pm\lambda_4^\sharp) = \tfrac{\pi}{4}\ .$$

Because of $\lambda \in \Delta(\mathfrak{m},\mathfrak{a}) = \{\pm\lambda_1,\dots,\pm\lambda_4\}$ we therefrom see that $\varphi(Z) = \varphi(\lambda^\sharp) \in \{0, \frac{\pi}{4}\}$ holds.

Now we suppose that α is composite. Then there exist $\lambda, \mu \in \Delta(\mathfrak{m},\mathfrak{a})$ with

$$\mu|\mathfrak{a}' = \alpha = \lambda|\mathfrak{a}' \tag{4.35}$$

and $\mu \neq \lambda$; we also have $\mu \neq -\lambda$ (because otherwise we would have $\alpha = 0 \notin \Delta(\mathfrak{m}',\mathfrak{a}')$). Therefore there exist $r, s \in \{1,\dots,4\}$ with $r \neq s$ so that $\lambda \in \{\pm\lambda_r\}$, $\mu \in \{\pm\lambda_s\}$ holds (where the λ_k form the positive root system of $\mathfrak{m}$ described in Theorem 3.15(b)), and thus we have for every $t \in \mathbb{R}$

$$\lambda(\cos(t)X + \sin(t)JY)^2 = \varkappa_r(t) \quad \text{and} \quad \mu(\cos(t)X + \sin(t)JY)^2 = \varkappa_s(t)$$

(where the $\varkappa_k$ are the eigenfunctions of the Jacobi operator as in Theorem 2.49; compare Equations (3.24) and (3.23) in the proof of Theorem 3.15). By plugging $t = \varphi(Z)$ in these equations, we obtain via Equation (4.34)

$$\lambda(Z)^2 = \varkappa_r(\varphi(Z)) \quad \text{and} \quad \mu(Z)^2 = \varkappa_s(\varphi(Z)) \tag{4.36}$$

and therefore

$$\varkappa_r(\varphi(Z)) \overset{(4.36)}{=} \lambda(Z)^2 \overset{(4.35)}{=} \mu(Z)^2 \overset{(4.36)}{=} \varkappa_s(\varphi(Z))\ .$$

The diagram of the graphs of the functions $\varkappa_k$ in Theorem 2.49 thus shows that $\varphi(Z) \in \{0, \arctan(\frac{1}{2}), \frac{\pi}{4}\}$ holds.

This completes the proof of (4.33). We also saw that if $Z \in \mathfrak{m}' \setminus \{0\}$ is such that $\Delta(\mathfrak{m}', \mathbb{R}Z)$ contains an elementary root, then $\varphi(Z) \neq \arctan(\frac{1}{2})$ holds.

Equation (4.33) shows that the function $\mathfrak{m}' \setminus \{0\} \to \mathbb{R},\ Z \mapsto \varphi(Z)$ attains only discrete values; because this function is continuous by Proposition 2.30, it follows that it is constant. It also follows from (4.33) that the constant value φ_0 of that function is a member of $\{0, \arctan(\frac{1}{2}), \frac{\pi}{4}\}$. Finally, in the case $\varphi_0 = \arctan(\frac{1}{2})$ we have for any Cartan subalgebra $\mathfrak{a}'$ of $\mathfrak{m}'$, say $\mathfrak{a}' = \mathbb{R}Z$ with some $Z \in \mathbb{S}(\mathfrak{m}')$, $\varphi(Z) = \arctan(\frac{1}{2})$ and therefore $\Delta(\mathfrak{m}',\mathfrak{a}')$ does not contain any elementary roots. □

We now classify the Lie triple systems $\mathfrak{m}'$ of rank 1 in $\mathfrak{m}$. For this purpose we fix $Z \in \mathbb{S}(\mathfrak{m}')$ and use the notation concerning the Cartan algebras $\mathfrak{a}' = \mathbb{R}Z$ and $\mathfrak{a}$ introduced at the beginning of the proof of Lemma 4.12. In particular we have the canonical decomposition of Z given in Equation (4.34). We abbreviate $\Delta' := \Delta(\mathfrak{m}',\mathfrak{a}')$ and $\Delta := \Delta(\mathfrak{m},\mathfrak{a})$ and fix a system of positive roots Δ'_+ in Δ'. Then we have by Proposition 4.6(a)

$$\Delta' \subset \{\, \lambda|\mathfrak{a}' \mid \lambda \in \Delta,\ \lambda(Z) \neq 0 \,\} \tag{4.37}$$

and
$$\mathfrak{m}' = \mathbb{R}Z \oplus \bigoplus_{\alpha \in \Delta'_+} \mathfrak{m}'_\alpha \tag{4.38}$$
with
$$\forall \alpha \in \Delta'_+ \; : \; \mathfrak{m}'_\alpha \; = \; \left(\bigoplus_{\substack{\lambda \in \Delta \\ \lambda(Z) = \alpha(Z)}} \mathfrak{m}_\lambda \right) \cap \mathfrak{m}' \; . \tag{4.39}$$

In the case $\dim \mathfrak{m}' = 1$, $\mathfrak{m}' = \mathbb{R}Z$ is of type $(\mathrm{Geo}, \varphi(Z))$. Thus we suppose in the sequel that $\dim \mathfrak{m}' \geq 2$ holds. Then Equation (4.38) shows that we have
$$\Delta' \neq \varnothing \; , \tag{4.40}$$
and on $\mathfrak{m}' \setminus \{0\}$ the $\mathfrak{A}$-angle function φ is equal to some constant $\varphi_0 \in \{0, \arctan(\frac{1}{2}), \frac{\pi}{4}\}$ by Lemma 4.12. To complete the classification, we now treat the three possible values for φ_0 individually.

The case $\varphi_0 = 0$. Then we have $Z = X$ by Equation (4.34). By Theorem 3.15(b) we have
$$\lambda_1(X) = 0 \quad \text{and} \quad \lambda_2(X) = \lambda_3(X) = \lambda_4(X) = \sqrt{2} \; ;$$
therefrom we conclude by (4.37) and (4.40)
$$\Delta' = \{\pm\alpha\} \quad \text{with} \quad \alpha(tZ) = \sqrt{2} \cdot t \text{ for } t \in \mathbb{R}$$
and by (4.38) and (4.39)
$$\mathfrak{m}' = \mathbb{R}X \oplus \mathfrak{m}'_\alpha \quad \text{with} \quad \{0\} \neq \mathfrak{m}'_\alpha \subset \mathfrak{m}_{\lambda_2} \oplus \mathfrak{m}_{\lambda_3} \oplus \mathfrak{m}_{\lambda_4} \; . \tag{4.41}$$

Immediately, we will show that
$$\text{either} \quad \mathfrak{m}'_\alpha \subset (\mathbb{R}X)^{\perp, V(A)} \quad \text{or} \quad \mathfrak{m}'_\alpha = \mathbb{R} \cdot JX \tag{4.42}$$
holds. Then we conclude: In the case $\mathfrak{m}'_\alpha \subset (\mathbb{R}X)^{\perp,V(A)}$ we have $\mathfrak{m}' = \mathfrak{a}' \oplus \mathfrak{m}'_\alpha \subset V(A)$, therefore $\mathfrak{m}'$ is of type $(\mathrm{P1}, 1 + \dim \mathfrak{m}'_\alpha)$. On the other hand, in the case $\mathfrak{m}'_\alpha = \mathbb{R} \cdot JX$ we have $\mathfrak{m}' = \mathfrak{a}' \oplus \mathfrak{m}'_\alpha = \mathbb{C}X$, therefore $\mathfrak{m}'$ is of type $(\mathrm{P2})$.

We now prove (4.42): Let $H \in \mathfrak{m}'_\alpha$ be given. Then we have by (4.41) and Theorem 3.15(b)
$$H \in \mathfrak{m}_{\lambda_2} \oplus \mathfrak{m}_{\lambda_3} \oplus \mathfrak{m}_{\lambda_4} = \mathbb{R} \cdot JX \oplus (\mathbb{R}X)^{\perp, V(A)}$$
and therefore there exist $t \in \mathbb{R}$ and $X' \in V(A)$ with $X' \perp X$ so that $H = t \cdot JX + X'$ holds. Via Proposition 2.47 we calculate (with the functions ρ and C defined there)
$$\rho(X, H) = -2t \quad \text{and} \quad C(X, H) = 2\, X \wedge X'$$
and therefore
$$\widetilde{H} := \tfrac{1}{2} R(X, H)H = (\|X'\|^2 + t^2) \cdot X - 2t \cdot JX' \; . \tag{4.43}$$

Because $\mathfrak{m}'$ is curvature-invariant, we have $\widetilde{H} \in \mathfrak{m}'$. As $\mathfrak{m}'$ is orthogonal to $\mathbb{R}JY \oplus \mathfrak{m}_{\lambda_1} = (\mathbb{R}JX)^{\perp,JV(A)}$ by Equation (4.41) and hence in particular to JX', we therefore have

$$0 = \langle \widetilde{H}, JX' \rangle_{\mathbb{R}} = (-2t) \cdot \langle JX', JX' \rangle_{\mathbb{R}} = (-2t) \cdot \|X'\|^2 \ .$$

Therefore we have either $t = 0$, implying $H = X' \in (\mathbb{R}X)^{\perp,V(A)}$; or else $\|X'\| = 0$, implying $H = t \cdot JX \in \mathbb{R}JX$. Thus, we have shown

$$\mathfrak{m}'_\alpha \subset (\mathbb{R}X)^{\perp,V(A)} \ \cup \ \mathbb{R} \cdot JX \ .$$

Because $\mathfrak{m}'_\alpha$ is a linear space, we in fact have

$$\text{either} \quad \mathfrak{m}'_\alpha \subset (\mathbb{R}X)^{\perp,V(A)} \quad \text{or} \quad \mathfrak{m}'_\alpha \subset \mathbb{R} \cdot JX \ ;$$

if the second case holds, then we actually have $\mathfrak{m}'_\alpha = \mathbb{R} \cdot JX$ because of $\mathfrak{m}'_\alpha \neq \{0\}$. Thus (4.42) is shown.

The case $\varphi_0 = \arctan(\frac{1}{2})$. By Equation (4.34) we then have

$$Z = \tfrac{2}{\sqrt{5}} X + \tfrac{1}{\sqrt{5}} JY \ , \tag{4.44}$$

and from Theorem 3.15(b) we thus obtain

$$\lambda_1(Z) = \tfrac{\sqrt{2}}{\sqrt{5}} \ , \quad \lambda_2(Z) = 2\,\tfrac{\sqrt{2}}{\sqrt{5}} \ , \quad \lambda_3(Z) = \tfrac{\sqrt{2}}{\sqrt{5}} \quad \text{and} \quad \lambda_4(Z) = 3\,\tfrac{\sqrt{2}}{\sqrt{5}} \ . \tag{4.45}$$

Because of $\varphi_0 = \arctan(\frac{1}{2})$ Lemma 4.12 shows that there do not exist any elementary roots in Δ'; therefore we conclude from Equations (4.45) by (4.37) and (4.40)

$$\Delta' = \{\pm\alpha\} \quad \text{with} \quad \alpha(tZ) = \tfrac{\sqrt{2}}{\sqrt{5}} \cdot t \text{ for } t \in \mathbb{R}$$

and by (4.38) and (4.39)

$$\mathfrak{m}' = \mathbb{R}Z \oplus \mathfrak{m}'_\alpha \quad \text{with} \quad \{0\} \neq \mathfrak{m}'_\alpha \subset \mathfrak{m}_{\lambda_1} \oplus \mathfrak{m}_{\lambda_3} \ . \tag{4.46}$$

We now show

$$\forall\, H \in \mathbb{S}(\mathfrak{m}'_\alpha)\ \exists\, U \in \mathbb{S}(V(A)) \ : \big(\ H = \pm\tfrac{1}{\sqrt{5}}(Y + JX + \sqrt{3}JU) \quad \text{and} \quad U \perp X, Y\ \big) \ . \tag{4.47}$$

Let $H \in \mathbb{S}(\mathfrak{m}'_\alpha)$ be given. Then we have by (4.46) and Theorem 3.15(b)

$$H \in \mathfrak{m}_{\lambda_1} \oplus \mathfrak{m}_{\lambda_3} = J(\mathbb{R}X \oplus \mathbb{R}Y)^{\perp,V(A)} \oplus \mathbb{R}(JX + Y) \ .$$

Consequently there exist $U' \in V(A)$ with $U' \perp X, Y$ and $t \in \mathbb{R}$ so that

$$H = JU' + t \cdot (JX + Y) = tY + J(U' + tX) \tag{4.48}$$

and therefore

$$\operatorname{Re}_A H = tY \quad \text{and} \quad \operatorname{Im}_A H = U' + tX \ , \tag{4.49}$$

hence

$$\|\operatorname{Re}_A H\|^2 = t^2 \quad \text{and} \quad \|\operatorname{Im}_A H\|^2 = \|U'\|^2 + t^2 \tag{4.50}$$

holds. Equations (4.49) show that $\mathrm{Re}_A H$ is orthogonal to $\mathrm{Im}_A H$, and therefore either A or $-A$ is adapted to H by Proposition 2.32(a).

In fact $-A$ is adapted to H: If A were adapted to H, then we would have by Theorem 2.28(c)

$$\|\,\mathrm{Re}_A H\|^2 = (\cos\varphi(H))^2 = (\cos\varphi_0)^2 = \tfrac{4}{5} \quad \text{and} \quad \|\,\mathrm{Im}_A H\|^2 = (\sin\varphi(H))^2 = (\sin\varphi_0)^2 = \tfrac{1}{5}$$

and thus $\|\,\mathrm{Re}_A H\|^2 = 4\,\|\,\mathrm{Im}_A H\|^2$. This equation implies via Equations (4.50) $-3t^2 = 4\|U'\|^2$ and therefore $t = \|U'\| = 0$. Because of Equation (4.48) $H = 0$ follows, which is a contradiction.

Because $-A$ is adapted to H, we have by Theorem 2.28(c)

$$\|\,\mathrm{Re}_{-A} H\|^2 = (\cos\varphi(H))^2 = (\cos\varphi_0)^2 = \tfrac{4}{5} \quad \text{and} \quad \|\,\mathrm{Im}_{-A} H\|^2 = (\sin\varphi(H))^2 = (\sin\varphi_0)^2 = \tfrac{1}{5}$$

By Proposition 2.3(e),(g) we have $\mathrm{Re}_A H = \mathrm{Im}_A(JH) = J\,\mathrm{Im}_{-A}(H)$ and $\mathrm{Im}_A H = -\,\mathrm{Re}_A(JH) = -J\,\mathrm{Im}_{-A}(H)$, and therefore it follows

$$\|\,\mathrm{Re}_A H\|^2 = \tfrac{1}{5} \quad \text{and} \quad \|\,\mathrm{Im}_A H\|^2 = \tfrac{4}{5}\,.$$

From Equations (4.50) we thus obtain $t^2 = \frac{1}{5}$ and $\|U'\|^2 + t^2 = \frac{4}{5}$, and hence there exists $\varepsilon \in \{\pm 1\}$ so that

$$t = \varepsilon\,\tfrac{1}{\sqrt{5}} \quad \text{and} \quad \|U'\| = \sqrt{\tfrac{3}{5}}$$

holds. Consequently, we have $U := \varepsilon\,\sqrt{5/3}\cdot U' \in \mathbb{S}(V(A))$. Equation (4.48) shows that we have $H = \varepsilon\,\frac{1}{\sqrt{5}}(Y + JX + \sqrt{3}\,JU)$, and therefore (4.47) is satisfied with this choice of U.

Next we prove $\dim \mathfrak{m}'_\alpha = 1$: Let $H_1, H_2 \in \mathbb{S}(\mathfrak{m}'_\alpha)$ be given; we will show $H_2 = \pm H_1$. By (4.47), there exist $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ and $U_1, U_2 \in \mathbb{S}(V(A))$ so that

$$H_k = \tfrac{\varepsilon_k}{\sqrt{5}} \cdot (Y + JX + \sqrt{3}JU_k)$$

holds for $k \in \{1,2\}$. Under the assumption $H_2 \neq \pm H_1$ we could suppose without loss of generality that $\varepsilon_1 = \varepsilon_2 = 1$ holds, and then $H_1 - H_2 = \sqrt{3/5}\cdot J(U_1 - U_2)$ would be a non-zero $\mathfrak{A}$-principal vector contained in $\mathfrak{m}'_\alpha \subset \mathfrak{m}'$. But this is a contradiction to $\forall H \in \mathfrak{m}' \setminus \{0\} : \varphi(H) = \arctan(\frac{1}{2})$.

Thus, $\mathfrak{m}'_\alpha$ is 1-dimensional, and therefore we have $\mathfrak{m}' = \mathfrak{a}' \oplus \mathfrak{m}'_\alpha = \mathbb{R}Z \oplus \mathbb{R}H$ with any $H \in \mathbb{S}(\mathfrak{m}'_\alpha)$. Equations (4.44) and (4.47) therefore show that $\mathfrak{m}'$ is a space of type (A).

The case $\varphi_0 = \frac{\pi}{4}$. $\mathfrak{m}'$ is an $\mathfrak{A}$-isotropic subspace of $\mathfrak{m}$ (see Proposition 2.29(b)); therefore the “complex closure” $\widehat{\mathfrak{m}}' := \mathfrak{m}' + J\mathfrak{m}' \subset \mathfrak{m}$ of $\mathfrak{m}'$ also is an $\mathfrak{A}$-isotropic subspace by Proposition 2.20(d), and hence a curvature-invariant subspace of $\mathfrak{m}$ of type $(\mathrm{I}1,k)$ with $k := \dim_{\mathbb{C}} \widehat{\mathfrak{m}}'$. Proposition 2.43(d) shows that the restriction of the curvature tensor of the $\mathbb{C}$Q-space $\mathfrak{m}$ to $\widehat{\mathfrak{m}}'$ is the curvature tensor of a complex projective space of constant holomorphic sectional curvature 4.

If $\mathfrak{m}'$ is a complex subspace of $\mathbb{V}$, we have $\mathfrak{m}' = \widehat{\mathfrak{m}}'$; therefore $\mathfrak{m}'$ then is of type $(\mathrm{I}1,k)$. Otherwise, $\mathfrak{m}'$ is a curvature-invariant subspace of $\widehat{\mathfrak{m}}'$; by the well-known classification of totally geodesic submanifolds in a complex projective space, it follows that $\mathfrak{m}'$ is a totally real subspace of $\widehat{\mathfrak{m}}'$, and therefore a k-dimensional totally real, isotropic subspace of $\mathbb{V}$. Consequently, $\mathfrak{m}'$ is of type $(\mathrm{I}2, k)$.

This completes the proof of Theorem 4.2. □

4.13 Remark. Chen and Nagano gave in their paper [CN77] (1977) a classification of the totally geodesic submanifolds of the complex quadric using a different approach. We briefly describe their strategy. They study the complex quadric Q^m in two different ways: On the one hand, they investigate the oriented real Grassmannian $G_2^+(\mathbb{R}^{m+2})$ (which is homothetic to Q^m, as we noted in Remark 2.24) as a submanifold of $\bigwedge^2\mathbb{R}^{m+2}$; it should be noted that $\bigwedge^2\mathbb{R}^{m+2}$ can be canonically identified with $\mathfrak{aut}_s(\mathbb{C}^{m+2}) \cong \mathfrak{o}(m+2)$. On the other hand, they regard Q^m as a Riemannian symmetric space isomorphic to $\mathrm{SO}(m+2)/(\mathrm{SO}(2)\times\mathrm{SO}(m))$ (see Remark 3.10(a)). Now they make the following approach: If M is a connected, complete, totally geodesic submanifold of Q, then M can be regarded as a symmetric subspace G'/K' of Q, where G' is a subgroup of $\mathrm{SO}(m+2)$ (see [KN69], Theorem XI.4.2, p. 235). In the usual way, the symmetric structure of M gives rise to a splitting $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{m}'$ of the Lie algebra of G, we have $\mathfrak{k}' \subset \mathfrak{o}(2)\oplus\mathfrak{o}(m)$, $\mathfrak{m}'$ is a Lie triple system canonically isomorphic to the tangent space of M and $\mathfrak{k}'$ acts on $\mathfrak{m}'$ by the adjoint representation. Chen/Nagano now distinguish three cases: (1) $\mathfrak{k}'$ acts irreducibly on $\mathfrak{m}'$ and $\mathfrak{k}' \subset \mathfrak{o}(m)$ holds (Lemmata 3.1–3.3), (2) $\mathfrak{k}'$ acts irreducibly on $\mathfrak{m}'$ and $\mathfrak{k}' \not\subset \mathfrak{o}(m)$ holds (Lemma 3.4), (3) $\mathfrak{k}'$ acts reducibly on $\mathfrak{m}'$ (Lemma 3.5). In the treatment of these cases, $\mathfrak{k}'$ and $\mathfrak{m}'$ are regarded as subsystems of $\mathfrak{o}(m+2) \cong \bigwedge^2\mathbb{R}^{m+2}$; explicit calculations of the Lie bracket of such elements play an important role.

Some of the arguments of [CN77] appear to be faulty, mainly in the proof of case (2) as described above; because case (3) is treated by reduction to the cases (1) and (2), this gap also concerns case (3). The fact that the spaces of our type (A) are missing from the paper seems to stem from an oversight in the proof of its Lemma 3.4. Also, there is an unfounded assumption in the proof of Lemma 3.5, which causes the totally geodesic submanifolds of type (G3) to be missed. Moreover, it is incorrectly stated that the totally geodesic submanifolds corresponding to our case $(\mathrm{I}1, k')$ are neither complex nor totally real submanifolds of Q.

We should also mention the older paper [CL75] by Chen and Lue (1975), where the real-2-dimensional curvature-invariant subspaces of T_pQ are classified. Chen and Lue find eight types of such subspaces, which they denote by I, ..., VIII; the correspondence between their types and the types of curvature-invariant subspaces in our notation is as follows:

Type of [CL75]	I	II	III	IV	V	VI	VII	VIII
Type of Thm. 4.2	$(\mathrm{I}2, 2)$	(A)	(P2)	$(\mathrm{I}1, 1)$	$(\mathrm{P}1, 2)$	$(\mathrm{G}2, 1, 1)$	$(\mathrm{G}2, 1, 1)$	$(\mathrm{G}2, 1, 1)$

Interestingly the spaces of type (A), which are missing from [CN77], can be found here. Also note that our type $(\mathrm{G}2, 1, 1)$ is divided in [CL75] into the three types VI, VII and VIII; this division is necessary because Chen/Lue do not have the concept of a conjugation adapted to a vector (or to a 2-flat) available.

In 1978, Chen and Nagano introduced their famous (M_+, M_-)-method for determining totally geodesic submanifolds of symmetric spaces, see [CN78]. This method is based on the following idea: Suppose M is a symmetric space of compact type and $p \in M$. To every closed geodesic $c : [0, \delta] \to M$ with $c(0) = c(\delta) = p$, it is associated a pair $(M_+(c), M_-(c))$ of totally geodesic submanifolds of M; we denote the set of isometry classes of such pairs by $P(M)$. It can be shown that $P(M)$ is finite. Now, let a totally geodesic embedding $f : B \to M$ of another symmetric space B of compact type be given. Then it can be shown that for every pair (B_+, B_-) of B there exists a pair (M_+, M_-) of M so that $f(B_\pm)$ is a totally geodesic submanifold of $M_\pm$. If one now tabulates $P(M)$ for the finitely many isometry classes of irreducible symmetric spaces M of compact type (this is done in [CN78]), one can use this information to exclude symmetric spaces which cannot occur as totally geodesic submanifolds of M. Frequently, among the finitely many types of symmetric spaces of compact type and rank $\leq \mathrm{rk}(M)$, only a few candidates B for totally geodesic submanifolds of M remain. However, not necessarily all these candidates occur as totally geodesic submanifolds of M. To complete the classification of the totally geodesic submanifolds of M, one therefore must for every candidate B either construct a totally geodesic embedding of B into M explicitly (thereby showing that B indeed occurs as a totally geodesic submanifold of M), or show by other means that B cannot occur as a totally geodesic submanifold of M.

As an application, Chen and Nagano give in [CN78] the maximal totally geodesic submanifolds of the irreducible symmetric spaces of compact type and rank 2. The manifolds of type (G3), which are maximal totally geodesic submanifolds of Q^2, and the manifolds of type (A), which are maximal totally geodesic submanifolds of Q^3, are again missing.

Chapter 5

Totally geodesic submanifolds

In the previous chapter, we classified the curvature-invariant subspaces of the tangent space T_pQ of an m-dimensional complex quadric Q. These subspaces of T_pQ are in bijective correspondence with the connected, complete, totally geodesic submanifolds of Q through p.

We now wish to find out which (connected, complete) totally geodesic submanifolds M_U of Q correspond to the various curvature-invariant subspaces $U \subset T_pQ$. The isometry type of the universal covering manifold $\widetilde{M_U}$ of M_U (and therefore the local isometry type of M_U) is easily determined via the theorem of Cartan/Ambrose/Hicks by computing the restriction of the curvature tensor R of Q to U; in this way one obtains the results of the following table. In this table we denote the universal cover of the sphere $\mathbb{S}^k_r$ (with $k \in \mathbb{N}$ and $r \in \mathbb{R}_+$) by $\widetilde{\mathbb{S}}^k_r$; we have $\widetilde{\mathbb{S}}^k_r = \mathbb{S}^k_r$ for $k \geq 2$ and $\widetilde{\mathbb{S}}^k_r = \mathbb{R}$ for $k = 1$.

type of U	with ...	isometry class of $\widetilde{M_U}$
(Geo, t)	$t \in [0, \frac{\pi}{4}]$	$\mathbb{R}$
$(\mathrm{G1}, k)$	$2 \leq k \leq m-1$	Q^k
$(\mathrm{G2}, k_1, k_2)$	$k_1, k_2 \geq 1,\ k_1 + k_2 \leq m$	$\widetilde{\mathbb{S}}^{k_1}_{1/\sqrt{2}} \times \widetilde{\mathbb{S}}^{k_2}_{1/\sqrt{2}}$
$(\mathrm{G3})$		$\mathbb{C}\mathrm{P}^1 \times \mathbb{R}$
$(\mathrm{P1}, k)$	$1 \leq k \leq m$	$\widetilde{\mathbb{S}}^k_{1/\sqrt{2}}$
$(\mathrm{P2})$		Q^1
(A)		$\mathbb{S}^2_{\sqrt{10}/2}$
$(\mathrm{I1}, k)$	$1 \leq k \leq \frac{m}{2}$	$\mathbb{C}\mathrm{P}^k$
$(\mathrm{I2}, k)$	$1 \leq k \leq \frac{m}{2}$	$\widetilde{\mathbb{S}}^k_1$

However, we want to know more: namely the exact global structure of M_U and how M_U lies in Q. For that we need to construct totally geodesic isometric embeddings of suitable Riemannian manifolds onto M_U explicitly; we will be successful for all types of curvature invariant subspaces U except for the type (A). Thereby we will prove in particular:

5.1 Theorem. *Let $p \in Q$ and $U \subset T_pQ$ be a curvature-invariant subspace. Then the global isometry class of the connected, complete, totally geodesic submanifold M_U of Q with $p \in M_U$ and $T_pM_U = U$ is given in the following table in dependence of the type of U.*

type of U	*with …*	*isometry class of M_U*	*M_U complex or totally real?*
(Geo, t)	$t \in [0, \frac{\pi}{4}],\ \tan(t) \in \mathbb{Q}$	$\mathbb{S}^1_{L/2\pi}$, *see* [10]	*totally real*
(Geo, t)	$t \in [0, \frac{\pi}{4}],\ \tan(t) \in \mathbb{R} \setminus \mathbb{Q}$	$\mathbb{R}$	*totally real*
$(\mathrm{G1}, k)$	$2 \leq k \leq m-1$	Q^k	*complex*
$(\mathrm{G2}, k_1, k_2)$	$k_1, k_2 \geq 1,\ k_1 + k_2 \leq m$	$(\mathbb{S}^{k_1}_{1/\sqrt{2}} \times \mathbb{S}^{k_2}_{1/\sqrt{2}})/\{\pm\mathrm{id}\}$	*totally real*
$(\mathrm{G3})$		$\mathbb{CP}^1 \times \mathbb{RP}^1$	*neither*
$(\mathrm{P1}, k)$	$1 \leq k \leq m$	$\mathbb{S}^k_{1/\sqrt{2}}$	*totally real*
$(\mathrm{P2})$		Q^1	*complex*
(A)		$\mathbb{S}^2_{\sqrt{10}/2}$	*neither*
$(\mathrm{I1}, k)$	$1 \leq k \leq \frac{m}{2}$	$\mathbb{CP}^k$	*complex*
$(\mathrm{I2}, k)$	$1 \leq k \leq \frac{m}{2}$	$\mathbb{RP}^k$	*totally real*

Here $\mathbb{CP}^k$ *is equipped with the Fubini-Study metric of constant holomorphic sectional curvature* 4 *as usual, and* $\mathbb{RP}^k$ *is equipped with a Riemannian metric of constant sectional curvature* 1.

From the construction of the embeddings we will also obtain some further results: In Section 5.4, an investigation of the geodesics of Q will show which of them are closed and what their minimal period is. It follows from this investigation that the diameter of any complex quadric is $\pi/\sqrt{2}$. In Section 5.5 we will obtain two foliations on the 2-dimensional quadric Q^2, one perpendicular to the other, by a natural construction. It will turn out that these foliations correspond via the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to Q^2$ (see Section 3.4) to the canonical foliations of the Riemannian product manifold $\mathbb{P}^1 \times \mathbb{P}^1$. Finally, in Section 5.8 we will derive from Theorem 5.1 the result that any two (connected, complete) totally geodesic submanifolds of real dimension ≥ 3 which are isometric to each other are already holomorphically congruent in Q and therefore of the same type.

As before, we suppose that $(\mathbb{V}, \mathfrak{A})$ is an $(m+2)$-dimensional $\mathbb{C}$Q-space and consider the m-dimensional complex quadric $Q := Q(\mathfrak{A})$. We will see in Section 8.1 that in the case $m = 1$, Q is isometric to a 2-dimensional sphere $\mathbb{S}$, and therefore the totally geodesic submanifolds of Q then correspond to great circles on $\mathbb{S}$. Thus we now suppose $m \geq 2$.

As in Chapter 1, we put $\widetilde{Q} := \widetilde{Q}(\mathfrak{A})$, we denote by $\pi : \mathbb{S}(\mathbb{V}) \to \mathbb{P}(\mathbb{V})$ the Hopf fibration and by $\mathcal{H}_z$ and $\mathcal{H}_z Q$ the horizontal lift at $z \in \mathbb{S}(\mathbb{V})$ of the tangent space $T_{\pi(z)}\mathbb{P}(\mathbb{V})$ resp. $T_{\pi(z)}Q$. As usual, we will take the liberty of denoting by $\langle \cdot, \cdot \rangle$ the inner product resp. the Riemannian metric of any of the euclidean spaces resp. Riemannian manifolds involved in the following constructions; also we will denote by J the complex structure of any unitary space or Hermitian manifold.

5.1 Preparations

As a preparation we establish that we can assign a type (in the sense of Theorem 4.2) not only to curvature-invariant subspaces of T_pQ, but also to the corresponding totally geodesic submanifolds:

[10] Here L is the minimal period of the geodesic $\gamma_v : \mathbb{R} \to Q$ with $\gamma_v(0) = p$, $\dot{\gamma}_v(0) = v$, see Proposition 5.18.

5.2 Proposition. *Let M be a connected, complete, totally geodesic submanifold of Q. Then all curvature-invariant subspaces T_pM (where $p \in M$) are of one and the same type in the sense of Theorem 4.2.*

In the sequel, we assign this type also to the totally geodesic submanifold M.

Proof. M is in particular a homogeneous subspace of the symmetric $\mathrm{Aut}_s(\mathfrak{A})_0$-space Q (see [KN69], Theorem XI.4.2, p. 235), meaning that there exists a Lie subgroup G of $\mathrm{Aut}_s(\mathfrak{A})_0$ whose elements leave M invariant and which acts transitively on M. Hence there exists for any given $p_1, p_2 \in M$ some $B \in G \subset \mathrm{Aut}(\mathfrak{A})$ with $\underline{B}(M) = M$ and $p_2 = \underline{B}(p_1)$, and therefore also $T_{p_2}M = \underline{B}_* T_{p_1}M$. Because $\underline{B}_*|T_{p_1}Q : T_{p_1}Q \to T_{p_2}Q$ is a $\mathbb{C}$Q-isomorphism, it follows that $T_{p_1}M$ and $T_{p_2}M$ are of the same type, see Theorem 4.2. $\square$

5.3 Corollary. *The subbundles $\{\, v \in \mathbb{S}(TM) \,|\, \varphi(v) = \varphi_0 \,\}$ (with $\varphi_0 \in [0, \frac{\pi}{4}]$) of the unit sphere bundle $\mathbb{S}(TM)$ are invariant under the geodesic flow of Q. This means more explicitly: For every non-stationary geodesic $\gamma : \mathbb{R} \to Q$ and every $t \in \mathbb{R}$, we have $\varphi(\dot\gamma(t)) = \varphi(\dot\gamma(0))$.*

Proof. Let $\gamma : \mathbb{R} \to Q$ be a non-stationary geodesic, then $M := \gamma(\mathbb{R})$ is a real-1-dimensional totally geodesic submanifold of Q, and for each $t \in \mathbb{R}$, the curvature-invariant subspace $T_{\gamma(t)}M$ is of type $(\mathrm{Geo}, \varphi(\dot\gamma(t)))$. Therefore the statement follows from Proposition 5.2. $\square$

We now show some very simple lemmas which will be of general use in the following constructions.

5.4 Lemma. *Let M be a Kähler manifold and N be a connected, totally geodesic submanifold of M. If there exists $p_0 \in N$ so that $T_{p_0}N$ is a totally real subspace of $T_{p_0}M$, then N already is a totally real submanifold of M (meaning that T_pN is totally real in T_pM for every $p \in N$).*

Proof. We denote the complex structure of M by J. Let $p \in N$ and $v, w \in T_pN$ be given. We have to show $\langle v, Jw \rangle = 0$.

N being connected, there exists a curve $\gamma : [0,1] \to N$ with $\gamma(0) = p_0$ and $\gamma(1) = p$. Moreover, there exist vector fields $X, Y \in \mathfrak{X}_\gamma(N)$ which are parallel with respect to the covariant derivative of N with $X_1 = v$ and $Y_1 = w$. We have $X_0, Y_0 \in T_{p_0}N$, and therefore by the hypothesis

$$\langle X_0, JY_0 \rangle = 0 \,. \tag{5.1}$$

Because N is a totally geodesic submanifold of M, X and Y are also parallel with respect to the covariant derivative of M; because the endomorphism field J of M is parallel, it follows that $J \circ Y$ is another parallel field of M. Because also the Riemannian metric $\langle \cdot, \cdot \rangle$ of M is parallel, it follows that the function $t \mapsto \langle X_t, JY_t \rangle$ is constant. We therefore conclude from Equation (5.1)

$$\langle v, Jw \rangle = \langle X_1, JY_1 \rangle = \langle X_0, JY_0 \rangle = 0 \,.$$

$\square$

5.5 Lemma. *Let $\overline{M}$ be a Riemannian manifold, M a (quasi-)regular submanifold of $\overline{M}$ and N a totally geodesic submanifold of $\overline{M}$. If $N \subset M$ holds, then N is also a totally geodesic submanifold of M.*

Proof. N is a submanifold of M because M is a (quasi-)regular submanifold of $\overline{M}$. We denote by $h^{N\hookrightarrow M}$, $h^{M\hookrightarrow \overline{M}}$ and $h^{N\hookrightarrow \overline{M}}$ the second fundamental forms of the respective inclusion maps. Because N is a totally geodesic submanifold of $\overline{M}$, we have for any $p \in N$ and $v, w \in T_pN$

$$0 = h^{N\hookrightarrow \overline{M}}(v,w) = \underbrace{h^{N\hookrightarrow M}(v,w)}_{\in T_pM} + \underbrace{h^{M\hookrightarrow \overline{M}}(v,w)}_{\perp_p(M\hookrightarrow \overline{M})},$$

whence $h^{N\hookrightarrow M}(v,w) = 0$ follows. This shows that N is a totally geodesic submanifold of M. Moreover, we see that $h^{M\hookrightarrow \overline{M}}$ vanishes on $T_pN \times T_pN$. □

5.6 Lemma. *Let N, $\widetilde{M}$ and M be Riemannian manifolds, $\pi : \widetilde{M} \to M$ a Riemannian submersion and $\widetilde{f} : N \to \widetilde{M}$ a horizontal[11] isometric immersion; we also consider the map $f := \pi \circ \widetilde{f} : N \to M$. In this situation we have:*

(a) f also is an isometric immersion. If we denote the second fundamental forms of the isometric immersions $\widetilde{f}$ and f by $\widetilde{h}$ and h respectively, we have $h = \pi_* \circ \widetilde{h}$.

(b) If $\widetilde{f}$ is totally geodesic, then f also is totally geodesic.

Proof. For (a). Because $\pi_*|\mathcal{H}_p : \mathcal{H}_p \to T_{\pi(p)}M$ is a linear isometry for every $p \in \widetilde{M}$ (where $\mathcal{H}_p = (\ker T_p\pi)^\perp$ denotes the horizontal space of the Riemannian submersion π at p), we see that f is an isometric immersion.

Now let vector fields $X, Y \in \mathfrak{X}(N)$ be given. Denoting by ∇^N, $\nabla^{\widetilde{M}}$ and ∇^M the Levi-Civita covariant derivatives of the respective Riemannian manifolds, we then have

$$\begin{aligned} f_*\nabla^N_X Y + h(X,Y) &\overset{(*)}{=} \nabla^M_X f_* Y = \nabla^M_X \pi_* \widetilde{f}_* Y \overset{(\dagger)}{=} \pi_* \nabla^{\widetilde{M}}_X \widetilde{f}_* Y \\ &\overset{(*)}{=} \pi_* \left(\widetilde{f}_* \nabla^N_X Y + \widetilde{h}(X,Y) \right) \\ &= f_* \nabla^N_X Y + \pi_* \widetilde{h}(X,Y) ; \end{aligned}$$

here the equals signs marked $(*)$ follow from the Gauss equation, and the equals sign marked $(\dagger)$ is a consequence of the fact that $\widetilde{f}$ is horizontal, see [O'N83], Lemma 7.45(3), p. 212. Thus we have shown

$$h(X,Y) = \pi_* \widetilde{h}(X,Y) .$$

For (b). If $\widetilde{f}$ is totally geodesic, we have $\widetilde{h} = 0$, wherefrom $h = 0$ follows by (a). Therefore f is then also totally geodesic. □

[11] The attribute "horizontal" here means that $\widetilde{f}_* T_q N$ is a subspace of the horizontal space $(\ker T_{\widetilde{f}(q)}\pi)^\perp$ for every $q \in N$.

5.7 Remark. In the situation of Lemma 5.6, in fact more can be said about the relation between $\widetilde{f}$ and f. In particular, it can be shown that $\widetilde{h}$ takes its values in the π-horizontal subbundle of $T\widetilde{M}$, and therefore $\widetilde{f}$ is actually totally geodesic if and only if f is totally geodesic; see [Rec85], Theorem 1 and Corollary 1, p. 266f.

5.2 Types (G1, k) and (P2)

5.8 Lemma. *Let $\widetilde{U}$ be a $(k+2)$-dimensional $\mathbb{C}$Q-subspace of $\mathbb{V}$, $1 \leq k < m$. Then $Q' := Q \cap [\widetilde{U}]$ is a k-dimensional complex quadric in $\mathbb{P}(\widetilde{U}) = [\widetilde{U}] \subset \mathbb{P}(\mathbb{V})$, and a totally geodesic, connected, compact Hermitian submanifold of Q. For any $p \in Q'$ and $z \in \pi^{-1}(\{p\})$ we have $T_pQ' = \pi_*(\mathcal{H}_zQ \cap T_z\widetilde{U})$; this curvature-invariant subspace of T_pQ is of type* (G1, k) *for $k \geq 2$ resp. of type* (P2) *for $k = 1$.*

5.9 Example. Let us denote by Q^k and Q^m the standard complex quadrics of the respective dimensions, see Example 1.1. The k-dimensional complex quadric

$$\{ [z_1, \dots, z_{k+2}, 0, \dots, 0] \in Q^m \mid [z_1, \dots, z_{k+2}] \in Q^k \}$$

is a totally geodesic submanifold of Q^m.

Proof of Lemma 5.8. Let $\mathfrak{A}_{\widetilde{U}}$ be the induced $\mathbb{C}$Q-structure of $\widetilde{U}$. Then we have $Q' = Q(\mathfrak{A}_{\widetilde{U}})$ and therefore Q' is a k-dimensional complex quadric in $\mathbb{P}(\widetilde{U})$. It follows from results of Chapters 1 and 3 that the quadric Q' (with its intrinsic Hermitian structure) is a connected, compact manifold.

Since Q' is a Hermitian submanifold of $\mathbb{P}(\widetilde{U})$, and $\widetilde{U}$ is a complex linear subspace of $\mathbb{V}$ and therefore $\mathbb{P}(\widetilde{U}) = [\widetilde{U}]$ a Hermitian submanifold of $\mathbb{P}(\mathbb{V})$, we see that Q' is a Hermitian submanifold of $\mathbb{P}(\mathbb{V})$. Because Q' is moreover contained in the Hermitian submanifold Q of $\mathbb{P}(\mathbb{V})$, we see that Q' is a Hermitian submanifold of Q.

To show that Q' is a totally geodesic submanifold of Q we use the well-known theorem that the connected components of the fixed point set of an isometry on a Riemannian manifold are regular, totally geodesic submanifolds (see [Kob72], Theorem II.5.1, p. 59).

Let $B : \mathbb{V} \to \mathbb{V}$ be the $\mathbb{C}$Q-automorphism characterized by $B|\widetilde{U} = \mathrm{id}_{\widetilde{U}}$ and $B|\widetilde{U}^\perp = -\mathrm{id}_{\widetilde{U}^\perp}$. By Proposition 3.2(a), $g := \underline{B}|Q : Q \to Q$ is a holomorphic isometry. If $p \in Q$ is given, say $p = \pi(z)$ with $z \in \widetilde{Q}$, we decompose z as $z = z_1 + z_2$ with $z_1 \in \widetilde{U}$ and $z_2 \in \widetilde{U}^\perp$. We also fix $A \in \mathfrak{A}$, then we have

$$\begin{aligned} g(p) = p &\iff \exists\, \lambda \in \mathbb{S}^1 \ :\ Bz = \lambda z \\ &\iff \exists\, \lambda \in \mathbb{S}^1 \ :\ z_1 - z_2 = \lambda(z_1 + z_2) \\ &\iff (\ z_2 = 0 \quad \text{or} \quad z_1 = 0\ ) \\ &\iff (\ z \in \widetilde{U} \cap \widetilde{Q} \quad \text{or} \quad z \in \widetilde{U}^\perp \cap \widetilde{Q}\ ) \\ &\iff (\ p \in Q(A|\widetilde{U}) \quad \text{or} \quad p \in Q(A|\widetilde{U}^\perp)\ )\ . \end{aligned}$$

This shows that the connected components of $\mathrm{Fix}(g)$ are exactly the disjoint subsets $Q(A|\widetilde{U}) = Q'$ and $Q(A|\widetilde{U}^\perp)$. Therefore Q' is a totally geodesic submanifold of Q.

Now let $p \in Q'$ and $z \in \pi^{-1}(\{p\})$ be given. Then we have

$$\begin{aligned} T_pQ' &= \{\, v \in T_pQ \,|\, g_* v = v \,\} \\ &= \pi_* \{\, w \in \mathcal{H}_z Q \,|\, B\overrightarrow{w} = \overrightarrow{w} \,\} \\ &= \pi_* \{\, w \in \mathcal{H}_z Q \,|\, \overrightarrow{w} \in \widetilde{U} \,\} = \pi_*(\mathcal{H}_z Q \cap T_z\widetilde{U}) \; . \end{aligned} \tag{5.2}$$

$T_z\widetilde{U}$ and $\mathcal{H}_zQ$ are $\mathbb{C}$Q-subspaces of the $\mathbb{C}$Q-space $T_z\mathbb{V}$ (see Theorem 2.26). Therefore $\mathcal{H}_zQ \cap T_z\widetilde{U}$ is a $\mathbb{C}$Q-subspace of the $\mathbb{C}$Q-space $\mathcal{H}_zQ$. Because $\pi_*|\mathcal{H}_zQ : \mathcal{H}_zQ \to T_pQ$ is a $\mathbb{C}$Q-isomorphism, we thus see from Equation (5.2) that T_pQ' is a $\mathbb{C}$Q-subspace of T_pQ, and hence a curvature-invariant subspace of type $(\mathrm{G1}, k)$ resp. $(\mathrm{P2})$. □

5.10 Proposition. *Let $p \in Q$ and a curvature-invariant subspace $U \subset T_pQ$ of type $(\mathrm{G1}, k)$ or of type $(\mathrm{P2})$ be given; in the latter case, we put $k := 1$.*

Fix $z \in \pi^{-1}(\{p\})$ arbitrarily. Then[12] $\widetilde{U} := \mathrm{span}_{\mathfrak{A}}\{z\} \oplus \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(U)}$ *is a $(k+2)$-dimensional $\mathbb{C}$Q-subspace of $(\mathbb{V}, \mathfrak{A})$. The complex quadric $Q' := Q \cap [\widetilde{U}]$ in $\mathbb{P}(\widetilde{U})$ is a totally geodesic, connected, compact Hermitian submanifold of Q with $p \in Q'$ and $T_pQ' = U$.*

Proof. By Theorems 1.16 and 2.26, $\overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(U)}$ is a k-dimensional $\mathbb{C}$Q-subspace of $\mathbb{V}$ which is orthogonal to $\mathrm{span}_{\mathfrak{A}}\{z\}$. Also, for a fixed $A \in \mathfrak{A}$, we have $\langle z, Az\rangle_{\mathbb{C}} = 0$ and therefore $\mathrm{span}_{\mathfrak{A}}\{z\} = \mathrm{span}_{\mathbb{C}}\{z, Az\}$ is a 2-dimensional $\mathbb{C}$Q-subspace of $\mathbb{V}$. It follows that $\widetilde{U}$ is a $(k+2)$-dimensional $\mathbb{C}$Q-subspace of $\mathbb{V}$.

Lemma 5.8 now shows that Q' is a k-dimensional complex quadric, and a totally geodesic, connected, compact Hermitian submanifold of Q. We have $p \in Q'$. Also, we have $\overrightarrow{\mathcal{H}_zQ} \cap \widetilde{U} = (\mathrm{span}_{\mathfrak{A}}\{z\})^\perp \cap \widetilde{U} = \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(U)}$ (see Equation (2.17)) and thus by Lemma 5.8 $T_pQ' = \pi_*(\mathcal{H}_zQ \cap T_z\widetilde{U}) = U$. □

5.3 Types $(\mathbf{G2}, \boldsymbol{k_1}, \boldsymbol{k_2})$ and $(\mathbf{P1}, \boldsymbol{k})$

In this section, we abbreviate $r := 1/\sqrt{2}$ and consider the sphere $\mathbb{S}_r(W)$, where $\{0\} \neq W \subset \mathbb{V}$ is any real linear subspace. Remember that $\mathbb{S}_r(W)$ is connected for $\dim_{\mathbb{R}} W \geq 2$, whereas it consists of exactly two points in the case $\dim_{\mathbb{R}} W = 1$. In the following constructions, sphere products $\mathbb{S}_r(W_1) \times \mathbb{S}_r(W_2)$ play an important role.

5.11 Proposition. *Let $p \in Q$ and a curvature-invariant subspace $U \subset T_pQ$ of type $(\mathrm{G2}, k_1, k_2)$ be given. Thus there exists $A' \in \mathfrak{A}(Q, p)$ and linear subspaces $W_1, W_2 \subset V(A')$ of dimension k_1 resp. k_2 with $W_1 \perp W_2$ so that $U = W_1 \oplus JW_2$ holds.*

[12] Concerning the definition of $\widetilde{U}$ remember that $\mathrm{span}_{\mathfrak{A}}\{z\} = \mathrm{span}_{\mathbb{C}}\{z, Az\}$ holds (with $A \in \mathfrak{A}$), compare Definition 2.10(e).

Let $z \in \pi^{-1}(\{p\})$ be given and let $A \in \mathfrak{A}$ be the lift of A' at z (meaning that the conjugation $A|\mathcal{H}_z Q : \mathcal{H}_z Q \to \mathcal{H}_z Q$ is conjugate to A' under the $\mathbb{C}$Q-isomorphism $\pi_*|\mathcal{H}_z Q : \mathcal{H}_z Q \to T_p Q$, see Theorem 2.25(b)). Then

$$\widetilde{V}_1 := \mathbb{R}(\mathrm{Re}_A\, z) \oplus \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(W_1)} \quad \text{and} \quad \widetilde{V}_2 := \mathbb{R}(\mathrm{Im}_A\, z) \oplus \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(W_2)} \tag{5.3}$$

are orthogonal subspaces of $V(A)$ of real dimension $k_1 + 1$ resp. $k_2 + 1$ and the map

$$f_U : \mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2) \to Q,\ (x, y) \mapsto \pi(x + Jy)$$

is a two-fold isometric covering map onto its image M with

$$\forall\, (x, y), (x', y') \in \mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2)\ :\ \big(\ f_U(x', y') = f_U(x, y) \Longleftrightarrow (x', y') = \pm(x, y)\ \big)\ . \tag{5.4}$$

M is a totally geodesic, totally real, connected, compact submanifold of Q with $p \in M$ and $T_p M = U$. Because of (5.4), f_U gives rise to an isometry $\underline{f}_U : (\mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2))/\{\pm \mathrm{id}\} \to M$.

5.12 Proposition. *Let $p \in Q$ and a curvature-invariant subspace $U \subset T_p Q$ of type $(\mathrm{P1}, k)$ be given. Thus there exists $A' \in \mathfrak{A}(Q, p)$ so that U is a k-dimensional subspace of $V(A')$.*

Let $z \in \pi^{-1}(\{p\})$ be given and let $A \in \mathfrak{A}$ be the lift of A' at z (as above). Then

$$\widetilde{V} := \mathbb{R}(\mathrm{Re}_A\, z) \oplus \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(U)} \tag{5.5}$$

is a linear subspace of $V(A)$ of real dimension $k + 1$ and the map

$$f_U : \mathbb{S}_r(\widetilde{V}) \to Q,\ x \mapsto \pi(x + J\ \mathrm{Im}_A\, z)$$

is an isometric embedding; its image M is a totally geodesic, totally real, connected, compact submanifold of Q with $p \in M$ and $T_p M = U$.

5.13 Example. For $k_1, k_2 \in \mathbb{N}_0$ with $1 \leq k_1 + k_2 \leq m$, the map

$$\begin{gathered} \mathbb{S}_r^{k_1} \times \mathbb{S}_r^{k_2} \to Q^m, \\ ((x_0, \ldots, x_{k_1}),\ (y_0, \ldots, y_{k_2})) \mapsto [x_0, \ldots, x_{k_1}, i \cdot y_0, \ldots, i \cdot y_{k_2}, 0, \ldots, 0] \end{gathered}$$

is an isometric immersion and a two-fold covering map onto its image. The latter is a totally geodesic submanifold of Q^m; it is of type $(\mathrm{G2}, k_1, k_2)$ (for $k_1, k_2 \neq 0$) resp. of type $(\mathrm{P1}, k_1)$ (for $k_2 = 0$).

Proof of Propositions 5.11 and 5.12. Let $p \in Q$ and a curvature-invariant subspace $U \subset T_p Q$ of type $(\mathrm{G2}, k_1, k_2)$ or of type $(\mathrm{P1}, k)$ be given; in the latter case, we put $k_1 := k$ and $k_2 := 0$. Then in either case there exist $A' \in \mathfrak{A}(Q, p)$ and linear subspaces $W_1, W_2 \subset V(A')$ of dimension k_1 resp. k_2 so that $U = W_1 \oplus J W_2$ and $W_1 \perp W_2$ holds. Let $z \in \pi^{-1}(\{p\})$ be given and let $A \in \mathfrak{A}$ be the lift of A' at z (Theorem 2.25(b)). Then we define the linear subspaces $\widetilde{V}_1$ and $\widetilde{V}_2$ of $V(A)$ by Equations (5.3).

Because $z \in \widetilde{Q}$ is isotropic, we have by Proposition 2.20(b),(c)

$$\|\,\mathrm{Re}_A\, z\| = \|\,\mathrm{Im}_A\, z\| = r \quad \text{and} \quad \mathrm{Re}_A\, z \perp \mathrm{Im}_A\, z\ . \tag{5.6}$$

For $\ell \in \{1,2\}$, we put $\widetilde{W}_\ell := \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(W_\ell)}$, then we have

$$\widetilde{W}_\ell \subset \overrightarrow{\mathcal{H}_z Q} \tag{5.7}$$

and consequently by Theorem 2.26

$$\mathrm{Re}_A\, z, \mathrm{Im}_A\, z \;\perp\; \widetilde{W}_1, \widetilde{W}_2 \;, \tag{5.8}$$

and because of Theorem 2.25(b), $W_\ell \subset V(A')$ implies

$$\widetilde{W}_\ell \subset V(A) \;. \tag{5.9}$$

Because of $W_1 \perp W_2$ we also have

$$\widetilde{W}_1 \perp \widetilde{W}_2 \;. \tag{5.10}$$

(5.8) shows that the sums in the definition (5.3) of $\widetilde{V}_\ell$ are indeed orthogonally direct, therefore we have $\dim_{\mathbb{R}} \widetilde{V}_\ell = k_\ell + 1$. (5.8) and (5.10) show

$$\widetilde{V}_1 \perp \widetilde{V}_2 \;, \tag{5.11}$$

and (5.9) shows

$$\widetilde{V}_\ell \subset V(A) \;. \tag{5.12}$$

We now consider the map

$$\widetilde{f} : \mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2) \to \widetilde{Q},\ (x,y) \mapsto x + Jy \;;$$

$\widetilde{f}$ indeed maps into $\widetilde{Q}$: For any $(x,y) \in \mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2) =: N$, we have $\|x\| = \|y\| = r$ and $x \perp y$, whence $\widetilde{f}(x,y) \in \widetilde{Q}$ follows by Proposition 2.23(b).

It should be kept in mind that we have

$$f_U = \pi \circ \widetilde{f} \quad \text{in the case of Proposition 5.11} \tag{5.13}$$

$$\text{and} \quad \forall x \in \mathbb{S}(\widetilde{V}) : f_U(x) = \pi(\widetilde{f}(x, \mathrm{Im}_A\, z)) \quad \text{in the case of Proposition 5.12.} \tag{5.14}$$

We next show that $\widetilde{f}$ is an isometric embedding. Let $(x,y) \in N$ and $u \cong (v,w) \in T_{(x,y)}N$ be given; here $\cong$ denotes the canonical isomorphy $T_{(x,y)}N \cong T_x\mathbb{S}_r(\widetilde{V}_1) \oplus T_y\mathbb{S}_r(\widetilde{V}_2)$. Then we have

$$\overrightarrow{T_{(x,y)}\widetilde{f}(u)} = \overrightarrow{v} + J\overrightarrow{w} \tag{5.15}$$

and consequently

$$\langle T_{(x,y)}\widetilde{f}(u), T_{(x,y)}\widetilde{f}(u)\rangle = \langle v + Jw, v + Jw\rangle = \langle v,v\rangle + \langle w,w\rangle = \langle u,u\rangle \;.$$

The latter equation shows that $\widetilde{f}$ is an isometric immersion into the regular submanifold $\widetilde{Q}$ of $\mathbb{V}$. Furthermore (5.15) implies $\widetilde{f}_* u \in \mathcal{H}_{\widetilde{f}(x,y)}Q$, see Equation (2.18) in Theorem 2.26, thus the map $\widetilde{f}$ is horizontal.

Note that we have $(\mathrm{Re}_A z, \mathrm{Im}_A z) \in N$ by (5.6) and $\widetilde{f}(\mathrm{Re}_A z, \mathrm{Im}_A z) = z$, also we have

$$\begin{aligned}\overrightarrow{\widetilde{f}_* T_{(\mathrm{Re}_A z, \mathrm{Im}_A z)} N} &\overset{(5.15)}{=} \{\, v + Jw \,|\, v \in T_{\mathrm{Re}_A z} \widetilde{V}_1,\ w \in T_{\mathrm{Im}_A z} \widetilde{V}_2,\ \overrightarrow{v} \perp \mathrm{Re}_A z,\ \overrightarrow{w} \perp \mathrm{Im}_A z \,\} \\ &= \widetilde{W}_1 \oplus J\widetilde{W}_2 = \overrightarrow{(\pi_* | \mathcal{H}_z)^{-1}(U)} \ . \end{aligned} \tag{5.16}$$

Because $\widetilde{f}$ is a horizontal isometric immersion, $f := \pi \circ \widetilde{f}$ is an isometric immersion by Lemma 5.6(a); moreover, we have for any $(x, y), (x', y') \in N$

$$\begin{aligned} f(x', y') = f(x, y) &\iff \exists\, \lambda \in \mathbb{S}^1 \ : \ x' + Jy' = \lambda \cdot (x + Jy) \\ &\iff x' + Jy' = \pm(x + Jy) \qquad \text{(note that } x, y, x', y' \in V(A) \text{ holds)} \\ &\iff (x', y') = \pm(x, y) \ . \end{aligned} \tag{5.17}$$

This shows that the fibres of f are exactly the fibres of the two-fold covering map $\tau : \mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2) \to (\mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2))/\{\pm \mathrm{id}\}$. Therefore f gives rise to an injective isometric immersion $\underline{f} : (\mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2))/\{\pm \mathrm{id}\} \to Q$ so that $\underline{f} \circ \tau = f$ holds. Because $(\mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2))/\{\pm \mathrm{id}\}$ is compact, $\underline{f}$ is in fact an isometric embedding, and therefore $M := \underline{f}((\mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2))/\{\pm \mathrm{id}\}) = f(\mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2))$ is a compact, and hence regular, submanifold of Q. It also follows that f is a two-fold covering map onto M. M is connected along with $(\mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2))/\{\pm \mathrm{id}\}$. Because of $z \in \widetilde{f}(N)$ and Equation (5.16), we have

$$p \in M \quad \text{and} \quad T_p M = U \ .$$

In order to show that M is a totally geodesic submanifold of Q, it suffices to show that the isometric immersion f is totally geodesic; because of Lemma 5.6(b), for this it is in turn sufficient to show that $\widetilde{f}$ is totally geodesic.

Because $\widetilde{f}$ is an injective immersion and N is compact, $\widetilde{f}$ is an embedding, and hence $\widetilde{M} := \widetilde{f}(N)$ is a regular submanifold of Q. To prove that the isometric immersion $\widetilde{f}$ is totally geodesic, it suffices to show that the submanifold $\widetilde{M}$ is totally geodesic, and for the proof of this fact we again use the theorem that the connected components of the common fixed point set of a set of isometries are totally geodesic submanifolds ([Kob72], Theorem II.5.1, p. 59):

$\widetilde{Y}_\ell := \widetilde{V}_\ell \oplus J\widetilde{V}_\ell$ is a $\mathbb{C}$Q-subspace of $\mathbb{V}$ for $\ell \in \{1, 2\}$. Let $B_1 : \mathbb{V} \to \mathbb{V}$ be the $\mathbb{C}$Q-automorphism characterized by

$$B_1 | (\widetilde{Y}_1 \oplus \widetilde{Y}_2) = \mathrm{id}_{\widetilde{Y}_1 \oplus \widetilde{Y}_2} \quad \text{and} \quad B_1 | (\widetilde{Y}_1 \oplus \widetilde{Y}_2)^\perp = -\mathrm{id}_{(\widetilde{Y}_1 \oplus \widetilde{Y}_2)^\perp}$$

and let $B_2 : \mathbb{V} \to \mathbb{V}$ be the $\mathbb{C}$Q-anti-automorphism characterized by

$$B_2 | \widetilde{Y}_1 = A | \widetilde{Y}_1 \quad \text{and} \quad B_2 | \widetilde{Y}_1^\perp = -A | \widetilde{Y}_1^\perp \ .$$

Then $g_1 := B_1 | \widetilde{Q}$ and $g_2 := B_2 | \widetilde{Q}$ are isometries of $\widetilde{Q}$.

Let $z' \in \widetilde{Q}$ be given; we represent z' in the form $z' = z'_Y + z'_\perp$ with $z'_Y \in \widetilde{Y}_1 \oplus \widetilde{Y}_2$ and $z'_\perp \in (\widetilde{Y}_1 \oplus \widetilde{Y}_2)^\perp$. Then we have

$$g_1(z') = z' \iff z'_Y - z'_\perp = z'_Y + z'_\perp \iff z'_\perp = 0 \iff z' \in \widetilde{Y}_1 \oplus \widetilde{Y}_2 \ ,$$

and if $g_1(z') = z'$ holds, say for $z' = z_1' + z_2'$ with $z_\ell' \in \widetilde{Y}_\ell$, we have

$$g_2(z') = z' \iff A(z_1' - z_2') = z_1' + z_2' \iff \big(z_1' \in V(A) \cap \widetilde{Y}_1 = \widetilde{V}_1 \text{ and } z_2' \in JV(A) \cap \widetilde{Y}_2 = J\widetilde{V}_2\big) .$$

Therefore we have

$$\begin{aligned} \mathrm{Fix}(\{g_1, g_2\}) &= \{\, z_1' + z_2' \,|\, z_1' \in \widetilde{V}_1,\ z_2' \in J\widetilde{V}_2,\ z_1' + z_2' \in \widetilde{Q} \,\} \\ &= \{\, x + Jy \,|\, x \in \widetilde{V}_1,\ y \in \widetilde{V}_2,\ x + Jy \in \widetilde{Q} \,\} \\ &= \{\, x + Jy \,|\, x \in \mathbb{S}_r(\widetilde{V}_1),\ y \in \mathbb{S}_r(\widetilde{V}_2) \,\} = \widetilde{M} . \end{aligned}$$

It follows by [Kob72], Theorem II.5.1, p. 59 that $\widetilde{M}$ is a totally geodesic submanifold of $\widetilde{Q}$, and therefore M is a totally geodesic submanifold of Q.

Because $T_pM = U$ is a totally real subspace of T_pQ, Lemma 5.4 shows that M is a totally real submanifold of Q.

In the situation of Proposition 5.11 we have by (5.13) $f_U = f$ and $\underline{f}_U = \underline{f}$; therefore all statements of Proposition 5.11 have been shown above.

In the situation of Proposition 5.12, we have $\mathbb{S}_r(\widetilde{V}_2) = \{\pm \mathrm{Im}_A\, z\}$, therefore N has exactly two connected components, and we have $f_U = f \circ \iota$ with the isometric embedding

$$\iota : \mathbb{S}_r(\widetilde{V}) \to N,\ x \mapsto (x, \mathrm{Im}_A\, z)$$

onto one of the connected components of N. Because of (5.17), we now see that the images of f_U and f coincide, therefore all statements of Proposition 5.12 have been shown above, with the exception of the fact that the isometric immersion f_U is an embedding. For the proof of this fact, we note that (5.17) also shows that f_U is injective. Because f_U is therefore an injective immersion defined on the compact manifold $\mathbb{S}_r(\widetilde{V})$, it is indeed an embedding. □

Among the totally geodesic submanifolds of a symmetric space, the maximal tori (i.e. the totally geodesic submanifolds whose tangent spaces are maximal flat subspaces) are of particular interest, for example because every geodesic runs in a maximal torus. In a $\mathbb{C}$Q-space, the maximal flat subspaces are exactly the curvature-invariant subspaces of type $(\mathrm{G2}, 1, 1)$ (see Theorem 2.54), and therefore the maximal tori of a complex quadric are the totally geodesic submanifolds of type $(\mathrm{G2}, 1, 1)$.

In the following proposition, we give a closer description of the geometry of these maximal tori. In particular, we describe a lattice $\Gamma \subset \mathbb{C}$ (i.e. a discrete subgroup Γ of the group $(\mathbb{C}, +)$) so that the maximal tori of Q are isometric to $\mathbb{C}/\Gamma$.

5.14 Proposition. *Let $p \in Q$ and a curvature-invariant subspace $U \subset T_pQ$ of type $(\mathrm{G2}, 1, 1)$ be given; we let M be the maximal torus of Q with $p \in M$ and $T_pM = U$.*

There exists $A' \in \mathfrak{A}(Q, p)$ and an orthonormal system (v_x, v_y) in $V(A')$ so that $U = \mathbb{R}v_x \oplus \mathbb{R}Jv_y$ holds. We fix $z \in \pi^{-1}(\{p\})$, denote by $A \in \mathfrak{A}$ the lift of A' at z (see Theorem 2.25(b)) *and put* $x := \sqrt{2}\,\mathrm{Re}_A\, z$, $y := \sqrt{2}\,\mathrm{Im}_A\, z$, $\widetilde{v}_x := \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(v_x)}$, $\widetilde{v}_y := \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(v_y)}$, $r := 1/\sqrt{2}$. *Then we consider the normal geodesics*

$$\begin{aligned} &\widetilde{\gamma}_1 : \mathbb{R} \to \mathbb{S}_r(\mathbb{V}),\, t \mapsto r\, \cos(\tfrac{t}{r}) \cdot x + r\, \sin(\tfrac{t}{r}) \cdot \widetilde{v}_x \\ \textit{and}\quad &\widetilde{\gamma}_2 : \mathbb{R} \to \mathbb{S}_r(\mathbb{V}),\, t \mapsto r\, \cos(\tfrac{t}{r}) \cdot y + r\, \sin(\tfrac{t}{r}) \cdot \widetilde{v}_y \end{aligned}$$

and the map

$$f : \mathbb{C} \to Q,\ t + is \mapsto \pi(\,\widetilde{\gamma}_1(t) + J\widetilde{\gamma}_2(s)\,)\ .$$

f *is an isometric covering map onto* M *; its deck transformation group is given by the translations in* $\mathbb{C}$ *by the elements of the lattice*

$$\Gamma := \mathbb{Z}\,\tfrac{\pi}{\sqrt{2}}(1+i) \oplus \mathbb{Z}\,\tfrac{\pi}{\sqrt{2}}(1-i)\ . \tag{5.18}$$

It follows that M *is isometric to the torus* $\mathbb{C}/\Gamma \cong \mathbb{S}^1_{1/2} \times \mathbb{S}^1_{1/2}$.

Moreover, we have $f(0) = p$ *and (identifying* $T_0\mathbb{C}$ *with* $\mathbb{C}$ *)*

$$\forall \tau, \sigma \in \mathbb{R}\ :\ T_0 f(\tau + i\sigma) = \tau\, v_x + \sigma\, Jv_y\ . \tag{5.19}$$

Proof. We put

$$\widetilde{V}_1 := \mathbb{R}x \oplus \mathbb{R}\widetilde{v}_x \quad \text{and} \quad \widetilde{V}_2 := \mathbb{R}y \oplus \mathbb{R}\widetilde{v}_y$$

and consider, as in Proposition 5.11, the isometric embedding

$$\widetilde{f}_U : \mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2) \to \widetilde{Q},\ (x', y') \mapsto x' + Jy'$$

onto $\widetilde{M} := \widetilde{f}_U(\mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2))$ and the two-fold isometric covering map

$$f_U := \pi \circ \widetilde{f}_U : \mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2) \to Q$$

onto M also described in Proposition 5.11.

The normal geodesic $\widetilde{\gamma}_k : \mathbb{R} \to \mathbb{S}_r(\widetilde{V}_k)$ is periodic with period $2r\pi$ and an isometric covering map of the circle $\mathbb{S}_r(\widetilde{V}_k)$, and hence

$$\chi : \mathbb{C} \to \mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2),\ t + is \mapsto (\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(s))$$

is an isometric covering map whose deck transformation group is given by the translations in $\mathbb{C}$ by the elements of the lattice

$$\widetilde{\Gamma} := \mathbb{Z}\, 2r\pi \oplus \mathbb{Z}\, 2r\pi\, i\ .$$

Therefore also

$$\widetilde{f} := \widetilde{f}_U \circ \chi : \mathbb{C} \to \widetilde{Q}$$

is an isometric covering map onto $\widetilde{M}$ with the same deck transformation group.

Because both $\widetilde{f}$ and $\pi|\widetilde{M} : \widetilde{M} \to M$ are isometric covering maps (as (5.4) shows, the latter is a two-fold covering map whose fibres are of the form $\{\pm z'\}$ with $z' \in \widetilde{M}$), we see that also

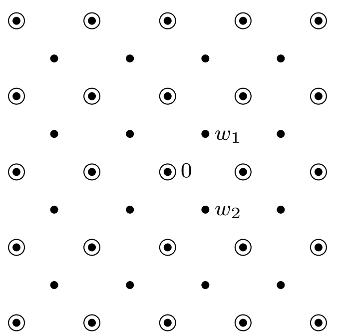
$f = (\pi|\widetilde{M}) \circ \widetilde{f}$ is an isometric covering map $\mathbb{C} \to M$. We obviously have $f(0) = p$, and Equation (5.19) follows easily from the facts $\widetilde{\gamma}_1'(0) = \widetilde{v}_x$ and $\widetilde{\gamma}_2'(0) = \widetilde{v}_y$.

It remains to show that the deck transformation group of f is indeed as given in Equation (5.18). For this we note that we have

$$\begin{aligned} f^{-1}(\{p\}) &= ((\pi|\widetilde{M}) \circ \widetilde{f})^{-1}(\{p\}) = \widetilde{f}^{-1}((\pi|\widetilde{M})^{-1}(\{p\})) = \widetilde{f}^{-1}(\{z\}) \,\dot\cup\, \widetilde{f}^{-1}(\{-z\}) \\ &= \widetilde{\Gamma} \,\dot\cup\, (\, w_1 + \widetilde{\Gamma}\,) =: \Gamma_1 \end{aligned}$$

with $w_1 := r\pi(1+i)$; for the last equals sign note that $\widetilde{f}(0) = z$ and $\widetilde{f}(r\pi + r\pi\, i) = -z$ holds.

In the following diagram we depict the lattices $\widetilde{\Gamma}$ and Γ_1, there the elements of $\widetilde{\Gamma}$ are marked by $\circ$ and the elements of Γ_1 are marked by $\bullet$:



with $w_2 := w_1 - 2r\pi i = r\pi(1-i)$.

This diagram shows that $\Gamma_1 = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2 = \Gamma$ holds, and therefore the deck transformation group of f is indeed given by the translations by the elements of Γ. $\square$

5.4 Types (Geo, t): Geodesics in Q

The totally geodesic submanifolds of Q of type (Geo, t) are exactly the images of normal geodesics $\gamma : \mathbb{R} \to Q$ whose tangent vector $\dot\gamma(s)$ has the $\mathfrak{A}(Q,\gamma(s))$-angle t for some (and then for every) $s \in \mathbb{R}$. To describe these submanifolds of Q it therefore suffices to give a description of the geodesics of Q.

Because every geodesic of Q runs in a maximal torus, we can combine the well-known facts about geodesics on a flat, 2-dimensional torus with Proposition 5.14 to obtain information on the geodesics of Q.

5.15 Proposition. *Let $p \in Q$, $v \in \mathbb{S}(T_pQ)$ and $z \in \pi^{-1}(\{p\})$ be given.*

Let $A' \in \mathfrak{A}(Q,p)$ be adapted to v in the sense of Theorem 2.28. Then we have the canonical representation

$$\begin{cases} v = \cos(\varphi) \cdot v_x + \sin(\varphi) \cdot Jv_y \\ \textit{with } \varphi := \varphi(v) \in [0, \frac{\pi}{4}]\,,\ v_x, v_y \in \mathbb{S}(V(A')) \textit{ and } v_x \perp v_y\,. \end{cases} \tag{5.20}$$

We let $A \in \mathfrak{A}$ be the lift of A' at z (Theorem 2.25(b)) and put $x := \mathrm{Re}_A\, z$, $y := \mathrm{Im}_A\, z$, $\widetilde{v}_x := \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(v_x)}$ and $\widetilde{v}_y := \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(v_y)}$. We consider the isometric covering map $f : \mathbb{C} \to Q$ onto a maximal torus of Q from Proposition 5.14 in this situation.

Then the curve

$$\gamma_v : \mathbb{R} \to \mathbb{C},\ t \mapsto f(e^{i\varphi} \cdot t)$$

is the maximal geodesic of Q with $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$.

Proof. Let $M = f(\mathbb{C})$ be the maximal torus of Q with $p \in M$ and $T_pM = \mathbb{R}v_x \oplus \mathbb{R}Jv_y$. It is clear that

$$\delta : \mathbb{R} \to \mathbb{C},\ t \mapsto e^{i\varphi} \cdot t$$

is a geodesic of $\mathbb{C}$; because $f : \mathbb{C} \to M$ is an isometric covering map onto the totally geodesic submanifold M of Q, it follows that $\gamma_v = f \circ \delta$ is a geodesic of Q.

Moreover we have $\gamma_v(0) = f(0) = p$ and by Equation (5.19)

$$\dot{\gamma}_v(0) = f_*(\dot{\delta}(0)) = f_*(e^{i\varphi}) = \cos(\varphi)v_x + \sin(\varphi)Jv_y \overset{(5.20)}{=} v\ ;$$

in this calculation we again identified $T_0\mathbb{C}$ with $\mathbb{C}$. □

5.16 Remark. If we have $v \in \mathbb{S}(T_pQ)$ with $\varphi(v) = \frac{\pi}{4}$, then the geodesic $\widehat{\gamma} : \mathbb{R} \to \mathbb{P}(\mathbb{V})$ of $\mathbb{P}(\mathbb{V})$ with $\dot{\widehat{\gamma}}(0) = v$ satisfies $\widehat{\gamma}(\mathbb{R}) \subset Q$, and therefore $\widehat{\gamma}$ also is a geodesic of Q.

Proof of Remark 5.16. Fix $z \in \pi^{-1}(\{p\})$ and $A \in \mathfrak{A}$, and let $w \in \mathcal{H}_zQ$ be the π-horizontal lift of v at z. Because of $z \in \widetilde{Q}$ and $\varphi(v) = \frac{\pi}{4}$ we have

$$\langle z, Az\rangle_{\mathbb{C}} = \langle \overrightarrow{w}, A\overrightarrow{w}\rangle_{\mathbb{C}} = 0 \tag{5.21}$$

and because of $\overrightarrow{w}, A\overrightarrow{w} \in \overrightarrow{\mathcal{H}_zQ}$ we have by Proposition 1.13(b)

$$\overrightarrow{w}, A\overrightarrow{w}\ \perp\ z, Az\ . \tag{5.22}$$

For $t \in \mathbb{R}$ we have $\widehat{\gamma}(t) = \pi(\widetilde{\gamma}(t))$ with

$$\widetilde{\gamma}(t) = \cos(t)\, z + \sin(t)\, \overrightarrow{w}\ .$$

By Equations (5.22) and (5.21),

$$\langle \widetilde{\gamma}(t), A(\widetilde{\gamma}(t))\rangle_{\mathbb{C}} = \cos(t)^2 \cdot \langle z, Az\rangle_{\mathbb{C}} + \sin(t)^2 \cdot \langle \overrightarrow{w}, A\overrightarrow{w}\rangle_{\mathbb{C}} = 0$$

holds; this shows that we have $\widetilde{\gamma}(\mathbb{R}) \subset \widetilde{Q}$ and therefore $\widehat{\gamma}(\mathbb{R}) \subset Q$. □

Our next aim is to calculate the length of closed geodesics in Q. Let us first recapitulate the corresponding well-known result for 2-dimensional tori corresponding to an orthogonal lattice:

5.17 Proposition. *Let* (w_1, w_2) *be an orthonormal basis of the real-2-dimensional euclidean space* $\mathbb{C}$, $\ell \in \mathbb{R}_+$ *and* $\Gamma := \mathbb{Z}\,\ell w_1 \oplus \mathbb{Z}\,\ell w_2$ *be the lattice in* $\mathbb{C}$ *generated by* ℓw_1 *and* ℓw_2. *Then we consider the flat torus* $\mathbb{T} := \mathbb{C}/\Gamma$ *along with the canonical projection* $\vartheta : \mathbb{C} \to \mathbb{T},\ z \mapsto z + \Gamma$. *The diameter of* $\mathbb{T}$ *is* $\ell/\sqrt{2}$.

Further let $z \in \mathbb{C}$ *be given and put* $p := \vartheta(z) \in \mathbb{T}$. *We identify* $T_p\mathbb{T}$ *with* $T_z\mathbb{C}$ *via the linear isomorphism* $T_z\vartheta$, *so that* $\overrightarrow{v} \in \mathbb{C}$ *is well-defined for* $v \in T_p\mathbb{T}$.

Now let $v \in \mathbb{S}(T_p\mathbb{T})$ *be given and let* $\gamma : \mathbb{R} \to \mathbb{T}$ *be the maximal geodesic of* $\mathbb{T}$ *with* $\gamma(0) = p$ *and* $\dot\gamma(0) = v$. *We denote by* $\alpha \in [0, \pi]$ *the (non-oriented) angle between* $\overrightarrow{v}$ *and* w_1 *and suppose that* $\alpha < \frac{\pi}{2}$ *holds.*[13]

(a) If $\alpha = 0$ *holds, then* γ *is closed and its minimal period is* ℓ.

(b) If $\alpha \neq 0$ *holds and* $\tan(\alpha)$ *is rational, say* $\tan(\alpha) = \frac{k_1}{k_2}$ *where* $k_1, k_2 \in \mathbb{N}$ *are relatively prime, then* γ *is also closed and its minimal period is* $\ell \cdot \sqrt{k_1^2 + k_2^2}$.

(c) If $\tan(\alpha)$ *is irrational, then* γ *is injective and* $\gamma(\mathbb{R})$ *is dense in* $\mathbb{T}$.

5.18 Proposition. *Let* $p \in Q$ *and* $v \in \mathbb{S}(T_pQ)$ *be given. As in Proposition 5.15, we denote by* $\gamma_v : \mathbb{R} \to Q$ *the maximal geodesic of* Q *with* $\gamma_v(0) = p$ *and* $\dot\gamma_v(0) = v$.

(a) If $\tan\varphi(v)$ *is rational, then* γ_v *is periodic.*

(i) If $\varphi(v) = 0$ *holds, then the minimal period of* γ_v *is* $L := \sqrt{2} \cdot \pi$.

(ii) If $\varphi(v) > 0$ *and* $\tan\varphi(v) = \frac{k_1}{k_2}$ *holds with* $k_1, k_2 \in \mathbb{N}$ *relatively prime, and* k_1 *and* k_2 *are both odd, then the minimal period of* γ_v *is*

$$L := \frac{\pi}{\sqrt{2}} \cdot \sqrt{k_1^2 + k_2^2}\,.$$

(iii) If $\varphi(v) > 0$ *and* $\tan\varphi(v) = \frac{k_1}{k_2}$ *holds with* $k_1, k_2 \in \mathbb{N}$ *relatively prime, and one of the numbers* k_1 *and* k_2 *is even, then the minimal period of* γ_v *is*

$$L := \sqrt{2} \cdot \pi \cdot \sqrt{k_1^2 + k_2^2}\,.$$

(b) If $\tan\varphi(v)$ *is irrational, then* γ_v *is injective. Let us denote by* U *a 2-flat of* T_pQ *containing* v, *and by* M *the maximal torus of* Q *with* $p \in M$ *and* $T_pM = U$. *Then* $\gamma_v(\mathbb{R})$ *is dense in* M.

5.19 Remark. In the cases of Proposition 5.18 where γ_v is periodic, say with minimal period L, we know from the general theory of symmetric spaces that $\gamma_v|[0, L[$ is injective.

[13] The case $\alpha = \frac{\pi}{2}$ can be reduced to the case $\alpha = 0$ by replacing (w_1, w_2) with $(\pm w_2, w_1)$. The case $\alpha > \frac{\pi}{2}$ can be reduced to the case $\alpha < \frac{\pi}{2}$ by replacing w_1 with $-w_1$.

Proof of Proposition 5.18. We let $A \in \mathfrak{A}(Q,p)$ be adapted to v (see Theorem 2.28) and consider a canonical representation

$$v = \cos(\varphi) \cdot v_x + \sin(\varphi) \cdot Jv_y$$

of v, i.e. we have $\varphi := \varphi(v)$ and (v_x, v_y) is an orthonormal system in $V(A)$. Then $U := \mathbb{R}v_x \oplus \mathbb{R}Jv_y$ is a 2-flat of T_pQ with $v \in U$. Let us denote by M the maximal torus of Q with $p \in M$ and $T_pM = U$. Then the geodesic $\gamma := \gamma_v$ runs entirely in M. Therefore the desired results can be obtained by application of Proposition 5.17.

As we saw in Proposition 5.14, M is isometric to the flat torus $\mathbb{C}/\Gamma$, where the lattice $\Gamma := \mathbb{Z}\,\ell w_1 \oplus \mathbb{Z}\,\ell w_2$ is given by $\ell := \pi$ and the orthonormal $\mathbb{R}$-basis (w_1, w_2) of $\mathbb{C}$ with $w_1 := \frac{1+i}{\sqrt{2}}$ and $w_2 := \frac{1-i}{\sqrt{2}}$. More specifically, the isometric covering map $f : \mathbb{C}/\Gamma \to M$ described in that proposition gives rise to an isometry $\underline{f} : \mathbb{C}/\Gamma \to M$ so that the following diagram commutes:

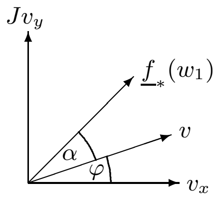
$$\begin{tikzcd} \mathbb{C} \arrow[r, "f"] \arrow[d] & M \\ \mathbb{C}/\Gamma \arrow[ur, "\underline{f}"'] & \end{tikzcd} .$$

As a consequence of Equation (5.19), $\underline{f}$ satisfies

$$\underline{f}_*(w_1) = \tfrac{1}{\sqrt{2}} \cdot (v_x + Jv_y) \; ;$$

here we again identify $T_0(\mathbb{C}/\Gamma) \cong T_0\mathbb{C} \cong \mathbb{C}$.

As Proposition 5.17 shows, the behaviour of the geodesic γ is controlled by the non-oriented angle α between $\widetilde{v} := \overrightarrow{(\underline{f}_*)^{-1}(v)}$ and w_1. As $\underline{f}$ is an isometry, the angle between v and $\underline{f}_* w_1$ is also equal to α, and therefore we have $\alpha = \frac{\pi}{4} - \varphi$, see the following diagram:



Because of this relation, we have

$$\tan(\alpha) = \frac{1 - \tan(\varphi)}{1 + \tan(\varphi)} \; . \tag{5.23}$$

For (a). If $\tan(\varphi)$ is rational, then Equation (5.23) shows that $\tan(\alpha)$ is also rational and therefore Proposition 5.17 shows that the geodesic γ is closed.

For (a)(i). If $\varphi = 0$ and hence $\tan(\alpha) = 0$ holds, then we have $\tan(\alpha) = 1$ by Equation (5.23), and therefore the minimal period of γ is $\ell \cdot \sqrt{1^2 + 1^2} = \sqrt{2} \cdot \pi$ by Proposition 5.17.

For (a)(ii),(iii). Suppose that $\varphi > 0$ holds and that $\tan(\varphi)$ is rational, say $\tan(\varphi) = \frac{k_1}{k_2}$, where $k_1, k_2 \in \mathbb{N}$ are relatively prime and $k_1 \leq k_2$ holds. In the case $k_1 = k_2 = 1$ we have $\tan(\alpha) = 0$ by Equation (5.23) and therefore the minimal period of γ is π by Proposition 5.17(a). Otherwise we have $k_1 < k_2$ and we obtain from Equation (5.23)

$$\tan(\alpha) = \frac{1 - \frac{k_1}{k_2}}{1 + \frac{k_1}{k_2}} = \frac{k_2 - k_1}{k_2 + k_1} \,. \tag{5.24}$$

Because k_1 and k_2 are relatively prime, the greatest common divisor of $k_2 - k_1$ and $k_2 + k_1$ is at most 2. (Any common divisor of $k_2 - k_1$ and $k_2 + k_1$ also divides $2k_2$ and $2k_1$.)

Thus we see that if both k_1 and k_2 are odd (hence $k_2 \pm k_1$ is even), the greatest common divisor of $k_2 - k_1$ and $k_2 + k_1$ is 2, and therefore we obtain by Proposition 5.17(b) and Equation (5.24) for the minimal period of γ

$$\ell \cdot \sqrt{\left(\tfrac{k_2-k_1}{2}\right)^2 + \left(\tfrac{k_2+k_1}{2}\right)^2} = \tfrac{\pi}{\sqrt{2}} \cdot \sqrt{k_2^2 + k_1^2} \,.$$

On the other hand, if either k_1 or k_2 is even (then the other of these two numbers is necessarily odd, and hence $k_2 \pm k_1$ is odd), then $k_2 - k_1$ and $k_2 + k_1$ are relatively prime, and therefore we obtain by Proposition 5.17(b) and Equation (5.24) for the minimal period of γ

$$\ell \cdot \sqrt{(k_2 - k_1)^2 + (k_2 + k_1)^2} = \sqrt{2}\,\pi \cdot \sqrt{k_2^2 + k_1^2} \,.$$

For (b). Suppose that $\tan(\varphi)$ is irrational. Then $\tan(\alpha)$ also is irrational. (It follows from Equation (5.23) that also $\tan(\varphi) = \frac{1-\tan(\alpha)}{1+\tan(\alpha)}$ holds, and therefore the rationality of $\tan(\alpha)$ would imply the rationality of $\tan(\varphi)$.) Therefore the statement follows from Proposition 5.17(c). $\square$

5.20 Proposition. *We denote by $d : Q \times Q \to \mathbb{R}$ the geodesic distance function of Q and by $\underline{A} : Q \to Q$ the antipode map of Q, see Remark 3.3. For any $p, q \in Q$ we have*

(a) $d(p,q) \leq \frac{\pi}{\sqrt{2}}$.

(b) $d(p,q) = \frac{\pi}{\sqrt{2}} \iff q = \underline{A}(p)$.

In particular, the diameter of Q is equal to $\frac{\pi}{\sqrt{2}}$. The preceding statements also justify the name "antipode map" for $\underline{A}$.

Proof. *For (a).* Let $p, q \in Q$ be given. By the Theorem of Hopf/Rinow (see [Lan99], Theorem VIII.6.6, p. 225) there exists a normal geodesic $\gamma : \mathbb{R} \to Q$ of Q with $\gamma(0) = p$ and $\gamma(t_0) = q$, where $t_0 := d(p,q)$. γ runs in a maximal torus M of Q, and therefore we have $t_0 \leq \operatorname{diam}(M)$. By Proposition 5.14, M is isometric to $\mathbb{S}^1_{1/2} \times \mathbb{S}^1_{1/2}$, whence it follows that $\operatorname{diam}(M) = \frac{\pi}{\sqrt{2}}$ holds. Thus, we have shown $d(p,q) \leq \frac{\pi}{\sqrt{2}}$.

For (b). In the situation discussed in the proof of (a), we now suppose that $d(p,q) = \frac{\pi}{\sqrt{2}}$ holds. Note that we have $p, q \in M$. We consider the isometry $\underline{f} : \mathbb{C}/\Gamma \to M$ induced by the isometric covering map $f : \mathbb{C} \to M$ from Proposition 5.14(b); here Γ is defined as in that proposition, and we use the point p also for the construction of Proposition 5.14, so that we have $\underline{f}(0+\Gamma) = p$. The only point in $\mathbb{C}/\Gamma$ which has distance $\frac{\pi}{\sqrt{2}}$ to $0+\Gamma = \underline{f}^{-1}(p)$ is $\frac{\pi}{\sqrt{2}} + \Gamma$; because $\underline{f}$ is an isometry, it follows that $q = \underline{f}(\frac{\pi}{\sqrt{2}} + \Gamma) = \underline{A}(p)$ holds.

Conversely, if we have $q = \underline{A}(p)$ in the situation of the proof of (a), then $\underline{f}^{-1}(p) = 0 + \Gamma$ and $\underline{f}^{-1}(q) = \frac{\pi}{\sqrt{2}} + \Gamma$ have distance $\frac{\pi}{\sqrt{2}}$ in M and therefore also in Q. □

5.5 Types $(\mathbf{I1}, \boldsymbol{k})$ and $(\mathbf{I2}, \boldsymbol{k})$

5.21 Proposition. *Let $p \in Q$, a curvature-invariant subspace $U \subset T_pQ$ and $z \in \pi^{-1}(\{p\})$ be given.*

(a) If U is of type $(\mathrm{I}1, k)$, then $\widetilde{V} := \mathbb{C}z \oplus \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(U)}$ is a $(k+1)$-dimensional complex isotropic subspace of $\mathbb{V}$. The k-dimensional complex projective subspace $M := [\widetilde{V}]$ of $\mathbb{P}(\mathbb{V})$ (equipped with a Hermitian metric of constant holomorphic sectional curvature 4) is contained in Q and therefore a totally geodesic, connected, compact Hermitian submanifold of Q. Also $p \in M$ and $T_pM = U$ holds.

(b) If U is of type $(\mathrm{I}2, k)$, then $\widetilde{V} := \mathbb{R}z \oplus \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(U)}$ is a $(k+1)$-dimensional totally real isotropic subspace of $\mathbb{V}$. $M := [\widetilde{V}] := \{\, \pi(v) \,|\, v \in \mathbb{S}(\widetilde{V}) \,\}$ is a totally geodesic, totally real submanifold of $\mathbb{P}(\mathbb{V})$ which is isometric to $\mathbb{RP}^k$ (equipped with a Riemannian metric of constant sectional curvature 1) and which is contained in Q; hence it is a totally geodesic, connected, compact totally real Riemannian submanifold of Q. Also $p \in M$ and $T_pM = U$ holds.

5.22 Example. Let $k \in \mathbb{N}$ with $k \leq \frac{m}{2}$ be given. We regard $\mathbb{C}^{m+2}$ as a $\mathbb{C}$Q-space in the usual way (Example 2.6) and denote the standard basis of $\mathbb{C}^{m+2}$ by $(e_1, \dots, e_{m+2})$.

(a) The complex $(k+1)$-dimensional linear subspace

$$\widetilde{V}_1 := \operatorname{span}_{\mathbb{C}}\{e_1 + Je_2, e_3 + Je_4, \dots, e_{2k+1} + Je_{2k+2}\}$$

of $\mathbb{C}^{m+2}$ is isotropic; therefore $[\widetilde{V}_1]$ is a totally geodesic Hermitian submanifold of the standard complex quadric Q^m.

(b) The totally real $(k+1)$-dimensional linear subspace

$$\widetilde{V}_2 := \operatorname{span}_{\mathbb{R}}\{e_1 + Je_2, e_3 + Je_4, \dots, e_{2k+1} + Je_{2k+2}\}\,,$$

of $\mathbb{C}^{m+2}$ is isotropic; therefore $[\widetilde{V}_2]$ is a totally geodesic, totally real submanifold of Q^m.

Proof of Proposition 5.21. We fix $A \in \mathfrak{A}$.

For (a). By Proposition 1.13(b), $\mathbb{C}z$ is orthogonal to $\overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(U)}$, therefore the sum in the definition of $\widetilde{V}$ is indeed orthogonally direct and we have $\dim \widetilde{V} = k+1$. For any $v \in \widetilde{V}$, say $v = \lambda z + u$ with $\lambda \in \mathbb{C}$ and $u \in \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(U)}$, we have

$$\langle v, Av\rangle_{\mathbb{C}} = \lambda^2 \langle z, Az\rangle_{\mathbb{C}} + 2\lambda\langle z, Au\rangle_{\mathbb{C}} + \langle u, Au\rangle_{\mathbb{C}} .$$

We have $\langle z, Az\rangle_{\mathbb{C}} = 0$ because of $z \in \widetilde{Q}$, $\langle z, Au\rangle_{\mathbb{C}} = 0$ by Proposition 1.13(b), and $\langle u, Au\rangle_{\mathbb{C}} = 0$ because U and therefore also $\overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(U)}$ is an isotropic subspace. Thus, we see $\langle v, Av\rangle_{\mathbb{C}} = 0$. This shows that $\widetilde{V}$ is an $\mathfrak{A}$-isotropic subspace of $\mathbb{V}$, and therefore we have $M := [\widetilde{V}] \subset Q$.

Obviously M is a connected, compact (and hence regular), totally geodesic submanifold of $\mathbb{P}(\mathbb{V})$; because of $M \subset Q$ it follows from Lemma 5.5 that M is a totally geodesic submanifold of Q. Because the Riemannian metric and the complex structure of both Q and M are inherited from $\mathbb{P}(\mathbb{V})$, we see that M is a Hermitian submanifold of Q. Finally, we have $z \in \widetilde{V}$, hence $p \in M$, and

$$T_pM = \pi_* T_z \mathbb{S}(\widetilde{V}) = \pi_*\big((\pi_*|\mathcal{H}_z)^{-1}(U)\big) = U .$$

For (b). $U' := U \oplus JU$ is a complex-k-dimensional subspace of T_pQ, which is isotropic by Proposition 2.20(d) and therefore a curvature-invariant subspace of type $(\mathrm{I}1, k)$. By (a), $\widetilde{V}' := \mathbb{C}z \oplus (\pi_*|\mathcal{H}_z)^{-1}(U)$ is a $(k+1)$-dimensional complex isotropic subspace of $\mathbb{V}$, and the k-dimensional complex projective subspace $M' := [\widetilde{V}']$ is a totally geodesic, Hermitian submanifold of Q.

$\widetilde{V}$ is a maximal totally real subspace of $\widetilde{V}'$, therefore $M = [\widetilde{V}]$ is a totally real, totally geodesic submanifold of the complex projective space $[\widetilde{V}']$, and hence of Q. Clearly, M is connected and compact, and we have $p \in M$ and $T_pM = U$. □

In the case $m = 2$, there exists a pair of foliations of Q by totally geodesic submanifolds of type $(\mathrm{I}1, 1)$, which intersect orthogonally at every point of Q, see also [Rec95], Remark 3. These foliations are the image under the Segre embedding $f : \mathbb{P}^1 \times \mathbb{P}^1 \to Q$ (see Section 3.4) of the two foliations of $\mathbb{P}^1 \times \mathbb{P}^1$ induced by the product structure. However, the foliations on Q can also be constructed without use of the Segre embedding via the following theorem.

5.23 Theorem. *Let $(M, \varphi, p_0, \sigma)$ be a Riemannian symmetric G-space and G_{p_0} the isotropy group of the action $\varphi : G \times M \to M$ at p_0. Suppose that a linear subspace $U \subset T_{p_0}M$ with*

$$\forall g \in G_{p_0} \;:\; (\varphi_g)_* U = U \tag{5.25}$$

is given. Then there exists one and only one vector subbundle $E \subset TM$ so that

$$E_{p_0} = U \quad \text{and} \quad \forall g \in G,\, p \in M \;:\; E_{\varphi_g(p)} = (\varphi_g)_* E_p \tag{5.26}$$

holds. E is a parallel subbundle of TM and therefore induces a foliation of M.

The proof of this theorem is given below. — To obtain the mentioned foliations on the 2-dimensional quadric Q, we fix $p_0 \in Q$. The $\mathbb{C}$Q-space $T_{p_0}Q$ contains exactly two complex 1-dimensional, isotropic subspaces U_1 and U_2, as was noted in Remark 2.40. As we already saw there, U_1 and U_2 are invariant under $\mathrm{Aut}(\mathfrak{A}(Q, p_0))_0$ and therefore under the isotropy action $(I(Q)_0)_{p_0} \times T_{p_0}Q \to T_{p_0}Q,\ (f, v) \mapsto f_* v$ (see Proposition 3.9(b)). By Theorem 5.23, U_1 and U_2 therefore induce two parallel subbundles E_1 and E_2 of TQ; because the Riemannian metric of Q is parallel, $T_{p_0}Q = U_1 \oplus U_2$ implies $TQ = E_1 \oplus E_2$. Hence the foliations induced by E_1 and E_2 intersect orthogonally at every point of Q.

We also note that because Q is simply connected and complete, the de Rham decomposition theorem shows anew that there exists an isometry $f : \mathbb{P}^1 \times \mathbb{P}^1 \to Q$ such that the leaves of the foliations induced by E_1 and E_2 are $(f(\{p\} \times \mathbb{P}^1))_{p\in\mathbb{P}^1}$ resp. $(f(\mathbb{P}^1 \times \{p\}))_{p\in\mathbb{P}^1}$.

Proof of Theorem 5.23. For any $g_1, g_2 \in G$ with $\varphi_{g_1}(p_0) = \varphi_{g_2}(p_0) =: p$, we have $g_2^{-1} \cdot g_1 \in G_{p_0}$ and hence

$$(\varphi_{g_1})_* U = (\varphi_{g_2} \circ \varphi_{g_2^{-1}\cdot g_1})_* U = (\varphi_{g_2})_* (\varphi_{g_2^{-1}\cdot g_1})_* U \overset{(5.25)}{=} (\varphi_{g_2})_* U \ .$$

Therefore E can be consistently defined by

$$\forall g \in G \ : \ E_{\varphi_g(p_0)} := (\varphi_g)_* U \ . \tag{5.27}$$

With this choice of E, (5.26) is satisfied: We have $E_{p_0} = (\varphi_e)_* U = U$ (where e denotes the neutral element of G); also if $g \in G$ and $p \in M$ are given, there exists $g_0 \in G$ with $\varphi_{g_0}(p_0) = p$, whence we obtain

$$E_{\varphi_g(p)} = E_{\varphi_{g\cdot g_0}(p_0)} \overset{(5.27)}{=} (\varphi_{g\cdot g_0})_* U = (\varphi_g)_* (\varphi_{g_0})_* U \overset{(5.27)}{=} (\varphi_g)_* E_{\varphi_{g_0}(p_0)} = (\varphi_g)_* E_p \ .$$

E is determined uniquely by (5.26) because G acts transitively on M.

Let us now consider the canonical splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G induced by the symmetric structure of M and the canonical isomorphism $\tau : \mathfrak{m} \to T_{p_0}M,\ X \mapsto (\varphi^{p_0})_* X_e$. We fix a basis $(v_1, \dots, v_k)$ of $E_{p_0} = U$, and consider for $j \in \{1, \dots, k\}$ the left-invariant vector field $X_j := \tau^{-1}(v_j) \in \mathfrak{m}$ and the vector field

$$Y_j := \big(\, g \mapsto (\varphi^{p_0})_* (X_j)_g \,\big) \ \in \ \mathfrak{X}_{\varphi^{p_0}}(M) \ .$$

Let us denote for any $g \in G$ by $L_g : G \to G$ the left translation with g. Then we have

$$\varphi^{p_0} \circ L_g = \varphi_g \circ \varphi^{p_0} \tag{5.28}$$

and therefore

$$(Y_j)_g = (\varphi^{p_0})_* (X_j)_g = (\varphi^{p_0})_* (L_g)_* (X_j)_e \overset{(5.28)}{=} (\varphi_g)_* (\varphi^{p_0})_* (X_j)_e = (\varphi_g)_* \tau(X_j) = (\varphi_g)_* v_j \ .$$

Because of Equation (5.26) it follows that $((Y_1)_g, \dots, (Y_k)_g)$ is a basis of $E_{\varphi_g(p_0)}$ for every $g \in G$. Thus, for every local section ϱ of the fibre bundle $\varphi^{p_0} : G \to M$, $(Y_1 \circ \varrho, \dots, Y_k \circ \varrho)$ is a local basis field of E. Therefore E is a differentiable vector subbundle of TM.

Next, we prove that E is parallel. For this we first note that the Levi-Civita covariant derivative ∇ of the Riemannian symmetric space M coincides with the canonical covariant derivative of the second kind in the sense of NOMIZU of M regarded as a naturally reductive homogeneous space (see Appendices A.1 and A.2) and therefore satisfies

$$\forall X, Z \in \mathfrak{m} \; : \; \nabla_Z((\varphi^{p_0})_* X) \equiv 0 \; . \tag{5.29}$$

Also, it can be shown that the horizontal structure $\mathcal{H}$ on the principal fibre bundle $\varphi^{p_0} : G \to M$ characterized by

$$\forall g \in G \; : \; \mathcal{H}_g = \{ \, X_g \, | \, X \in \mathfrak{m} \, \}$$

is a G_{p_0}-invariant connection in the sense of EHRESMANN, meaning that every curve in M can be globally lifted horizontally with respect to $\mathcal{H}$.

Now, let a curve $\alpha : I \to M$ be given, and let $\widetilde{\alpha} : I \to G$ be an $\mathcal{H}$-horizontal lift of α in the bundle $\varphi^{p_0} : G \to M$. Then we have for every $j \in \{1, \dots, k\}$: $Y_j \circ \widetilde{\alpha} \in \mathfrak{X}_\alpha(M)$ and

$$\nabla_\partial (Y_j \circ \widetilde{\alpha}) = \nabla_{\widetilde{\alpha}_* \partial} Y_j = \nabla_{\widetilde{\alpha}_* \partial} (\varphi^{p_0})_* X_j = 0 \; ,$$

where the last equality is justified by Equation (5.29) and the $\mathcal{H}$-horizontality of $\widetilde{\alpha}$. Because $(Y_1 \circ \widetilde{\alpha}, \dots, Y_k \circ \widetilde{\alpha})$ is a basis field of E along α, this shows that E is invariant under parallel displacement along α.

Now let $X, Y \in \Gamma(E)$ be given. Because ∇ is torsion-free, we have

$$[X, Y] = \nabla_X Y - \nabla_Y X \; ;$$

because E is parallel, it follows that $[X, Y] \in \Gamma(E)$ holds. Thus, E is involutive. By the global version of the theorem of Frobenius, there exists a foliation of M whose leaves are integral manifolds of E. □

5.6 Type (G3)

5.24 Proposition. *Let $p \in Q$ and a curvature-invariant subspace $U \subset T_pQ$ of type* (G3) *be given. Then U is contained in a 2-dimensional $\mathbb{C}$Q-subspace $U' \subset T_pQ$; except for the case $(m = 2,\ U' = T_pQ)$, the subspace U' of T_pQ is curvature-invariant of type* (G1, 2)*. We let Q' be the connected, complete, totally geodesic submanifold of Q with $p \in Q'$ and $T_pQ' = U'$; Q' is a 2-dimensional complex quadric (see Proposition 5.10).*

Then there exists a holomorphic isometry $f : \mathbb{P}^1 \times \mathbb{P}^1 \to Q'$ such that the connected, compact, totally geodesic Riemannian submanifold $M := f(\mathbb{P}^1 \times \mathbb{R}\mathrm{P}^1)$ of Q satisfies $p \in M$ and $T_pM = U$.

More specifically, if W is a 2-dimensional unitary space, then f can be chosen conjugate to the Segre embedding $\mathbb{P}(W) \times \mathbb{P}(W) \to Q(\mathfrak{A}_{\mathrm{End}(W)})$ described in Section 3.4 under suitable holomorphic isometries $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}(W) \times \mathbb{P}(W)$ and $Q' \to Q(\mathfrak{A}_{\mathrm{End}(W)})$.

Proof. By definition of the type (G3) there exist $A \in \mathfrak{A}(Q, p)$ and an orthonormal system (x, y) in $V(A)$ so that

$$U = \mathbb{C}\,(x - Jy) \oplus \mathbb{R}\,(x + Jy)$$

holds. U is contained in the 2-dimensional $\mathbb{C}$Q-subspace $U' := \mathbb{C}x \oplus \mathbb{C}y$ of T_pQ. U' contains exactly two complex-1-dimensional, isotropic subspaces, namely

$$\mathbb{C}\,(x + Jy) \quad \text{and} \quad \mathbb{C}\,(x - Jy)$$

(see Remark 2.40). We let Q' be the connected, complete, totally geodesic submanifold of Q with $p \in Q'$ and $T_pQ' = U'$; Q' is a 2-dimensional complex quadric by Proposition 5.10.

We now let W be a 2-dimensional unitary space. We regard $\mathrm{End}(W)$ as a $\mathbb{C}$Q-space with the $\mathbb{C}$Q-structure which was denoted by $\mathfrak{A}$ in Section 3.4 and which we now denote by $\mathfrak{A}_{\mathrm{End}(W)}$, and consider the Segre embedding $f_0 : \mathbb{P}(W) \times \mathbb{P}(W) \to \mathbb{P}(\mathrm{End}(W))$ also described in Section 3.4, which in fact is an isometry onto $Q(\mathfrak{A}_{\mathrm{End}(W)})$. We fix $q \in \mathbb{P}(W)$. Then $N := f_0(\mathbb{P}(W) \times \{q\})$ is a totally geodesic, complex-1-dimensional submanifold of $Q(\mathfrak{A}_{\mathrm{End}(W)})$ of constant holomorphic sectional curvature 4 with $p_0 := f_0(q, q) \in N$. It follows that $Y_1 := (f_0)_* T_{(q,q)}(\mathbb{P}(W) \times \{q\}) = T_{p_0}N$ is a complex-1-dimensional, curvature-invariant subspace of the $\mathbb{C}$Q-space $T_{p_0}Q(\mathfrak{A}_{\mathrm{End}(W)})$. By Theorem 4.2 it follows that Y_1 is either of type (P2) or of type $(\mathrm{I}1,1)$. However, Y_1 cannot be of type (P2), because then N would be isometric to Q^1 and therefore of constant curvature 2.[14] Therefore Y_1 is of type $(\mathrm{I}1,1)$ and hence an isotropic subspace of $T_{p_0}Q(\mathfrak{A}_{\mathrm{End}(W)})$. By the same arguments one sees that also $Y_2 := (f_0)_* T_{(q,q)}(\{q\} \times \mathbb{P}(W))$ is a complex-1-dimensional isotropic subspace of $T_{p_0}Q(\mathfrak{A}_{\mathrm{End}(W)})$, and obviously $T_{p_0}Q(\mathfrak{A}_{\mathrm{End}(W)}) = Y_1 \oplus Y_2$ holds.

Both $Q(\mathfrak{A}_{\mathrm{End}(W)})$ and Q' are 2-dimensional complex quadrics, therefore there exists a holomorphic isometry $g : Q(\mathfrak{A}_{\mathrm{End}(W)}) \to Q'$ with $g(p_0) = p$. $T_{p_0}g : T_{p_0}Q(\mathfrak{A}_{\mathrm{End}(W)}) \to T_pQ'$ is an isomorphism of $\mathbb{C}$Q-spaces, and therefore maps isotropic subspaces onto isotropic subspaces, hence $\{Y_1, Y_2\}$ onto $\{\mathbb{C}(x - Jy), \mathbb{C}(x + Jy)\}$. g can be chosen such that

$$g_* Y_1 = \mathbb{C}(x - Jy) \quad \text{and} \quad g_* Y_2 = \mathbb{C}(x + Jy)$$

holds.

Further we fix an arbitrary holomorphic isometry $h_1 : \mathbb{P}^1 \to \mathbb{P}(W)$. Moreover, we regard $\mathbb{R}\mathbb{P}^1$ as a submanifold of $\mathbb{P}^1$ (then $\mathbb{R}\mathbb{P}^1$ is a geodesic circle in $\mathbb{P}^1$) and consider the geodesic circle $C \subset \mathbb{P}(W)$ so that $q \in C$ and $(g \circ f_0)_*(T_{(q,q)}(\{q\} \times C)) = \mathbb{R}(x + Jy)$ holds. Then there exists a holomorphic isometry $h_2 : \mathbb{P}^1 \to \mathbb{P}(W)$ with $h_2(\mathbb{R}\mathbb{P}^1) = C$.

$f := g \circ f_0 \circ (h_1 \times h_2) : \mathbb{P}^1 \times \mathbb{P}^1 \to Q'$ is a holomorphic isometry, and it follows from the construction of g and h_k that f has the properties stated in the proposition. $\square$

[14] As will be shown in Proposition 8.1, Q^1 is isometric to $\mathbb{S}^2_{1/\sqrt{2}}$.

5.7 Type (A)

In this section we suppose $m \geq 3$.

5.25 Proposition. *Let $p \in Q$ and a curvature-invariant subspace $U \subset T_pQ$ of type* (A) *be given. Then the connected, complete, totally geodesic submanifold M of Q with $p \in M$ and $T_pM = U$ is isometric to the sphere $\mathbb{S}^2_{r=\sqrt{10}/2}$.*

Proof. By definition of the type (A) there exist $A \in \mathfrak{A}(Q,p)$ and an orthonormal system (x,y,z) in $V(A)$ so that with

$$a := \tfrac{1}{\sqrt{5}}(2x + Jy) \quad \text{and} \quad b := \tfrac{1}{\sqrt{5}}(y + Jx + \sqrt{3}\,Jz)\ ,$$

(a,b) is an orthonormal basis of U. As was already mentioned in the proof of Theorem 4.2 (see Equation (4.1)), we have

$$\langle\, R(a,b)b\,,\ a\,\rangle = \tfrac{2}{5}$$

where R denotes the curvature tensor of Q. Because the curvature tensor of the Riemannian symmetric space M is parallel, it follows that M is a space of constant curvature $\frac{2}{5} = \frac{1}{r^2}$ with $r := \frac{\sqrt{10}}{2}$, and therefore M is locally isometric to the sphere $\mathbb{S}^2_r$. Hence M is isometric either to the sphere $\mathbb{S}^2_r$, or to the real projective space $\mathbb{R}\mathrm{P}^2$ equipped with a Riemannian metric of constant sectional curvature $\frac{2}{5}$. To decide between these two cases, we calculate the length of closed geodesics in M: Let $v \in \mathbb{S}(T_pM)$ be given. Because M is a complete, totally geodesic submanifold of Q, the maximal geodesic $\gamma_v : \mathbb{R} \to Q$ of Q with $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$ runs completely in M and also is a geodesic of M. We have $\varphi(v) = \arctan(\frac{1}{2})$, therefore it follows from Proposition 5.18(a)(iii) that γ_v is periodic and that its minimal period is

$$\sqrt{10}\cdot\pi = 2\pi r\ .$$

This shows that M is isometric to $\mathbb{S}^2_r$. □

5.26 Remarks. (a) In the situation of Proposition 5.25 one would like to construct a totally geodesic, isometric embedding $f : \mathbb{S}^2_{\sqrt{10}/2} \to Q$ onto M explicitly, as we did for the other types of totally geodesic submanifolds. Such an embedding can be constructed via the fact that $M = \exp(U)$ holds, where $\exp : T_pQ \to Q$ denotes the exponential map of the complete Riemannian manifold Q. However, this construction results in a very complicated formula for f, which does not provide any insight into the geometry of M. At the moment, I am unable to give a clearer, more informative description of M.

(b) In the situation described in Proposition 5.25 there does not exist a horizontal submanifold $\widetilde{M}$ of $\widetilde{Q}$ with $\pi(\widetilde{M}) = M$, because U is not totally real. This follows by combination of several results from [Rec85]: Theorem 5, Theorem 6 and Theorem 4(a).

(c) Note that the diameter of M is $\pi \cdot \sqrt{10}/2$, which is strictly larger than the diameter $\pi/\sqrt{2}$ of Q (see Proposition 5.20).

5.8 Isometric totally geodesic submanifolds

Theorem 5.1 follows from Propositions 5.10, 5.11, 5.12, 5.18, 5.21, 5.24 and 5.25. It has the following corollary:

5.27 Corollary. *Let M_1, M_2 be two connected, complete, totally geodesic submanifolds of Q.*

(a) M_1 and M_2 are holomorphically congruent in Q if and only if they are of the same type (see Proposition 5.2, also note the identifications of types stated in Theorem 4.2).

(b) If M_1 and M_2 are of real dimension ≥ 3, then they are of the same type if and only if they are isometric to each other.[15]

Proof. For (a). If there exists $f \in I_h(Q)$ so that $M_2 = f(M_1)$ holds, we fix $p \in M_1$. Then we have $f_* T_p M_1 = T_{f(p)} M_2$. Because $T_p f : T_p Q \to T_{f(p)} Q$ is a $\mathbb{C}$Q-isomorphism by Proposition 3.2(a), it follows by Theorem 4.2 that $T_p M_1$ and $T_{f(p)} M_2$, hence M_1 and M_2, are of the same type.

Conversely, if M_1 and M_2 are of the same type, we fix $p_k \in M_k$ for $k \in \{1, 2\}$. Because $T_{p_1} M_1$ and $T_{p_2} M_2$ are curvature-invariant subspaces of $T_{p_1} Q$ resp. $T_{p_2} Q$ of the same type, Theorem 4.2 shows that there exists a $\mathbb{C}$Q-isomorphism $L : T_{p_1} Q \to T_{p_2} Q$ with $L(T_{p_1} M_1) = T_{p_2} M_2$. By Theorem 3.5(a) there exists a holomorphic isometry $f : Q \to Q$ with $f(p_1) = p_2$ and $T_{p_1} f = L$. We thus have $f_* T_{p_1} M_1 = T_{p_2} M_2$, and therefore $f(M_1) = M_2$ because of the rigidity of totally geodesic submanifolds. Hence M_1 and M_2 are holomorphically congruent in Q.

For (b). We now suppose that M_1 and M_2 are of real dimension ≥ 3. If they are of the same type, it has already been shown in (a) that they are holomorphically congruent in Q; in particular they are isometric to each other. Conversely, if M_1 and M_2 are isometric to each other, then an inspection of the table of isometry classes in Theorem 5.1 shows that M_1 and M_2 have to be of the same type. $\square$

[15] Note that this statement is false if M_1 and M_2 are of dimension 1 or 2. For the case of dimension 1, the totally geodesic submanifolds of type (Geo, t) are isometric to $\mathbb{R}$ for all $t \in [0, \frac{\pi}{4}]$ with $\tan(t) \in \mathbb{R} \setminus \mathbb{Q}$. For the case of dimension 2: We will see in Section 8.1 that Q^1 is isometric to $\mathbb{S}^2_{1/\sqrt{2}}$ and therefore both the submanifolds of type $(\mathrm{P1}, 2)$ and of type $(\mathrm{P2})$ are isometric to $\mathbb{S}^2_{1/\sqrt{2}}$.

Chapter 6

Subquadrics

Let $Q \subset \mathbb{P}(\mathbb{V})$ be an m-dimensional complex quadric. Then for every $k < m$, Q contains k-dimensional, complex submanifolds Q' which are complex quadrics in the following sense: For each Q' there exists a $(k+1)$-dimensional complex projective subspace Λ of $\mathbb{P}(\mathbb{V})$ such that Q' is a complex quadric in Λ in the sense of Chapter 1. We call such submanifolds of Q *subquadrics* of Q, and they are the subject of study of the present chapter.

The totally geodesic submanifolds of Q of type $(\mathrm{G}1, k)$ are subquadrics of Q. However, for $k \leq \frac{m}{2} - 1$, not every k-dimensional subquadric of Q is a totally geodesic submanifold. In fact, it will turn out that then there exists an infinite multitude of congruence classes of k-dimensional subquadrics of Q, and that the set of these congruence classes can be parameterized by an angle $t \in [0, \frac{\pi}{4}]$, where the totally geodesic subquadrics constitute the congruence class with $t = 0$. The subquadrics corresponding to the angle t are obtained from the totally geodesic ones by a "rotation" by this angle, compare Theorem 6.13(c) and Remark 6.14. — On the other hand, for $k > \frac{m}{2} - 1$ all k-dimensional subquadrics of Q are totally geodesic submanifolds of type $(\mathrm{G}1, k)$.

In Section 6.1 we prove a classification of the subquadrics in a given complex quadric $Q \subset \mathbb{P}(\mathbb{V})$. In particular, we characterize those complex subspaces U of $\mathbb{V}$ for which there exists a complex quadric in $[U] = \mathbb{P}(U)$ which is a subquadric of Q. We call the complex subspaces of $\mathbb{V}$ with this property *complex t-subspaces*; here the parameter $t \in [0, \frac{\pi}{4}]$ corresponds to a congruence class of subquadrics as described above. The objective of Section 6.2 is to further study the properties of t-subspaces, in particular see Theorem 6.13. In Section 6.3 we study the extrinsic geometry of subquadrics Q' regarded as submanifolds of Q; it will turn out that the geometry depends strongly on the parameter t.

As before, we fix $m \in \mathbb{N}$, let $(\mathbb{V}, \mathfrak{A})$ be a $\mathbb{C}$Q-space of dimension $n := m + 2$ and consider the m-dimensional complex quadric $Q := Q(\mathfrak{A})$. For any unitary space U we denote the set of conjugations on U by $\mathrm{Con}(U)$.

6.1 Complex subquadrics of a complex quadric

6.1 Definition. *Suppose* $k \in \{1, \dots, m-1\}$.

(a) We call $Q' \subset \mathbb{P}(\mathbb{V})$ *a* k-dimensional complex quadric *if there exists a* $(k+1)$*-dimensional complex projective subspace* Λ *of* $\mathbb{P}(\mathbb{V})$ *such that* Q' *is a (symmetric) complex quadric in* Λ *in the sense of Chapter 1.*

(b) We call a k*-dimensional complex quadric* Q' *a* (complex) subquadric *of* Q *if* $Q' \subset Q$ *holds.*

6.2 Examples. (a) For $k < m$, the totally geodesic submanifolds of Q of type $(\mathrm{G}1, k)$ are k-dimensional complex subquadrics of Q.

(b) Suppose $k \leq \frac{m}{2} - 1$ and let Λ be a totally geodesic submanifold of Q of type $(\mathrm{I}1, k+1)$. Then Λ is a $(k+1)$-dimensional complex projective subspace of $\mathbb{P}(\mathbb{V})$ contained in Q (see Proposition 5.21(a)), and every complex quadric Q' in Λ is a k-dimensional complex subquadric of Q. However, Q' is not totally geodesic in Q (because otherwise it would also be totally geodesic in Λ, which is impossible).

The aim of the present section is to classify all complex subquadrics of Q. As we already mentioned in the introduction of the chapter, it will turn out that there are many more congruence classes of subquadrics of dimension $\leq \frac{m}{2} - 1$ besides the two described in Example 6.2.

In the sequel, we denote for any complex linear subspace $U \subset \mathbb{V}$ by $P_U : \mathbb{V} \to \mathbb{V}$ the orthogonal projection of $\mathbb{V}$ onto U. It should be noted that P_U is $\mathbb{C}$-linear and that

$$\forall\, u \in U,\ v \in \mathbb{V}\ :\ \langle u, v \rangle_{\mathbb{C}} = \langle u, P_U v \rangle_{\mathbb{C}} \tag{6.1}$$

holds.

6.3 Definition. *Let a complex linear subspace* $U \subset \mathbb{V}$, $t \in [0, \frac{\pi}{4}]$ *and* $A \in \mathfrak{A}$ *be given. We then call* U *a* complex t-subspace *of* $\mathbb{V}$ *if*

$$\forall u \in U \setminus \{0\}\ :\ \sphericalangle(Au, U) = 2t$$

holds. Here we denote for $v \in \mathbb{V} \setminus \{0\}$ *by* $\sphericalangle(v, U) \in [0, \frac{\pi}{2}]$ *the angle between* v *and* U, *i.e.* $\sphericalangle(v, U) := \min_{u \in \mathbb{S}(U)} \sphericalangle(v, u)$ *; this angle is also given by* $\cos(\sphericalangle(v, U)) = \|P_U(v)\| / \|v\|$.

It is clear that the definition of a complex t*-subspace does not depend on the choice of* $A \in \mathfrak{A}$ *.*

6.4 Examples. Let $U \subset \mathbb{V}$ be a complex subspace.

(a) U is a complex 0-subspace of $\mathbb{V}$ if and only if it is a $\mathbb{C}$Q-subspace.

Proof. A complex subspace $U \subset \mathbb{V}$ is a $\mathbb{C}$Q-subspace if and only if $A(U) = U$ holds for $A \in \mathfrak{A}$, and this is equivalent to $\forall u \in U : P_U(Au) = Au$, which is in turn equivalent to U being a complex 0-subspace. $\square$

(b) U is a complex $\frac{\pi}{4}$-subspace if and only if it is $\mathfrak{A}$-isotropic.

Proof. U being $\mathfrak{A}$-isotropic means that $A(U) \subset U^\perp$ holds (see Proposition 2.20(a)), and this is equivalent to $P_U|A(U) = 0$, which is in turn equivalent to U being a $\frac{\pi}{4}$-subspace. □

6.5 Remark. Not every complex subspace $U \subset \mathbb{V}$ of dimension ≥ 2 is a complex t-subspace for some $t \in [0, \frac{\pi}{4}]$.

For example, let $A \in \mathfrak{A}$ and (x_1, x_2, x_3) be an orthonormal system in $V(A)$ (remember that $\dim_{\mathbb{R}} V(A) = \dim_{\mathbb{C}} \mathbb{V} = n \geq 3$ holds). Further fix some $\varphi \in]0, \frac{\pi}{4}]$ and consider

$$v_1 := \cos(\varphi)x_1 + \sin(\varphi)Jx_2 \quad \text{and} \quad v_2 := x_3 .$$

Then $U := \mathbb{C}v_1 \oplus \mathbb{C}v_2$ is a complex-2-dimensional subspace of $\mathbb{V}$. We have $\sphericalangle(Av_1, U) = 2\varphi \neq 0$, but $\sphericalangle(Av_2, U) = 0$, and therefore U is not a complex t-subspace of $\mathbb{V}$ for any $t \in [0, \frac{\pi}{4}]$.

6.6 Theorem. *(a) Let U be a complex t-subspace of $\mathbb{V}$ of dimension ≥ 3 with $t \in [0, \frac{\pi}{4}[$. Then $Q' := Q \cap [U]$ is a complex subquadric of Q (like in the case of totally geodesic subquadrics studied in Section 5.3), and for any $A \in \mathfrak{A}$*

$$A' := \frac{1}{\cos(2t)} (P_U \circ A)|U \tag{6.2}$$

is a conjugation on the unitary space U with $Q' = Q(A')$. We call a conjugation A' obtained from some $A \in \mathfrak{A}$ in this way an adapted conjugation on U (corresponding to A).

(b) Let U be a complex $\frac{\pi}{4}$-subspace of $\mathbb{V}$ of dimension ≥ 3. Then we have $[U] \subset Q$, and therefore every complex quadric in the complex projective space $[U] = \mathbb{P}(U)$ is a subquadric of Q. In this situation we call every conjugation A' on U an adapted conjugation on U (corresponding to every $A \in \mathfrak{A}$).

(c) Let Q' be a k-dimensional subquadric of Q. Then there exists a unique $t \in [0, \frac{\pi}{4}]$ and a unique complex t-subspace U of $\mathbb{V}$ of dimension $k + 2 \geq 3$ so that Q' is obtained by the construction of (a) (for $t < \frac{\pi}{4}$) or (b) (for $t = \frac{\pi}{4}$) with these data. In this setting, we call Q' a (complex) t-subquadric *of Q.*

6.7 Definition. *If U is a complex t-subspace of $\mathbb{V}$ ($t \in [0, \frac{\pi}{4}]$) and $A' \in \mathrm{Con}(U)$ is an adapted conjugation on U, then we call the $\mathbb{C}$Q-structure $\mathfrak{A}' := \mathbb{S}^1 \cdot A'$ an* adapted $\mathbb{C}$Q-structure on U. *Note that then every element of $\mathfrak{A}'$ is an adapted conjugation on U and that for $t < \frac{\pi}{4}$, $\mathfrak{A}'$ is unique.*

For the proof of Theorem 6.6 we shall need the following lemma:

6.8 Lemma. *Let $U \subset \mathbb{V}$ be a complex linear subspace and $t \in [0, \frac{\pi}{4}]$. Then U is a complex t-subspace if and only if*

$$\forall\, A \in \mathfrak{A}\ \exists\, A' \in \mathrm{Con}(U) \ :\ \cos(2t) \cdot A' = (P_U \circ A)|U \tag{6.3}$$

holds. In this case, Equation (6.3) is satisfied for a pair (A, A') if and only if A' is an adapted conjugation on U corresponding to A.

Proof of Lemma 6.8. Let us first suppose that (6.3) holds. We fix $A \in \mathfrak{A}$ and choose $A' \in \mathrm{Con}(U)$ as in (6.3). Then we have for every $u \in U \setminus \{0\}$

$$\cos(\sphericalangle(Au, U)) = \tfrac{\|P_U(Au)\|}{\|Au\|} \overset{(6.3)}{=} \cos(2t)\, \tfrac{\|A'u\|}{\|Au\|} \overset{(*)}{=} \cos(2t)$$

(where the equality marked $(*)$ follows from the fact that both A and A' are conjugations, and therefore $\|A'u\| = \|u\| = \|Au\|$ holds), whence $\sphericalangle(Au, U) = 2t$ follows. Thus we have shown that U is a complex t-subspace.

Let us now suppose conversely that U is a complex t-subspace. If $t = \frac{\pi}{4}$ holds, then we have $\cos(2t) = 0$ and $(P_U \circ A)|U = 0$ (U is $\mathfrak{A}$-isotropic by Example 6.4(b), and therefore we have $A(U) \subset U^\perp$), which shows that (6.3) is satisfied with arbitrary $A' \in \mathrm{Con}(U)$ in this case. Thus we may now suppose that $t < \frac{\pi}{4}$ holds. We let $A \in \mathfrak{A}$ be given and put

$$A' := \frac{1}{\cos(2t)}\, (P_U \circ A)|U \ .$$

We will show immediately that $A' \in \mathrm{Con}(U)$ holds; then it is obvious that (6.3) holds with this choice of A'.

It is clear that A' is anti-linear. We have for every $u \in U$

$$\|A'u\| = \tfrac{1}{\cos(2t)}\, \|P_U(Au)\| = \tfrac{1}{\cos(2t)}\, \cos(\underbrace{\sphericalangle(Au, U)}_{=2t})\, \underbrace{\|Au\|}_{=\|u\|} = \|u\| \ ,$$

which shows that A' is orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathbb{R}}$, and for every $u, v \in U$

$$\begin{aligned} \langle A'u, v \rangle_{\mathbb{R}} &= \tfrac{1}{\cos(2t)}\, \langle P_U(Au), v \rangle_{\mathbb{R}} \overset{(6.1)}{=} \tfrac{1}{\cos(2t)}\, \langle Au, v \rangle_{\mathbb{R}} \\ &= \tfrac{1}{\cos(2t)}\, \langle u, Av \rangle_{\mathbb{R}} \overset{(6.1)}{=} \tfrac{1}{\cos(2t)}\, \langle u, P_U(Av) \rangle_{\mathbb{R}} = \langle u, A'v \rangle_{\mathbb{R}} \ , \end{aligned}$$

which shows that A' is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathbb{R}}$. Therefore A' is a conjugation on U.

The statement that Equation (6.3) is satisfied for a pair (A, A') if and only if A' is an adapted conjugation on U corresponding to A is obvious from the definition of adaptedness in Theorem 6.6(a),(b). □

Proof of Theorem 6.6. For (a). Let $A \in \mathfrak{A}$ be given. Then Lemma 6.3 shows that the endomorphism A' defined by Equation (6.2) is a conjugation on U, therefore $Q(A')$ is a k-dimensional complex quadric in $\mathbb{P}(\mathbb{V})$. We will show immediately that

$$Q(A') = Q \cap [U] \tag{6.4}$$

holds; then it also follows that $Q \cap [U]$ is a subquadric of Q.

For the proof of (6.4) it should first be noted that both the left-hand side and the right-hand side of that equation are contained in $[U]$. Moreover, we have for every $u \in \mathbb{S}(U)$

$$\begin{aligned} u \in \widetilde{Q}(A') &\iff \langle u, A'u \rangle_{\mathbb{C}} = 0 \iff \tfrac{1}{\cos(2t)}\, \langle u, P_U\, Au \rangle_{\mathbb{C}} = 0 \\ &\overset{(6.1)}{\iff} \langle u, Au \rangle_{\mathbb{C}} = 0 \iff u \in \widetilde{Q} \ , \end{aligned}$$

whence (6.4) follows.

For (b). If U is a complex $\frac{\pi}{4}$-subspace of $\mathbb{V}$, then it is an $\mathfrak{A}$-isotropic subspace by Example 6.4(b), and therefore we have $[U] \subset Q$. Consequently every complex quadric in $[U]$ is a subquadric of Q.

For (c). Let Q' be a subquadric of Q; this means that there exists a complex subspace $U \subset \mathbb{V}$ and a conjugation $A' \in \mathrm{Con}(U)$ so that $Q' = Q(A') \subset Q$ holds.

We tentatively choose an arbitrary $A \in \mathfrak{A}$ and consider the symmetric $\mathbb{C}$-bilinear forms on U:

$$\beta : U \times U \to \mathbb{C},\ (v,w) \mapsto \langle v, Aw\rangle_{\mathbb{C}}$$
$$\text{and} \quad \beta' : U \times U \to \mathbb{C},\ (v,w) \mapsto \langle v, A'w\rangle_{\mathbb{C}} \;.$$

We have

$$\beta^\sharp = P_U \circ A|U \quad \text{and} \quad (\beta')^\sharp = A' \;, \tag{6.5}$$

where the Riesz endomorphisms are constructed in the unitary space U. We see from (6.5) that β' is non-degenerate. However, it should be noted that it is possible for β to be degenerate.

Because of the hypothesis $Q' \subset Q$, we get

$$\forall v \in U \ : \big(\beta'(v,v) = 0 \implies \beta(v,v) = 0\big) \;. \tag{6.6}$$

Let us first consider the case where β is non-degenerate. Then $Q' = Q(\beta')$ and $Q(\beta)$ are algebraic complex quadrics in the sense of Chapter 1 in the complex projective space $\mathbb{P}(U)$, and because of (6.6) we have $Q' \subset Q(\beta)$. Because Q' is compact, $Q(\beta)$ is connected, and these two manifolds are of the same dimension, we in fact have $Q' = Q(\beta)$. By Proposition 1.3, it follows that $\beta' = \lambda \cdot \beta$ holds for some $\lambda \in \mathbb{C}^\times$. By appropriately modifying the tentative choice of $A \in \mathfrak{A}$ made above, we can ensure $\lambda \in \mathbb{R}_+$. From Equations (6.5) it follows that

$$A' = \lambda \cdot P_U \circ (A|U) \tag{6.7}$$

holds. Choosing some $v \in \mathbb{S}(U)$ we see that

$$1 = \|v\| = \|A'v\| \overset{(6.7)}{=} \|\lambda\, P_U(Av)\| = \lambda \cdot \|P_U(Av)\| \le \lambda \cdot \|Av\| = \lambda \;,$$

and hence $\lambda \geq 1$ holds. Therefore, there exists $t \in [0, \frac{\pi}{4}[$ with $\lambda = \frac{1}{\cos(2t)}$. Now Lemma 6.8 shows that U is a complex t-subspace, and (a) shows that the quadric $Q' = Q(A')$ coincides with $Q \cap [U]$.

Let us now consider the case where β is degenerate. We will show that then $\beta = 0$ holds; from this it follows that $P_U \circ A|U \equiv 0$ holds and therefore U is $\mathfrak{A}$-isotropic and hence a complex $\frac{\pi}{4}$-subspace by Example 6.4(b). Thus Q' is obtained by the construction of (b).

To show $\beta = 0$ we first note that because β is degenerate, there is some $v_0 \in U \setminus \{0\}$ so that

$$\forall v \in U \ : \ \beta(v, v_0) = 0 \tag{6.8}$$

holds.

Now consider the complex hyperplane $L := \{ v \in U \,|\, \beta'(v, v_0) = 0 \}$ of U and let $v \in U \setminus L$ be given. If we define $w_\lambda := \lambda\, v_0 + v$ for $\lambda \in \mathbb{C}$, we have

$$\beta'(w_\lambda, w_\lambda) = \beta'(v_0, v_0) \cdot \lambda^2 + 2\beta'(v, v_0) \cdot \lambda + \beta'(v, v) \tag{6.9}$$

and by Equation (6.8)

$$\beta(w_\lambda, w_\lambda) = \beta(v, v)\ . \tag{6.10}$$

Because of $v \notin L$ we have $\beta'(v, v_0) \neq 0$. Therefore Equation (6.9) implies the existence of some $\lambda_0 \in \mathbb{C}$ with $\beta'(w_{\lambda_0}, w_{\lambda_0}) = 0$. By Equations (6.6) and (6.10) it follows that $\beta(v, v) = 0$ holds. Thus, we have shown

$$\forall v \in U \setminus L \ :\ \beta(v, v) = 0\ . \tag{6.11}$$

Because L is a proper linear subspace of U, $U \setminus L$ is dense in U. Therefore (6.11) implies

$$\forall v \in U \ :\ \beta(v, v) = 0\ .$$

Because β is symmetric, we conclude $\beta = 0$. □

6.2 Properties of complex t-subspaces

We saw in Theorem 6.6 that for any complex t-subspace $U \subset \mathbb{V}$ with $t < \frac{\pi}{4}$, $Q \cap [U]$ is a subquadric of Q, and besides the subquadrics of Q which are contained in a complex projective subspace $\Lambda \subset \mathbb{P}(\mathbb{V})$ contained entirely in Q, all subquadrics of Q are obtained in this way. For this reason it is of interest to study the properties of complex t-subspaces, which we do in the present section.

The complex 0-subspaces and the complex $\frac{\pi}{4}$-subspaces of $\mathbb{V}$ are exactly the $\mathbb{C}$Q-subspaces resp. the complex isotropic subspaces of $\mathbb{V}$, as we already noted in Example 6.4; the properties of these spaces have been studied extensively in Sections 2.2 and 2.3. Thus we will restrict the following investigations to complex t-subspaces with $0 < t < \frac{\pi}{4}$. Where analogous statements for the cases $t = 0$ or $t = \frac{\pi}{4}$ give additional insight, we take note of this fact in a remark.

We continue to use the notations of the previous section. In particular $(\mathbb{V}, \mathfrak{A})$ is an n-dimensional $\mathbb{C}$Q-space and $Q := Q(\mathfrak{A})$ is the corresponding, $(m = n - 2)$-dimensional complex quadric.

6.9 Proposition. *Suppose* $0 < t < \frac{\pi}{4}$.

(a) Any complex t-subspace U of $\mathbb{V}$ is of complex dimension $\leq \frac{n}{2}$.

(b) Any complex t-subquadric Q' of Q is of complex dimension $\leq \frac{m}{2} - 1$.

6.10 Remark. Note that the statement of Proposition 6.9 is also true for $t = \frac{\pi}{4}$ (see Corollary 2.22), but not for $t = 0$.

Proof of Proposition 6.9. For (a). Let a a complex t-subspace U be given and fix $A \in \mathfrak{A}$. Then we have for every $u \in U \setminus \{0\}$: $\sphericalangle(Au, U) = 2t$ and therefore $Au \notin U$. This shows that $A(U) \cap U = \{0\}$ holds, and therefore we have $2 \dim U = \dim A(U) + \dim U = \dim(A(U) \oplus U) \leq \dim \mathbb{V} = n$.

For (b). Let Q' be a complex t-subquadric of Q. By Theorem 6.6, Q' is contained in $[U]$, where U is some complex t-subspace of $\mathbb{V}$. Therefore we have by (a): $\dim Q' = \dim U - 2 \leq \frac{m}{2} - 1$. □

In the sequel, we will consider the characteristic angle (in the sense of Section 2.5) of vectors with respect to varied $\mathbb{C}$Q-structures. Wherever $(\widetilde{\mathbb{V}}, \widetilde{\mathfrak{A}})$ is a $\mathbb{C}$Q-space and $v \in \widetilde{\mathbb{V}}$, we now denote the $\widetilde{\mathfrak{A}}$-angle of v by $\varphi_{\widetilde{\mathfrak{A}}}(v)$ for this reason.

6.11 Proposition. *Let $U \subset \mathbb{V}$ be a complex t-subspace with $0 < t < \frac{\pi}{4}$ and $\mathfrak{A}'$ the adapted $\mathbb{C}$Q-structure for U (see Definition 6.7). Then we have*

$$\forall v \in U \setminus \{0\} \ : \ \cos(2\varphi_{\mathfrak{A}}(v)) = \cos(2t) \cdot \cos(2\varphi_{\mathfrak{A}'}(v)) \ . \tag{6.12}$$

Proof. Without loss of generality, we may suppose $v \in \mathbb{S}(U)$. Fix $A \in \mathfrak{A}$ and let $A' \in \mathfrak{A}'$ be the adapted conjugation on U corresponding to A. We then have by Theorem 2.28(a) and Lemma 6.8

$$\begin{aligned} \cos(2\varphi_{\mathfrak{A}}(v)) &= |\langle v, Av\rangle_{\mathbb{C}}| \overset{(6.1)}{=} |\langle v, P_U Av\rangle_{\mathbb{C}}| \\ &= |\langle v, \cos(2t)A'v\rangle_{\mathbb{C}}| = \cos(2t) \cdot |\langle v, A'v\rangle_{\mathbb{C}}| = \cos(2t) \cdot \cos(2\varphi_{\mathfrak{A}'}(v)) \ . \end{aligned}$$

□

6.12 Corollary. *Let $U \subset \mathbb{V}$ be a complex t-subspace with $0 < t < \frac{\pi}{4}$ and $\mathfrak{A}'$ the adapted $\mathbb{C}$Q-structure for U. Then we have for every $v \in U \setminus \{0\}$*

$$\varphi_{\mathfrak{A}}(v) \geq t \tag{6.13}$$

and

$$\varphi_{\mathfrak{A}}(v) = t \iff \varphi_{\mathfrak{A}'}(v) = 0 \ , \tag{6.14}$$

$$\varphi_{\mathfrak{A}}(v) = \tfrac{\pi}{4} \iff \varphi_{\mathfrak{A}'}(v) = \tfrac{\pi}{4} \ . \tag{6.15}$$

Proof. Let $v \in U \setminus \{0\}$ be given. By Proposition 6.11 we have

$$\cos(2\varphi_{\mathfrak{A}}(v)) = \cos(2t) \cdot \cos(2\varphi_{\mathfrak{A}'}(v)) \leq \cos(2t) \ ,$$

therefrom (6.13) follows. Because of $\cos(2t) \neq 0$, also (6.14) and (6.15) are obvious consequences of Proposition 6.11. □

6.13 Theorem. *Let $k \in \mathbb{N}$ with $k \leq \frac{n}{2}$ and $0 < t < \frac{\pi}{4}$ be given. Then the following statements are equivalent for any k-dimensional complex subspace $U \subset \mathbb{V}$:*

(a) U is a complex t-subspace of $\mathbb{V}$.

(b) $t = \min_{v \in \mathbb{S}(U)} \varphi_{\mathfrak{A}}(v)$ and there exists a totally real k-dimensional subspace W of U with $\mathbb{S}(W) \subset M_t$.[16] (Note that $U = W \oplus JW$ holds in this situation.)

(c) There exist $A \in \mathfrak{A}$, a $2k$-dimensional linear subspace $W \subset V(A)$, an orthogonal complex structure $\tau : W \to W$ and a k-dimensional, totally real subspace Y of the unitary space (W, τ) so that $U = g_t(Y \oplus JY)$ holds with the $\mathbb{C}$-linear map

$$g_t : Y \oplus JY \to \mathbb{V}\,,\ z \mapsto \cos(t) z + \sin(t) J \tau^{\mathbb{C}} z\,;$$

here, $\tau^{\mathbb{C}} : W \oplus JW \to W \oplus JW$ denotes the complexification of the $\mathbb{R}$-linear endomorphism $\tau : W \to W$ with respect to the orthogonal complex structure $J|(W \oplus JW)$. Regarded as a map onto U, g_t is a $\mathbb{C}$-linear isometry.

Moreover, we have:

(i) U is tangential to M_t along $M_t \cap U$, meaning that for any $v \in M_t \cap U$ we have $T_v U \subset T_v M_t$.

(ii) The map $g_t : Y \oplus JY \to U$ from (c) is a $\mathbb{C}$Q-isomorphism; here we regard $Y \oplus JY$ as a $\mathbb{C}$Q-subspace of $\mathbb{V}$, and the complex t-subspace U as a $\mathbb{C}$Q-space with its (unique) adapted $\mathbb{C}$Q-structure $\mathfrak{A}'$.

6.14 Remark. The equivalence (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) in Theorem 6.13 is also true for $t = 0$ and $t = \frac{\pi}{4}$.

Proof for $t = 0$. Note that the complex 0-subspaces are exactly the $\mathbb{C}$Q-subspaces of $\mathbb{V}$, see Example 6.4(a). *For (a) $\Longrightarrow$ (b),* choose $W := U \cap V(A)$ with any $A \in \mathfrak{A}$. *For (b) $\Longrightarrow$ (a):* We show that under the hypothesis of (b), U is invariant under any given $A \in \mathfrak{A}$ and thus a $\mathbb{C}$Q-subspace. Let $v \in U = W \oplus JW$ be given, say $v = x_1 + Jx_2$ with $x_1, x_2 \in W$. Because of $\mathbb{S}(W) \subset M_0$ there exists $\lambda_\ell \in \mathbb{S}^1$ with $x_\ell \in V(\lambda_\ell A)$ for $\ell \in \{1,2\}$. We have $Ax_1 = \overline{\lambda_1}\,\lambda_1\,Ax_1 = \overline{\lambda_1}\,x_1 \in W \oplus JW = U$ and analogously $AJx_2 \in U$, therefrom $Av \in U$ follows. *For (a) $\Longrightarrow$ (c),* choose $A \in \mathfrak{A}$ arbitrarily, put $Y := V(A) \cap U$, let Y' be another k-dimensional subspace of $V(A)$ with $Y' \perp Y$ (remember $\dim V(A) = n \geq 2k$), put $W := Y \oplus Y'$ and let $\tau : W \to W$ be an orthogonal complex structure on W with $\tau(Y) = Y'$. In the setting of (c) we then have $Y \oplus JY = U$ and $g_{t=0} = \mathrm{id}_U$, hence $g_{t=0}(Y \oplus JY) = U$. *For (c) $\Longrightarrow$ (a):* In this situation we have $U = Y \oplus JY$ and therefore U is a $\mathbb{C}$Q-subspace of $\mathbb{V}$.

Proof for $t = \frac{\pi}{4}$. Note that U is a complex $\frac{\pi}{4}$-subspace if and only if it is an $\mathfrak{A}$-isotropic subspace (Example 6.4(b)), which is in turn the case if and only if $\varphi_{\mathfrak{A}}(v) = \frac{\pi}{4}$ holds for every $v \in U \setminus \{0\}$ (Proposition 2.29(b)). Therefrom the equivalence (a) $\Longleftrightarrow$ (b) follows (for (a) $\Longrightarrow$ (b), W can be chosen as any maximal totally real subspace of U). (a) $\Longleftrightarrow$ (c) follows from Proposition 2.20(e),(f). □

Therefore every selection of data (A, W, τ, Y) as in Theorem 6.13(c) gives rise to a series $(U_t := g_t(Y \oplus JY))_{t \in [0, \frac{\pi}{4}]}$ of k-dimensional complex t-subspaces, where g_t is defined as in the theorem.

[16] Here M_t denotes the orbit of the action of $\mathrm{Aut}(\mathfrak{A})$ on $\mathbb{S}(\mathbb{V})$ consisting of the unit vectors of $\mathfrak{A}$-angle t, see Proposition 2.36.

Moreover every k-dimensional complex t-subspace $U \subset \mathbb{V}$ is a member of such a series (by virtue of the implication (a) $\Longrightarrow$ (c)).

A similar statement holds on the level of complex subquadrics of Q: If $k \geq 3$ holds in the previous situation, then $Q'_0 := Q \cap [U_0]$ is a $(k-2)$-dimensional complex 0-subquadric of Q, and via the linear isometry g_t we obtain the series $(Q'_t := \underline{g_t}(Q'_0))_{t\in[0,\frac{\pi}{4}]}$ of $(k-2)$-dimensional complex t-subquadrics of Q. Note that the subquadrics Q'_t with $t < \frac{\pi}{4}$ can also be described as $Q'_t = Q \cap [U_t]$ without explicit reference to g_t, however $Q'_{\pi/4}$ cannot be obtained in this way. Moreover every $(k-2)$-dimensional subquadric of Q is a member of such a series.

Proof of the last statement. Let a $(k-2)$-dimensional subquadric Q' of Q be given; by Theorem 6.6(c) there exists $t_0 \in [0, \frac{\pi}{4}]$ and a k-dimensional complex t_0-subspace $U \subset \mathbb{V}$ so that Q' is a complex quadric in $[U]$. In the case $t_0 < \frac{\pi}{4}$ we have $Q' = Q \cap [U]$ by Theorem 6.6(a). It follows that if we define the data (A, W, τ, Y) such that the corresponding series $(U_t)_{t\in[0,\frac{\pi}{4}]}$ of complex t-subspaces satisfies $U_{t_0} = U$, then the series $(Q'_t)_{t\in[0,\frac{\pi}{4}]}$ of complex t-subquadrics satisfies $Q'_{t_0} = Q'$.

It remains to consider the case $t_0 = \frac{\pi}{4}$. Then U is $\mathfrak{A}$-isotropic (Example 6.4(b)) and by Theorem 6.6(b) there exists a conjugation $A' \in \mathrm{Con}(U)$ so that $Q' = Q(A')$ holds. $V(A')$ is a totally real, k-dimensional, $\mathfrak{A}$-isotropic subspace of $\mathbb{V}$; it follows by Proposition 2.20(e),(f) that after a choice of some $A \in \mathfrak{A}$ there exist orthogonal, k-dimensional subspaces $Y, Y' \subset V(A)$ and an $\mathbb{R}$-linear isometry $\tau : Y \to Y'$ so that

$$V(A') = \{\, x + J\tau x \,|\, x \in Y \,\} \tag{6.16}$$

holds. We put $W := Y \oplus Y'$ and denote the unique extension of τ to an orthogonal complex structure on W also by $\tau : W \to W$. If we now define the functions g_t from Theorem 6.13(c) relative to this situation, we have $g_{\frac{\pi}{4}}(Y) = V(A')$ by Equation (6.16), therefrom $g_{\frac{\pi}{4}}(Y \oplus JY) = U$ and $g_{\frac{\pi}{4}} \circ (A|(Y \oplus JY)) = A' \circ g_{\frac{\pi}{4}}$ follows. This shows that if we define the series $(Q'_t)_{t\in[0,\frac{\pi}{4}]}$ with respect to the data (A, W, τ, Y), we have $Q'_{\frac{\pi}{4}} = Q(A') = Q'$. □

For the proof of Theorem 6.13 we will need the following lemma:

6.15 Lemma. *Let a k-dimensional complex subspace $U \subset \mathbb{V}$ and $0 < t < \frac{\pi}{4}$ be given. Then U is a complex t-subspace of $\mathbb{V}$ if and only if there exists a unitary basis $(v_1, \ldots, v_k)$ of U so that*

$$\forall \ell \;:\; \varphi_{\mathfrak{A}}(v_\ell) = t \quad \textit{and} \quad \forall \ell \neq \ell' \;:\; \langle v_\ell, A v_{\ell'} \rangle_{\mathbb{C}} = 0 \tag{6.17}$$

holds for some (and then for every) $A \in \mathfrak{A}$.

If this is the case, the v_ℓ can be chosen such that a fixed $A \in \mathfrak{A}$ is adapted to all v_ℓ in the sense of Theorem 2.28(b), and then $(x_1, \ldots, x_k, y_1, \ldots, y_k)$ with $x_\ell := \frac{\mathrm{Re}_A v_\ell}{\cos t}$ and $y_\ell := \frac{\mathrm{Im}_A v_\ell}{\sin t}$ is an orthonormal system in $V(A)$.

Proof of Lemma 6.15. First, let us suppose that U is a complex t-space; we denote its adapted $\mathbb{C}Q$-structure by $\mathfrak{A}'$. We let $A \in \mathfrak{A}$ be fixed, let $A' \in \mathfrak{A}'$ be the adapted conjugation on U corresponding to A, and let $(v_1, \ldots, v_k)$ be an orthonormal basis of $V(A')$. Then $(v_1, \ldots, v_k)$ also is a unitary basis of U, and we will show that it satisfies (6.17).

We have $\varphi_{\mathfrak{A}'}(v_\ell) = 0$ (where $\ell \in \{1, \ldots, k\}$) and therefore $\varphi_{\mathfrak{A}}(v_\ell) = t$ by Corollary 6.12. Moreover, for $\ell \neq \ell'$ we have by Lemma 6.8

$$\langle v_\ell, A v_{\ell'} \rangle_{\mathbb{C}} \overset{(6.1)}{=} \langle v_\ell, P_U\, A v_{\ell'} \rangle_{\mathbb{C}} = \cos(2t) \cdot \langle v_\ell, A' v_{\ell'} \rangle_{\mathbb{C}} \overset{v_{\ell'} \in V(A')}{=} \cos(2t) \cdot \langle v_\ell, v_{\ell'} \rangle_{\mathbb{C}} = 0 \,.$$

This shows that (6.17) holds.

Conversely, we suppose that $(v_1, \dots, v_k)$ is a unitary basis of U so that (6.17) holds. We fix $A \in \mathfrak{A}$, then we have for every $j \in \{1, \dots, k\}$

$$\|P_U(Av_j)\|^2 = \sum_{\ell=1}^{k} |\langle Av_j, v_\ell \rangle_{\mathbb{C}}|^2 \overset{(6.17)}{=} |\langle Av_j, v_j \rangle_{\mathbb{C}}|^2 = |\langle v_j, Av_j \rangle_{\mathbb{C}}|^2 \overset{(2.19)}{=} \cos(2t)^2 \tag{6.18}$$

and for every $j, j' \in \{1, \dots, k\}$ with $j \neq j'$

$$\begin{aligned} \langle P_U(Av_j), P_U(Av_{j'}) \rangle_{\mathbb{C}} &= \left\langle \sum_{\ell=1}^{k} \langle Av_j, v_\ell \rangle_{\mathbb{C}} v_\ell \,,\, \sum_{\ell'=1}^{k} \langle Av_{j'}, v_{\ell'} \rangle_{\mathbb{C}} v_{\ell'} \right\rangle \\ &= \sum_{\ell,\ell'=1}^{k} \underbrace{\langle Av_j, v_\ell \rangle_{\mathbb{C}}}_{=0 \text{ for } \ell \neq j} \cdot \underbrace{\overline{\langle Av_{j'}, v_{\ell'} \rangle_{\mathbb{C}}}}_{=0 \text{ for } \ell' \neq j'} \cdot \underbrace{\langle v_\ell, v_{\ell'} \rangle_{\mathbb{C}}}_{=\delta_{\ell,\ell'}} \overset{j \neq j'}{=} 0 \,. \end{aligned} \tag{6.19}$$

We now let $v \in U \setminus \{0\}$ be given, say $v = \sum_{j=1}^{k} \lambda_j \, v_j$ with $\lambda_j := \langle v, v_j \rangle_{\mathbb{C}}$. Then we have

$$P_U(Av) = \sum_{j=1}^{k} \overline{\lambda_j} \, P_U(Av_j)$$

and therefore

$$\|P_U(Av)\|^2 \overset{(6.19)}{=} \sum_{j=1}^{k} |\lambda_j|^2 \, \|P_U(Av_j)\|^2 \overset{(6.18)}{=} \cos(2t)^2 \cdot \sum_{j=1}^{k} |\lambda_j|^2 = \cos(2t)^2 \cdot \|v\|^2 \,,$$

whence $\sphericalangle(Av, U) = 2t$ follows. This shows that U is a complex t-subspace, see Definition 6.3.

Next we note that if some $A \in \mathfrak{A}$ is adapted to some $v \in \mathbb{V} \setminus \{0\}$ and $\mu \in \mathbb{S}^1$ is given, then $\mu^2 A$ is adapted to μv by Corollary 2.35. From this fact it follows that for every fixed $A \in \mathfrak{A}$ and any unitary basis $(v_1, \dots, v_k)$ of U there exist $\lambda_1, \dots, \lambda_k \in \mathbb{S}^1$ so that A is adapted to all $\lambda_\ell \, v_\ell$. Moreover, if the basis $(v_1, \dots, v_k)$ satisfies (6.17), then $(\lambda_1 v_1, \dots, \lambda_k v_k)$ also satisfies (6.17). Thus we see that if U is a complex t-subspace, then the unitary basis $(v_1, \dots, v_k)$ satisfying (6.17) can be chosen such that A is adapted to all v_ℓ. In the following we suppose that this situation is present.

We put $x_\ell := \frac{\mathrm{Re}_A \, v_\ell}{\cos t}$ and $y_\ell := \frac{\mathrm{Im}_A \, v_\ell}{\sin t}$; then we get with Proposition 2.3(e)

$$v_\ell = \cos(t) x_\ell + \sin(t) J y_\ell \,; \tag{6.20}$$

moreover from (6.17) and Theorem 2.28(c) we obtain

$$\|x_\ell\| = \|y_\ell\| = 1 \tag{6.21}$$

and (where we abbreviate $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\mathbb{R}}$)

$$\langle x_\ell, y_\ell \rangle = 0 \,. \tag{6.22}$$

This shows that Equation (6.20) is a canonical representation for v in the sense of Theorem 2.28(c).

Now let ℓ, ℓ' with $\ell' \neq \ell$ be given. Because $(v_1, \dots, v_k)$ is a unitary basis, we have

$$0 = \langle v_\ell, v_{\ell'} \rangle_{\mathbb{C}} = \cos(t)^2 \langle x_\ell, x_{\ell'} \rangle + \sin(t)^2 \langle y_\ell, y_{\ell'} \rangle \ + \ i \cdot \cos(t) \sin(t) \left(\langle y_\ell, x_{\ell'} \rangle - \langle x_\ell, y_{\ell'} \rangle \right) ,$$

and thus (note $\cos(t) \sin(t) \neq 0$)

$$\cos(t)^2 \langle x_\ell, x_{\ell'} \rangle + \sin(t)^2 \langle y_\ell, y_{\ell'} \rangle = 0 , \tag{6.23}$$

$$\langle y_\ell, x_{\ell'} \rangle - \langle x_\ell, y_{\ell'} \rangle = 0 . \tag{6.24}$$

From (6.17) we further obtain

$$0 = \langle v_\ell, A v_{\ell'} \rangle_{\mathbb{C}} = \cos(t)^2 \langle x_\ell, x_{\ell'} \rangle - \sin(t)^2 \langle y_\ell, y_{\ell'} \rangle \ + \ i \cdot \cos(t) \sin(t) \left(\langle y_\ell, x_{\ell'} \rangle + \langle x_\ell, y_{\ell'} \rangle \right) ,$$

and this equation implies

$$\cos(t)^2 \langle x_\ell, x_{\ell'} \rangle - \sin(t)^2 \langle y_\ell, y_{\ell'} \rangle = 0 , \tag{6.25}$$

$$\langle y_\ell, x_{\ell'} \rangle + \langle x_\ell, y_{\ell'} \rangle = 0 . \tag{6.26}$$

From Equations (6.23) and (6.25) we see

$$\langle x_\ell, x_{\ell'} \rangle = \langle y_\ell, y_{\ell'} \rangle = 0 , \tag{6.27}$$

whereas from Equations (6.24) and (6.26) we conclude

$$\langle x_\ell, y_{\ell'} \rangle = 0 . \tag{6.28}$$

Equations (6.21), (6.22), (6.27) and (6.28) exactly state the fact that $(x_1, \dots, x_k, y_1, \dots, y_k)$ is an orthonormal system in $V(A)$. □

Proof of Theorem 6.13. We prove the theorem in the following order: (a) ⇒ (c), (ii), (c) ⇒ (b), (i), (b) ⇒ (a).

For (a) ⇒ (c). Let U be a complex t-subspace and fix $A \in \mathfrak{A}$. By Lemma 6.15 there exists a unitary basis $(v_1, \dots, v_k)$ of U which satisfies (6.17) and such that A is adapted to all v_ℓ. We put $x_\ell := \frac{\mathrm{Re}_A v_\ell}{\cos t}$, $y_\ell := \frac{\mathrm{Im}_A v_\ell}{\sin t}$. Again by Lemma 6.15, $(x_1, \dots, x_k, y_1, \dots, y_k)$ is an orthonormal system in $V(A)$, and hence

$$W := \mathrm{span}_{\mathbb{R}}\{x_1, \dots, x_k, y_1, \dots, y_k\}$$

is a $2k$-dimensional linear subspace of $V(A)$. The linear map $\tau : W \to W$ defined by

$$\forall \ell \ : \ \tau(x_\ell) = y_\ell, \ \ \tau(y_\ell) = -x_\ell$$

is an orthogonal complex structure on W and

$$Y := \mathrm{span}_{\mathbb{R}}\{x_1, \dots, x_k\}$$

is a k-dimensional, totally real subspace of the unitary space (W, τ). Using these data, we define g_t as in the theorem. It is clear that g_t is $\mathbb{C}$-linear. Moreover, it transforms the unitary basis $(x_1, \dots, x_k)$ of $Y \oplus JY$ into the unitary basis $(v_1, \dots, v_k)$ of U. This shows that $g_t(Y \oplus JY) = U$ holds and that $g_t : Y \oplus JY \to U$ is a $\mathbb{C}$-linear isometry.

For (ii). Remaining in the situation of the proof of (c), it suffices to show that $A' := g_t \circ A \circ g_t^{-1} : U \to U$ is an adapted conjugation for the complex t-space U corresponding to $A \in \mathfrak{A}$, meaning that

$$\forall v \in U \; : \; \cos(2t) \cdot A'v = P_U(Av) \tag{6.29}$$

holds, see Lemma 6.8. Because both sides of this equation are anti-linear, it suffices to verify it for $v = v_j$, $j \in \{1, \dots, k\}$. We have $v_j = g_t(x_j)$ and therefore

$$A'v_j = g_t(Ax_j) = g_t(x_j) = v_j \; ;$$

on the other hand we have

$$P_U(Av_j) = \sum_{\ell=1}^{k} \langle Av_j, v_\ell \rangle_{\mathbb{C}} \, v_\ell \overset{(6.17)}{=} \langle Av_j, v_j \rangle_{\mathbb{C}} \, v_j \overset{(*)}{=} \cos(2t) \, v_j \, ,$$

for the equals sign marked $(*)$ we use the canonical representation (6.20) of v_j. This shows that Equation (6.29) holds.

For (c) $\Rightarrow$ (b). We suppose that the situation of (c) is present. Then we have for every $z \in \mathbb{S}(Y \oplus JY)$ by Theorem 2.28(a)

$$\begin{aligned}
\cos(2\varphi_{\mathfrak{A}}(g_t(z))) &= |\langle g_t(z), A(g_t(z)) \rangle_{\mathbb{C}}| = |\langle \cos(t) z + \sin(t) J\tau^{\mathbb{C}} z \, , \, \cos(t) Az - \sin(t) J\tau^{\mathbb{C}} Az \rangle_{\mathbb{C}}| \\
&= |\cos(t)^2 \, \langle z, Az \rangle_{\mathbb{C}} + \cos(t)\sin(t) \cdot \underbrace{(-\langle z, J\tau^{\mathbb{C}} Az \rangle_{\mathbb{C}} + \langle J\tau^{\mathbb{C}} z, Az \rangle_{\mathbb{C}})}_{=0} \\
&\qquad - \sin(t)^2 \, \underbrace{\langle J\tau^{\mathbb{C}} z, J\tau^{\mathbb{C}} Az \rangle_{\mathbb{C}}}_{=\langle z, Az \rangle_{\mathbb{C}}} | \\
&= |(\cos(t)^2 - \sin(t)^2) \cdot \langle z, Az \rangle_{\mathbb{C}}| = \cos(2t) \cdot |\langle z, Az \rangle_{\mathbb{C}}| = \cos(2t) \cdot \cos(2\varphi_{\mathfrak{A}}(z)) \, .
\end{aligned}$$

It follows from this calculation that we have on one hand

$$\forall z \in \mathbb{S}(Y \oplus JY) \; : \; \cos(2\varphi_{\mathfrak{A}}(g_t(z)) \leq \cos(2t)$$

and therefore because g_t is a linear isometry onto U:

$$\forall v \in \mathbb{S}(U) \; : \; \varphi_{\mathfrak{A}}(v) \geq t \; ; \tag{6.30}$$

on the other hand, for every $z \in \mathbb{S}(Y)$ we have $\varphi_{\mathfrak{A}}(z) = 0$ and therefore by the same calculation $\varphi_{\mathfrak{A}}(g_t(z)) = t$. This shows that the k-dimensional totally real subspace $g_t(Y)$ of U has the property $\mathbb{S}(g_t(Y)) \subset M_t$. This fact, together with (6.30), also implies $\min_{v \in \mathbb{S}(U)} \varphi_{\mathfrak{A}}(v) = t$.

For (i). Suppose that the situation of (b) holds. Let $v \in M_t \cap U$ and $w \in T_v U$ be given. Then the line parametrization $\gamma : \mathbb{R} \to U, t \mapsto v + t \cdot \overrightarrow{w}$ satisfies $\gamma(0) = v$ and $\dot{\gamma}(0) = w$. Furthermore, the function $f := \varphi_{\mathfrak{A}} \circ \gamma :]-\varepsilon, \varepsilon[\to [0, \frac{\pi}{4}]$ satisfies $f(0) = t$ and is differentiable in 0 by Proposition 2.30. Because of $t = \min_{v \in \mathbb{S}(U)} \varphi_{\mathfrak{A}}(v) = \min_{v \in U \setminus \{0\}} \varphi_{\mathfrak{A}}(v)$, f attains a local minimum in 0 and therefore we have

$$0 = f'(0) = \overrightarrow{T_v \varphi_{\mathfrak{A}}(\dot{\gamma}(0))} = \overrightarrow{T_v \varphi_{\mathfrak{A}}(w)}$$

and hence $w \in \ker(T_v \varphi_{\mathfrak{A}}) = T_v M_t$ (see Proposition 2.38).

For (b) ⇒ (a). We now suppose that $t = \min_{v\in\mathbb{S}(U)} \varphi_{\mathfrak{A}}(v)$ holds and that there exists a k-dimensional totally real subspace W of U with $\mathbb{S}(W) \subset M_t$. Consequently, we have

$$\forall v \in U \setminus \{0\} \;:\; \varphi_{\mathfrak{A}}(v) \geq t \;,$$

whence it follows that

$$\forall v \in U \setminus \{0\} \;:\; \cos(2\,\varphi_{\mathfrak{A}}(v)) \leq \cos(2t) \tag{6.31}$$

holds.

Let us fix an orthonormal basis $(v_1, \dots, v_k)$ of W, then $(v_1, \dots, v_k)$ also is a unitary basis of U. We will show that this unitary basis satisfies (6.17); thus U is a complex t-subspace of $\mathbb{V}$ according to Lemma 6.15.

Because of $\mathbb{S}(W) \subset M_t$ we have

$$\forall \ell \;:\; \varphi_{\mathfrak{A}}(v_\ell) = t \;. \tag{6.32}$$

Now suppose ℓ, ℓ' are given with $\ell \neq \ell'$ and choose $\lambda, \mu \in \mathbb{S}^1$ so that some $A \in \mathfrak{A}$ is adapted to both λv_ℓ and $\mu v_{\ell'}$. Then we have because of Theorem 2.28(b)

$$\langle \lambda v_\ell, A\lambda v_\ell \rangle_{\mathbb{C}} = \langle \mu v_{\ell'}, A\mu v_{\ell'} \rangle_{\mathbb{C}} = \cos(2t) \;. \tag{6.33}$$

Furthermore, we choose $\varepsilon \in \{\pm 1\}$ so that

$$\varepsilon \cdot \langle \lambda v_\ell, A\mu v_{\ell'} \rangle_{\mathbb{R}} \geq 0 \tag{6.34}$$

holds. Then the vector $w := \lambda v_\ell + \varepsilon \mu v_{\ell'} \in U$ satisfies $\|w\|^2 = 2$ and by means of Theorem 2.28 we have (for $(*)$ note that the $\mathbb{C}$-bilinear form $\langle\,\cdot\,, A\,\cdot\,\rangle_{\mathbb{C}}$ is symmetric)

$$\begin{aligned}
\langle w, Aw \rangle_{\mathbb{C}} &\overset{(*)}{=} \langle \lambda v_\ell, A\lambda v_\ell \rangle_{\mathbb{C}} + 2 \cdot \langle \lambda v_\ell, \varepsilon A\mu v_{\ell'} \rangle_{\mathbb{C}} + \varepsilon^2 \cdot \langle \mu v_{\ell'}, A\mu v_{\ell'} \rangle_{\mathbb{C}} \\
&\overset{(6.33)}{=} 2\cdot\big(\cos(2t) + \varepsilon \cdot \langle \lambda v_\ell, A\mu v_{\ell'} \rangle_{\mathbb{C}} \big) \\
&= 2\cdot\big(\cos(2t) + \varepsilon \cdot \langle \lambda v_\ell, A\mu v_{\ell'} \rangle_{\mathbb{R}} + i\varepsilon \cdot \langle \lambda v_\ell, JA\mu v_{\ell'} \rangle_{\mathbb{R}} \big) \;.
\end{aligned} \tag{6.35}$$

Consequently, we obtain (again using Theorem 2.28 and the fact that $\cos(2t) \geq 0$ holds)

$$\begin{aligned}
(2 \cdot \cos(2t))^2 &\overset{(6.31)}{\geq} (2 \cdot \cos(2\varphi_{\mathfrak{A}}(w)))^2 = |\langle w, Aw \rangle_{\mathbb{C}}|^2 \\
&\overset{(6.35)}{=} 4\cdot\big| \cos(2t) + \varepsilon\, \langle \lambda v_\ell, A\mu v_{\ell'} \rangle_{\mathbb{R}} + i\, \varepsilon \langle \lambda v_\ell, JA\mu v_{\ell'} \rangle_{\mathbb{R}} \big|^2 \\
&= 4\cdot\big((\cos(2t) + \underbrace{\varepsilon\, \langle \lambda v_\ell, A\mu v_{\ell'} \rangle_{\mathbb{R}}}_{\underset{(6.34)}{\geq}\, 0})^2 + \underbrace{(\varepsilon\, \langle \lambda v_\ell, JA\mu v_{\ell'} \rangle_{\mathbb{R}})^2}_{\geq 0}\big) \\
&\geq 4 \cdot \cos(2t)^2 = (2 \cdot \cos(2t))^2 \;.
\end{aligned}$$

Therefore all inequalities in the above chain of inequalities are in fact equalities. It follows that

$$\langle \lambda v_\ell, A\mu v_{\ell'} \rangle_{\mathbb{R}} = \langle \lambda v_\ell, JA\mu v_{\ell'} \rangle_{\mathbb{R}} = 0$$

and thus

$$\lambda\mu \cdot \langle v_\ell, Av_{\ell'} \rangle_{\mathbb{C}} = \langle \lambda v_\ell, A\mu v_{\ell'} \rangle_{\mathbb{C}} = \langle \lambda v_\ell, A\mu v_{\ell'} \rangle_{\mathbb{R}} + i \cdot \langle \lambda v_\ell, JA\mu v_{\ell'} \rangle_{\mathbb{R}} = 0 \tag{6.36}$$

holds. Equations (6.32) and (6.36) show that the unitary basis $(v_1, \dots, v_k)$ satisfies (6.17). □

In the following proposition we see that for fixed $0 < t < \frac{\pi}{4}$ and $k \leq \frac{n}{2}$, the set of k-dimensional complex t-subspaces of $\mathbb{V}$ is an orbit of the canonical action of the group $\mathrm{Aut}(\mathfrak{A}) \cup \overline{\mathrm{Aut}}(\mathfrak{A})$ (see Definition 2.10(a),(c) and Remark 2.12(b)) on the complex Grassmannian $G_k(\mathbb{V})$; here $\mathrm{Aut}(\mathfrak{A})$ is the subgroup of $\mathbb{C}$Q-isomorphisms of $\mathbb{V}$ and $\overline{\mathrm{Aut}}(\mathfrak{A})$ is the coset of $\mathbb{C}$Q-anti-isomorphisms of $\mathbb{V}$. Note that $\overline{\mathrm{Aut}}(\mathfrak{A}) = \{\, A \circ B \,|\, B \in \mathrm{Aut}(\mathfrak{A})\,\}$ holds for $A \in \mathfrak{A}$. In fact already $\mathrm{Aut}(\mathfrak{A})$ acts transitively on the mentioned orbit.

Also, we will see in Corollary 6.17 that for $m \neq 2$ the set of $(k-2)$-dimensional complex t-subquadrics of Q is an orbit of the canonical action of the isometry group $I(Q)$ on the set of all subquadrics of Q, and already $I_h(Q)$ acts transitively on this orbit.

6.16 Proposition. *Suppose $0 < t < \frac{\pi}{4}$ and $k \leq \frac{n}{2}$.*

(a) Let a k-dimensional complex t-subspace U of $\mathbb{V}$ and $B \in \mathrm{Aut}(\mathfrak{A}) \cup \overline{\mathrm{Aut}}(\mathfrak{A})$ be given. Then $B(U)$ is another complex t-subspace of $\mathbb{V}$, and if $A' \in \mathrm{Con}(U)$ is an adapted conjugation for U corresponding to $A \in \mathfrak{A}$, then $BA'B^{-1}|B(U) \in \mathrm{Con}(B(U))$ is an adapted conjugation for $B(U)$ corresponding to $BAB^{-1} \in \mathfrak{A}$.

(b) If U_1 and U_2 are two k-dimensional complex t-subspaces of $\mathbb{V}$, then there exists $B \in \mathrm{Aut}(\mathfrak{A})$ so that $U_2 = B(U_1)$ holds. If A_1' and A_2' are adapted conjugations for U_1 and U_2 respectively, B can be chosen so that $A_2' = BA_1'B^{-1}|U_2$ holds.

Proof. For (a). Let $A \in \mathfrak{A}$ be given and let A' be the adapted conjugation for U corresponding to A, then we have

$$\cos(2t) \cdot A' = (P_U \circ A)|U \tag{6.37}$$

(see Lemma 6.8). We also let $B \in \mathrm{Aut}(\mathfrak{A}) \cup \overline{\mathrm{Aut}}(\mathfrak{A})$ be given and put $A'' := BA'B^{-1}|B(U)$. Then we have $A'' \in \mathrm{Con}(B(U))$ (this is true even for $B \in \overline{\mathrm{Aut}}(\mathfrak{A})$), and for every $v \in B(U)$:

$$\cos(2t) \cdot A''(v) = B\big(\cos(2t) \cdot A'(\underbrace{B^{-1}v}_{\in U})\big) \overset{(6.37)}{=} B\big(P_U \circ A(B^{-1}v)\big) = P_{B(U)}(BAB^{-1}(v)) .$$

We now show $BAB^{-1} \in \mathfrak{A}$; from Lemma 6.8 it follows that $B(U)$ is a complex t-subspace and A'' is an adapted conjugation corresponding to BAB^{-1}. In the case $B \in \mathrm{Aut}(\mathfrak{A})$, $BAB^{-1} \in \mathfrak{A}$ follows from the definition of a $\mathbb{C}$Q-isomorphism. In the case $B \in \overline{\mathrm{Aut}}(\mathfrak{A})$, there exists $B' \in \mathrm{Aut}(\mathfrak{A})$ so that $B = AB'$ holds. Because of $B'A(B')^{-1} \in \mathfrak{A}$ there exists $\lambda \in \mathbb{S}^1$ so that $B'A(B')^{-1} = \lambda A$ holds, and then we have

$$BAB^{-1} = (AB')\,A\,(AB')^{-1} = AB'A(B')^{-1}A = A(\lambda A)A = \overline{\lambda}A \in \mathfrak{A} .$$

For (b). Let two k-dimensional complex t-subspaces U_1 and U_2 be given, and let A_1' and A_2' be adapted conjugations for U_1 resp. U_2. Thus there exist $A_1, A_2 \in \mathfrak{A}$ so that

$$\cos(2t) \cdot A_\ell' = (P_{U_\ell} \circ A_\ell)|U_\ell \tag{6.38}$$

holds for $\ell \in \{1, 2\}$.

Now we fix orthonormal bases $(u_1, \dots, u_k)$ of $V(A_1')$ and $(u_1', \dots, u_k')$ of $V(A_2')$, and we put for $\ell \in \{1, \dots, k\}$

$$x_\ell := \frac{\mathrm{Re}_{A_1} u_\ell}{\cos t}\,, \quad y_\ell := \frac{\mathrm{Im}_{A_1} u_\ell}{\sin t}\,, \quad x_\ell' := \frac{\mathrm{Re}_{A_2} u_\ell'}{\cos t} \quad \text{and} \quad y_\ell' := \frac{\mathrm{Im}_{A_2} u_\ell'}{\sin t}\,.$$

By Lemma 6.15 we see that $(x_1, \dots, x_k, y_1, \dots, y_k)$ and $(x_1', \dots, x_k', y_1', \dots, y_k')$ are orthonormal systems in $V(A_1)$ resp. $V(A_2)$. We choose an $\mathbb{R}$-linear isometry $L : V(A_1) \to V(A_2)$ with

$$L(x_\ell) = x_\ell' \quad \text{and} \quad L(y_\ell) = y_\ell'$$

for every ℓ, then the complexification $B := L^{\mathbb{C}} : \mathbb{V} \to \mathbb{V}$ of L is a $\mathbb{C}$Q-automorphism with $B(u_\ell) = u_\ell'$, consequently $B(U_1) = U_2$ and $BA_1'B^{-1}|U_2 = B_2'$ holds. □

6.17 Corollary. *Suppose $0 < t < \frac{\pi}{4}$ and $k \leq \frac{n}{2}$.*

(a) For any complex t-subquadric Q' of Q and any $f \in I_h(Q) \cup I_{ah}(Q)$, $f(Q')$ is another complex t-subquadric of Q.[17]

(b) If Q_1' and Q_2' are two $(k-2)$-dimensional complex t-subquadrics of Q, then there exists $f \in I_h(Q)$ with $Q_2' = f(Q_1')$.

Proof. For (a). By Theorem 6.6(c),(a) there exists a complex t-subspace $U \subset \mathbb{V}$ so that $Q' = Q \cap [U]$ holds. For any given $f \in I_h(Q) \cup I_{ah}(Q)$ there exists $B \in \mathrm{Aut}(\mathfrak{A}) \cup \overline{\mathrm{Aut}}(\mathfrak{A})$ with $f = \underline{B}|Q$ by Theorem 3.23(a),(b). Then $B(U)$ is another complex t-subspace by Proposition 6.16(a), and therefore Theorem 6.6(a) shows that $f(Q') = \underline{B}(Q') = \underline{B}(Q \cap [U]) = Q \cap [B(U)]$ is another complex t-subquadric of Q.

For (b). Again by Theorem 6.6(c),(a) there exist k-dimensional complex t-subspaces $U_1, U_2 \subset \mathbb{V}$ so that $Q_\ell' = Q \cap [U_\ell]$ holds for $\ell \in \{1, 2\}$. By Proposition 6.16(b) there exists $B \in \mathrm{Aut}(\mathfrak{A})$ with $U_2 = B(U_1)$, then we have $f := \underline{B}|Q \in I_h(Q)$ by Proposition 3.2(a) and $f(Q_1') = \underline{B}(Q \cap [U_1]) = Q \cap [B(U_1)] = Q \cap [U_2] = Q_2'$. □

6.18 Remark. The statements of Proposition 6.16 and Corollary 6.17 are also true for $t = 0$ and $t = \frac{\pi}{4}$.

6.3 Extrinsic geometry of subquadrics

As before, we suppose that $(\mathbb{V}, \mathfrak{A})$ is an $(n = m + 2)$-dimensional $\mathbb{C}$Q-space. Moreover, we let a k-dimensional subquadric Q' of the m-dimensional complex quadric $Q := Q(\mathfrak{A})$ be given. Then Theorem 6.6(c) shows that Q' is a complex t-subquadric for some $0 \leq t \leq \frac{\pi}{4}$, meaning that there exists a $(k+2)$-dimensional complex t-subspace $U \subset \mathbb{V}$ so that Q' is a (symmetric) complex quadric in $\mathbb{P}(U)$ in the sense of Chapter 1.

For $t \in \{0, \frac{\pi}{4}\}$ we already have a complete overview of the extrinsic geometry of Q' as a submanifold of Q:

[17] Note that for $m \neq 2$, $I_h(Q) \cup I_{ah}(Q) = I(Q)$ holds by Theorem 3.23(c).

(a) In the case $t = 0$, U is a $\mathbb{C}$Q-subspace of $\mathbb{V}$ and $Q' = Q \cap [U]$ is a totally geodesic submanifold of Q of type $(\mathrm{G}1, k)$ by Lemma 5.8.

(b) In the case $t = \frac{\pi}{4}$, U is $\mathfrak{A}$-isotropic, and therefore the complex projective space $[U]$ is a totally geodesic submanifold of Q of type $(\mathrm{I}1, k+1)$. Therefore the embedding $Q' \hookrightarrow Q$ is equal to the composition of the embedding $Q' \hookrightarrow \mathbb{P}(U) = [U]$ which has been studied in Chapter 1 and the totally geodesic embedding $[U] \hookrightarrow Q$. In this way we see in particular that the submanifold Q' of Q has parallel second fundamental form, but is not totally geodesic.

For this reason we again restrict the following investigations to the case

$$\boxed{0 < t < \tfrac{\pi}{4}}\,.$$

Then we have $Q' = Q \cap [U]$ by Theorem 6.6(a).

In the sequel, we denote for any Riemannian manifold M by ∇^M its Levi-Civita covariant derivative. Also, if N is another Riemannian manifold and $f : M \to N$ an isometric immersion, we denote the shape operator of f by A^f and the second fundamental form of f by h^f.

Q' is a complex quadric in $\mathbb{P}(U)$ in the sense of Chapter 1, therefore the results of that chapter concerning the extrinsic geometry of complex quadrics are applicable to Q' as a submanifold of $\mathbb{P}(U) = [U]$. Because $[U]$ is a totally geodesic submanifold of $\mathbb{P}(\mathbb{V})$ we thereby also understand the geometry of Q' as a submanifold of $\mathbb{P}(\mathbb{V})$. In particular we have

$$h^{Q' \hookrightarrow [U]} = h^{Q' \hookrightarrow \mathbb{P}(\mathbb{V})}\,. \tag{6.39}$$

But now, we wish to study the geometry of Q' regarded as a submanifold of Q.

For this purpose we fix some $A \in \mathfrak{A}$ and consider the adapted conjugation A' of the complex t-subspace U corresponding to A (see Theorem 6.6(a)); then we have $Q = Q(A)$ and $Q' = Q(A')$. Furthermore, we consider the following objects (see also Sections 1.2 and 1.3):

- The manifolds $\widetilde{Q} := \widetilde{Q}(A)$ and $\widetilde{Q}' := \widetilde{Q}(A')$.
- The Hopf fibration $\pi : \mathbb{S}(\mathbb{V}) \to \mathbb{P}(\mathbb{V}),\ z \mapsto [z]$, its horizontal space $\mathcal{H}_z$ at $z \in \mathbb{S}(\mathbb{V})$ and the horizontal lift $\mathcal{H}_z Q = (\pi_*|\mathcal{H}_z)^{-1}(T_{\pi(z)}Q)$ for $z \in \widetilde{Q}$. Analogously, we consider the horizontal lift $\mathcal{H}_z Q' = (\pi_*|\mathcal{H}_z)^{-1}(T_{\pi(z)}Q')$ for $z \in \widetilde{Q}'$.
- The tensor field C of type (1,1) on $\mathbb{V}$ characterized by

$$\forall u \in T\mathbb{V} \;:\; \overrightarrow{Cu} = A(\overrightarrow{u})$$

and the analogous tensor field C' on U defined by

$$\forall u \in TU \;:\; \overrightarrow{C'u} = A'(\overrightarrow{u})\,.$$

Note that for any $z \in \widetilde{Q}$, C_z is a conjugation on $T_z\mathbb{V}$ and in the sequel we will view $T_z\mathbb{V}$ as a $\mathbb{C}$Q-space with the $\mathbb{C}$Q-structure induced by C_z. In this regard, $\mathcal{H}_zQ$ is a $\mathbb{C}$Q-subspace of $T_z\mathbb{V}$. Moreover, for any $z \in \widetilde{Q}'$, T_zU is a complex t-subspace of $T_z\mathbb{V}$ and C'_z is the adapted conjugation on T_zU corresponding to C_z.

- The unit vector fields η, $\widetilde{\xi}$ and ξ defined (via C) in Section 1.3. C' gives rise to analogous unit vector fields $\widetilde{\xi}' := -C' \circ \eta|\widetilde{Q}'$ and $\xi' := \pi_*\widetilde{\xi}'$. The latter is a unit vector field in the normal bundle of the inclusion $Q' \hookrightarrow [U] \hookrightarrow \mathbb{P}(\mathbb{V})$.

6.19 Proposition. *Let $p \in Q'$ and $z \in \pi^{-1}(\{p\})$ be given. Then T_pQ' is a complex t-subspace of T_pQ and the shape operator $A^{Q' \hookrightarrow [U]}_{\xi'(z)} : T_pQ' \to T_pQ'$ is an adapted conjugation for T_pQ' corresponding to $A^{Q \hookrightarrow \mathbb{P}(\mathbb{V})}_{\xi(z)} \in \mathfrak{A}(Q,p)$.*

Proof. As we saw above, T_zU is a complex t-subspace of $T_z\mathbb{V}$, and C'_z is an adapted conjugation for T_zU corresponding to C_z. Because $\mathcal{H}_zQ'$ is a C'_z-invariant subspace of T_zU and $\mathcal{H}_zQ$ is a C_z-invariant subspace of $T_z\mathbb{V}$ (see Theorem 2.26) which contains $\mathcal{H}_zQ'$, it follows that $\mathcal{H}_zQ'$ is a complex t-subspace of $\mathcal{H}_zQ$ and $C'_z|\mathcal{H}_zQ'$ is an adapted conjugation for $\mathcal{H}_zQ'$ corresponding to $C_z|\mathcal{H}_zQ$.

By Theorem 1.16, $C_z|\mathcal{H}_zQ$ is conjugate to $A^{Q \hookrightarrow \mathbb{P}(\mathbb{V})}_{\xi(z)}$ under the $\mathbb{C}$-linear isometry $\pi_*|\mathcal{H}_zQ : \mathcal{H}_zQ \to T_pQ$, and $C'_z|\mathcal{H}_zQ'$ is conjugate to $A^{Q' \hookrightarrow [U]}_{\xi'(z)}$ under the $\mathbb{C}$-linear isometry $\pi_*|\mathcal{H}_zQ' : \mathcal{H}_zQ' \to T_pQ'$. Thus it follows that T_pQ' is a complex t-subspace of T_pQ and that $A^{Q' \hookrightarrow [U]}_{\xi'(z)}$ is an adapted conjugation for T_pQ' corresponding to $A^{Q \hookrightarrow \mathbb{P}(\mathbb{V})}_{\xi(z)}$. $\square$

In the next proposition we calculate the shape operator of the inclusion map $Q' \hookrightarrow Q$.

For $p \in Q'$ we denote by $P, P^\perp : T_pQ \to T_pQ$ the unitary projections of T_pQ onto T_pQ' resp. onto $(T_pQ')^\perp$. Also, we denote by $\nabla^\perp$ the covariant derivative of the normal bundle of $Q' \hookrightarrow Q$.

6.20 Proposition. *(a) The vector field ζ along $\pi|\widetilde{Q}' : \widetilde{Q}' \to Q'$ obtained by orthogonally projecting ξ' onto TQ satisfies*

$$\zeta = \xi' - \cos(2t) \cdot (\xi|Q') \tag{6.40}$$

and is therefore a normal field with respect to $Q' \hookrightarrow Q$. It satisfies $\|\zeta\| = \sin(2t)$ and for any $\widetilde{v} \in \mathcal{H}_zQ'$

$$\nabla^\perp_{\widetilde{v}}\zeta = \cos(2t) \cdot P^\perp A^{Q \hookrightarrow \mathbb{P}(\mathbb{V})}_{\xi(z)} \pi_*\widetilde{v}\,, \tag{6.41}$$

$$\|\nabla^\perp_{\widetilde{v}}\zeta\| = \tfrac{1}{2}\sin(4t) \cdot \|\widetilde{v}\| \tag{6.42}$$

In particular, ζ is not *a parallel field with respect to $\nabla^\perp$.*

(b) Via ζ the fundamental geometric entities of the isometric embedding $Q' \hookrightarrow Q$ can be expressed. More specifically, we have for any $v, w \in T_p Q'$ and $\nu \in \perp_p(Q' \hookrightarrow Q)$

$$h^{Q' \hookrightarrow Q}(v,w) = \langle v, A^{Q' \hookrightarrow [U]}_{\xi'(z)} w \rangle_{\mathbb{C}} \cdot \zeta_z \tag{6.43}$$

$$A^{Q' \hookrightarrow Q}_{\nu} v = \langle \nu, \zeta_z \rangle_{\mathbb{C}} \cdot A^{Q' \hookrightarrow [U]}_{\xi'(z)} v \ , \tag{6.44}$$

in particular

$$A^{Q' \hookrightarrow Q}_{\zeta(z)} v = \sin(2t)^2 \cdot A^{Q' \hookrightarrow [U]}_{\xi'(z)} v \ . \tag{6.45}$$

Proof. For (a). For every $z \in \widetilde{Q}'$ we have

$$\begin{aligned} \langle \xi'_z, \xi_z \rangle_{\mathbb{C}} &= \langle -\pi_* C' \eta_z, -\pi_* C \eta_z \rangle_{\mathbb{C}} = \langle \underbrace{A' z}_{\in U}, A z \rangle_{\mathbb{C}} \overset{(6.1)}{=} \langle A' z, P_U \, A z \rangle_{\mathbb{C}} \\ &\overset{(*)}{=} \cos(2t) \cdot \langle A' z, A' z \rangle_{\mathbb{C}} = \cos(2t) \ ; \end{aligned} \tag{6.46}$$

here we used for the equals sign marked $(*)$ Lemma 6.8 for the complex t-subspace U of $\mathbb{V}$. Because $(T_{\pi(z)} Q)^{\perp, T_{\pi(z)} \mathbb{P}(\mathbb{V})}$ is $\mathbb{C}$-spanned by the unit vector ξ_z, the orthogonal projection of ξ'_z onto $T_{\pi(z)} Q$ is given by

$$\zeta_z = \xi'_z - \langle \xi'_z, \xi_z \rangle_{\mathbb{C}} \cdot \xi_z \overset{(6.46)}{=} \xi'_z - \cos(2t) \cdot \xi_z \ ,$$

whence Equation (6.40) follows. It is clear by definition that ζ is tangential to Q; moreover Equation (6.40) shows that ζ is normal to Q'. We also obtain from Equation (6.40) for every $z \in \widetilde{Q}'$

$$\begin{aligned} \|\zeta_z\|^2 &= \underbrace{\langle \xi'_z, \xi'_z \rangle_{\mathbb{C}}}_{=1} - \cos(2t) \cdot (\underbrace{\langle \xi'_z, \xi_z \rangle_{\mathbb{C}}}_{\underset{(6.46)}{=} \cos(2t)} + \underbrace{\langle \xi_z, \xi'_z \rangle_{\mathbb{C}}}_{\underset{(6.46)}{=} \cos(2t)}) + \cos(2t)^2 \cdot \underbrace{\langle \xi_z, \xi_z \rangle_{\mathbb{C}}}_{=1} \\ &= 1 - \cos(2t)^2 = \sin(2t)^2 \ , \end{aligned}$$

and therefore $\|\zeta_z\| = \sin(2t)$.

As a consequence of Equation (1.19) in the proof of Theorem 1.16 we have for $\widetilde{v} \in \mathcal{H}_z Q'$

$$\nabla^{\mathbb{P}(\mathbb{V})}_{\widetilde{v}} \xi = -\pi_* C \widetilde{v}$$

and also, because $[U]$ is a totally geodesic submanifold of $\mathbb{P}(\mathbb{V})$,

$$\nabla^{\mathbb{P}(\mathbb{V})}_{\widetilde{v}} \xi' = \nabla^{[U]}_{\widetilde{v}} \xi' = -\pi_* C' \widetilde{v} \ .$$

It follows via Equation (6.40) that

$$\nabla^{\mathbb{P}(\mathbb{V})}_{\widetilde{v}} \zeta = -\pi_* C' \widetilde{v} + \cos(2t) \cdot \pi_* C \widetilde{v} \tag{6.47}$$

holds. We have $C' \widetilde{v} \in \mathcal{H}_z Q' \subset \mathcal{H}_z Q$ and $C \widetilde{v} \in \mathcal{H}_z Q$, and therefore Equation (6.47) shows that $\nabla^{\mathbb{P}(\mathbb{V})}_{\widetilde{v}} \zeta \in T_p Q$ holds. Therefore, the Gauss equation implies

$$\nabla^{Q}_{\widetilde{v}} \zeta = -\pi_* C' \widetilde{v} + \cos(2t) \cdot \pi_* C \widetilde{v}$$

and hence we obtain via the Weingarten equation $\nabla^Q_{\widetilde{v}}\zeta = -A^{Q'\hookrightarrow Q}_{\zeta(z)}\pi_*\widetilde{v} + \nabla^\perp_{\widetilde{v}}\zeta$

$$\begin{aligned}\nabla^\perp_{\widetilde{v}}\zeta &= P^\perp \nabla^Q_{\widetilde{v}}\zeta \\ &= -P^\perp \underbrace{\pi_* C'\widetilde{v}}_{\in T_pQ'} + \cos(2t)\cdot P^\perp \pi_* C\widetilde{v} \\ &= \cos(2t)\cdot P^\perp A^{Q\hookrightarrow \mathbb{P}(\mathbb{V})}_{\xi(z)}\pi_*\widetilde{v}\;,\end{aligned}$$

completing the proof of Equation (6.41). We now abbreviate $w := A^{Q\hookrightarrow \mathbb{P}(\mathbb{V})}_{\xi(z)}\pi_*\widetilde{v}$ and $w' := A^{Q'\hookrightarrow [U]}_{\xi'(z)}\pi_*\widetilde{v}$. With Proposition 6.19 and Lemma 6.8 we get

$$\cos(2t)\cdot w' = Pw \tag{6.48}$$

and therefore

$$\begin{aligned}\|\nabla^\perp_{\widetilde{v}}\zeta\|^2 &\overset{(6.41)}{=} \cos(2t)^2\cdot\|P^\perp w\|^2 = \cos(2t)^2\cdot(\|w\|^2 - \|Pw\|^2) \\ &\overset{(6.48)}{=} \cos(2t)^2\cdot(\|w\|^2 - \cos(2t)^2\cdot\|w'\|^2) = \cos(2t)^2\cdot(\|\widetilde{v}\|^2 - \cos(2t)^2\cdot\|\widetilde{v}\|^2) \\ &= (\tfrac{1}{2}\sin(4t))^2\cdot\|\widetilde{v}\|^2\;,\end{aligned}$$

which proves Equation (6.42). Because of $\sin(4t)\neq 0$ it also follows that ζ is not a parallel field.

For (b). Let $v,w\in T_pQ'$ be given. Because $[U]$ is a totally geodesic submanifold of $\mathbb{P}(\mathbb{V})$, we then have

$$h^{Q'\hookrightarrow [U]}(v,w) = h^{Q'\hookrightarrow \mathbb{P}(\mathbb{V})}(v,w) = \underbrace{h^{Q'\hookrightarrow Q}(v,w)}_{\in T_pQ} + \underbrace{h^{Q\hookrightarrow \mathbb{P}(\mathbb{V})}(v,w)}_{\perp T_pQ}\;. \tag{6.49}$$

Denoting by $P_{T_pQ}: T_p\mathbb{P}(\mathbb{V})\to T_p\mathbb{P}(\mathbb{V})$ the orthogonal projection onto T_pQ, we now obtain

$$h^{Q'\hookrightarrow Q}(v,w) \overset{(6.49)}{=} P_{T_pQ}(h^{Q'\hookrightarrow [U]}(v,w)) \overset{(*)}{=} \langle v, A^{Q'\hookrightarrow [U]}_{\xi'_z} w\rangle_{\mathbb{C}}\cdot P_{T_pQ}(\xi'_z) = \langle v, A^{Q'\hookrightarrow [U]}_{\xi'_z} w\rangle_{\mathbb{C}}\cdot \zeta_z\;,$$

where $(*)$ follows from Proposition 1.19. Thus we have shown Equation (6.43).

For the proof of Equation (6.44) we abbreviate $h := h^{Q'\hookrightarrow Q}$ and $A := A^{Q'\hookrightarrow Q}$. By definition of the shape operator A we have

$$\forall u,v\in T_pQ',\ \nu\in\perp_p(Q'\hookrightarrow Q)\ :\ \langle u, A_\nu v\rangle_{\mathbb{R}} = \langle h(u,v),\nu\rangle_{\mathbb{R}}\;. \tag{6.50}$$

Because of the parallelity of the complex structure J of Q and the fact that Q' is a complex submanifold of Q, h is $\mathbb{C}$-linear in both entries (see [KN69], Proposition IX.9.1, p. 175); from this fact and Equation (6.50) one obtains by use of Equation (2.1)

$$\forall u,v\in T_pQ',\ \nu\in\perp_p(Q'\hookrightarrow Q)\ :\ \langle u, A_\nu v\rangle_{\mathbb{C}} = \langle h(u,v),\nu\rangle_{\mathbb{C}}\;. \tag{6.51}$$

Now, let $u,v\in T_pQ'$ and $\nu\in\perp_p(Q'\hookrightarrow Q)$ be given. Then we have

$$\begin{aligned}\langle u, A_\nu v\rangle_{\mathbb{C}} &\overset{(6.51)}{=} \langle h(u,v),\nu\rangle_{\mathbb{C}} \overset{(6.43)}{=} \left\langle\, \langle u, A^{Q'\hookrightarrow [U]}_{\xi'(z)} v\rangle_{\mathbb{C}}\,\zeta_z\,,\,\nu\,\right\rangle_{\mathbb{C}} = \langle u, A^{Q'\hookrightarrow [U]}_{\xi'(z)} v\rangle_{\mathbb{C}}\cdot\langle\zeta_z,\nu\rangle_{\mathbb{C}} \\ &= \left\langle\, u\,,\,\langle \nu,\zeta_z\rangle_{\mathbb{C}}\, A^{Q'\hookrightarrow [U]}_{\xi'(z)} v\,\right\rangle\;,\end{aligned}$$

whence Equation (6.44) follows by variation of u. Equation (6.45) is an immediate consequence of (6.44) because of $\|\zeta\| = \sin(2t)$. $\square$

6.21 Corollary. *The submanifold Q' of Q does not have parallel second fundamental form. In particular, it is not totally geodesic.*

Proof. Let $X, Y \in \mathfrak{X}_c(Q')$ be parallel vector fields along some curve $c : I \to Q'$ and $\widetilde{c} : I \to \widetilde{Q}'$ be a horizontal lift of c with respect to π. Then Theorem 1.18 (applied to Q') shows that $A^{Q' \hookrightarrow [U]}_{\xi' \circ \widetilde{c}} \circ Y$ is another parallel vector field; because Q' is a Kähler manifold, it follows that the function $s \mapsto \langle X_s, A^{Q' \hookrightarrow [U]}_{\xi'(\widetilde{c}(s))} Y_s \rangle_{\mathbb{C}}$ is constant. Denoting this constant by $\alpha \in \mathbb{C}$, we see by Proposition 6.20(b) that

$$\nabla^{\perp}_{\partial} h^{Q' \hookrightarrow Q}(X, Y) = \alpha \cdot \nabla^{\perp}_{\dot{c}} \zeta \tag{6.52}$$

holds.

We now fix $s_0 \in I$, suppose $\dot{c}(s_0) \neq 0$ and choose $X = Y \in \mathfrak{X}_c(Q')$ so that $X_{s_0} \in \mathbb{S}(V(A^{Q' \hookrightarrow [U]}_{\xi'(\widetilde{c}(s_0))}))$ holds. Then we have

$$|\alpha| = |\langle X_{s_0}, A^{Q' \hookrightarrow [U]}_{\xi'(\widetilde{c}(s_0))} X_{s_0} \rangle_{\mathbb{C}}| = 1$$

and by Equation (6.42): $\|\nabla^{\perp}_{\dot{\widetilde{c}}(s_0)} \zeta\| = \frac{1}{2} \sin(4t) \cdot \|\dot{\widetilde{c}}(s_0)\| \neq 0$. From Equation (6.52) we thus see that $h^{Q' \hookrightarrow Q}(X, X)$ is not parallel at s_0. □

We now wish to study how far Q' is from being a totally geodesic submanifold of Q. For this purpose we study the behaviour of geodesics of Q which start tangential to Q'.

6.22 Proposition. *Let $p \in Q'$ and $v \in \mathbb{S}(T_p Q)$ be given, moreover we let $\gamma_v : \mathbb{R} \to Q$ be the maximal geodesic of Q with $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$.*

Then we have:

(a) For $\varphi_{\mathfrak{A}(Q',p)}(v) \neq \frac{\pi}{4}$, $\gamma_v(\mathbb{R}) \cap Q' = \{p\}$ holds.

(b) For $\varphi_{\mathfrak{A}(Q',p)}(v) = \frac{\pi}{4}$ we have $\gamma_v(\mathbb{R}) \subset Q'$ and therefore γ_v is also a geodesic of Q'.

In fact, the following lemma describes the situation in more detail:

6.23 Lemma. *Let $\mathbb{T}$ be a maximal torus of Q (i.e. a totally geodesic submanifold of Q of type* (G2, 1, 1)*, see Section 5.3) with $\mathbb{T} \cap Q' \neq \varnothing$. We let $p \in \mathbb{T} \cap Q'$ be given. Then we have* $\dim(T_p\mathbb{T} \cap T_p Q') \leq 1$ *(in particular, $\mathbb{T}$ is not tangential to Q'), and in the case* $\dim(T_p\mathbb{T} \cap T_p Q') = 1$*:*

(a) If $T_p\mathbb{T} \cap T_p Q'$ is not $\mathfrak{A}(Q',p)$-isotropic, then we have $\mathbb{T} \cap Q' = \{p\}$.

(b) If $T_p\mathbb{T} \cap T_p Q'$ is $\mathfrak{A}(Q',p)$-isotropic, then $\mathbb{T} \cap Q'$ is a circle of radius $\frac{1}{2}$; more precisely we have $\mathbb{T} \cap Q' = \gamma_v(\mathbb{R})$ for any $v \in (T_p\mathbb{T} \cap T_p Q') \setminus \{0\}$. (Here $\gamma_v : \mathbb{R} \to Q$ again denotes the maximal geodesic of Q with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.)

Proof of Proposition 6.22. For (a). We base the proof of Proposition 6.22(a) on Lemma 6.23(a), which is proven below. — It follows from Theorem 3.15(c) that there exists a maximal torus $\mathbb{T}$ of Q with $p \in \mathbb{T}$ and $v \in T_p\mathbb{T}$. Because $\mathbb{T}$ is a complete, totally geodesic submanifold of Q, we then have $\gamma_v(\mathrm{I\!R}) \subset \mathbb{T}$, and because of $\varphi_{\mathfrak{A}(Q',p)}(v) \neq \frac{\pi}{4}$, Lemma 6.23(a) shows

$$\{p\} \subset \gamma_v(\mathrm{I\!R}) \cap Q' \subset \mathbb{T} \cap Q' = \{p\}$$

and hence $\gamma_v(\mathrm{I\!R}) \cap Q' = \{p\}$.

For (b). We have $\varphi_{\mathfrak{A}(Q',p)}(v) = \frac{\pi}{4}$ and therefore by Corollary 6.12 also $\varphi_{\mathfrak{A}(Q,p)}(v) = \frac{\pi}{4}$. By Remark 5.16 (applied to the quadric $Q \subset \mathrm{I\!P}(\mathbb{V})$) it follows that $\gamma_v : \mathrm{I\!R} \to Q$ is also a geodesic in $\mathrm{I\!P}(\mathbb{V})$. Because v is tangential to the totally geodesic submanifold $[U]$ of $\mathrm{I\!P}(\mathbb{V})$, we see that γ_v runs completely in and is a geodesic of $[U]$; moreover Remark 5.16 (this time applied to $Q' \subset [U]$) shows that γ_v is a geodesic of Q'. In particular we have $\gamma_v(\mathrm{I\!R}) \subset Q'$. □

Proof of Lemma 6.23. We first note that because U and T_pQ' are complex t-subspaces of $\mathbb{V}$ resp. of T_pQ (see Proposition 6.19), we have by Corollary 6.12

$$\forall v \in U \setminus \{0\} \ : \ \varphi_{\mathfrak{A}}(v) \geq t > 0 \,, \tag{6.53}$$

$$\forall v \in T_pQ' \setminus \{0\} \ : \ \varphi_{\mathfrak{A}(Q,p)}(v) \geq t > 0 \,. \tag{6.54}$$

We consider the maximal flat subspace $\mathfrak{a} := T_p\mathbb{T}$ of T_pQ; then it follows from (6.54) that $\dim(\mathfrak{a} \cap T_pQ') \leq 1$ holds: Because of $\dim(\mathfrak{a}) = 2$, we would otherwise have $\mathfrak{a} \subset T_pQ'$; because $\mathfrak{a}$ contains vectors of $\mathfrak{A}(Q,p)$-angle 0, this would be a contradiction to (6.54).

We now suppose that $\dim(\mathfrak{a} \cap T_pQ') = 1$ holds and fix $v \in \mathbb{S}(\mathfrak{a} \cap T_pQ')$.

Then we introduce the data necessary for an explicit description of $\mathbb{T}$. From Theorem 2.54 it follows that there exists $A_1 \in \mathfrak{A}(Q,p)$ and an orthonormal system (v_1, v_2) in $V(A_1)$ so that $\mathfrak{a} = \mathrm{I\!R}v_1 \oplus \mathrm{I\!R}Jv_2$ holds. By reversing the signs of v_1, v_2 and A_1 where necessary (if the sign of A_1 is reversed, one also have to replace v_1 by Jv_2 and v_2 by Jv_1), one can ensure that A_1 is adapted to v and $v = \cos(\varphi_{\mathfrak{A}(Q,p)}(v))\, v_1 + \sin(\varphi_{\mathfrak{A}(Q,p)}(v))\, Jv_2$ is a canonical representation of v in the sense of Theorem 2.28(c).

We choose $z \in \pi^{-1}(\{p\})$ so that $A^{Q \hookrightarrow \mathrm{I\!P}(\mathbb{V})}_{\xi(z)} = A_1$ holds (see Proposition 1.15) and put

$$\widetilde{v} := \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(v)}, \quad \widetilde{v}_1 := \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(v_1)}, \quad \text{and} \quad \widetilde{v}_2 := \overrightarrow{(\pi_*|\mathcal{H}_z)^{-1}(v_2)}\,.$$

Then $(\widetilde{v}_1, \widetilde{v}_2)$ is an orthonormal system in $V(A)$, and if we denote by $\mathfrak{A}'$ the adapted $\mathbb{C}$Q-structure for the complex t-subspace $U \subset \mathbb{V}$, we have

$$\varphi_{\mathfrak{A}'}(\widetilde{v}) = \varphi_{\mathfrak{A}(Q',p)}(v) \tag{6.55}$$

(because $\pi_*|\mathcal{H}_zQ' : (\mathcal{H}_zQ', \mathfrak{A}') \to (T_pQ', \mathfrak{A}(Q',p))$ is a $\mathbb{C}$Q-isomorphism).

Further, we put

$$\widetilde{V}_1 := \mathrm{I\!R}(\mathrm{Re}_A\, z) \oplus \mathrm{I\!R}\widetilde{v}_1 \quad \text{and} \quad \widetilde{V}_2 := \mathrm{I\!R}(\mathrm{Im}_A\, z) \oplus \mathrm{I\!R}\widetilde{v}_2$$

and consider the map

$$\widetilde{f} : \mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2) \to \widetilde{Q},\ (x,y) \mapsto x + Jy\ ,$$

where we abbreviate $r := 1/\sqrt{2}$. By Proposition 5.11, $\widetilde{f}$ is an isometric embedding onto $\widetilde{\mathbb{T}} := \widetilde{f}(\mathbb{S}(\widetilde{V}_1) \times \mathbb{S}(\widetilde{V}_2))$, and $\pi|\widetilde{\mathbb{T}} : \widetilde{\mathbb{T}} \to \mathbb{T}$ is a two-fold covering map onto the maximal torus $\mathbb{T}$ with

$$\forall\,(x,y),\ (x',y') \in \mathbb{S}_r(\widetilde{V}_1) \times \mathbb{S}_r(\widetilde{V}_2)\ :\ \big(\pi(\widetilde{f}(x,y)) = \pi(\widetilde{f}(x',y')) \iff (x,y) = \pm(x',y')\big)\ . \tag{6.56}$$

We further put

$$\widetilde{W} := \widetilde{V}_1 \oplus J(\widetilde{V}_2) = \mathrm{span}_{\mathbb{R}}\{\mathrm{Re}_A\, z, J\,\mathrm{Im}_A\, z, \widetilde{v}_1, J\widetilde{v}_2\}\ , \tag{6.57}$$

note that $\widetilde{\mathbb{T}} \subset \widetilde{W}$ holds. We have $\mathrm{span}_{\mathbb{R}}\{\mathrm{Re}_A\, z, J\,\mathrm{Im}_A\, z\} = \mathrm{span}_{\mathbb{R}}\{z, Az\}$ and $\mathrm{span}_{\mathbb{R}}\{\widetilde{v}_1, J\widetilde{v}_2\} = \mathrm{span}_{\mathbb{R}}\{\widetilde{v}, A\widetilde{v}\}$ (because of $\varphi_{\mathfrak{A}}(z), \varphi_{\mathfrak{A}}(\widetilde{v}) \neq 0$, see (6.53)) and therefore

$$\widetilde{W} = \mathrm{span}_{\mathbb{R}}\{z, Az, \widetilde{v}, A\widetilde{v}\}\ . \tag{6.58}$$

We now show

$$\widetilde{W} \cap U = \mathbb{R}z \oplus \mathbb{R}\widetilde{v}\ . \tag{6.59}$$

By Equation (6.58) $z, \widetilde{v} \in \widetilde{W}$ holds, and we have $z \in \widetilde{Q}' \subset U$ and $\widetilde{v} \in \overrightarrow{\mathcal{H}_z Q'} \subset U$. This shows "$\supset$" in Equation (6.59). Conversely, let $u \in \widetilde{W} \cap U$ be given. Equation (6.58) shows that there exist $a,b,c,d \in \mathbb{R}$ so that $u = az + bAz + c\widetilde{v} + dA\widetilde{v}$ holds. By the inclusion "$\supset$" of Equation (6.59) we also have

$$u_1 := az + c\widetilde{v} \in \widetilde{W} \cap U$$

and consequently

$$\widetilde{W} \cap U \ni u_2 := u - u_1 = bAz + dA\widetilde{v}\ .$$

Also by the inclusion "$\supset$" of Equation (6.59) we have $Au_2 = bz + d\widetilde{v} \in \widetilde{W} \cap U$ and therefore

$$u_3 := u_2 + Au_2 \in \widetilde{W} \cap U \quad \text{and} \quad u_4 := u_2 - Au_2 \in \widetilde{W} \cap U\ .$$

We have $Au_3 = u_3$ and thus $u_3 \in V(A)$. If $u_3 \neq 0$ were the case, we would therefore have $\varphi_{\mathfrak{A}}(u_3) = 0$ in contradiction to (6.53). Hence we have $u_3 = 0$ and by the analogous argument also $u_4 = 0$. But $u_3 = u_4 = 0$ implies $u_2 = 0$ and therefore $b = d = 0$. This shows $u = az + c\widetilde{v} \in \mathbb{R}z \oplus \mathbb{R}\widetilde{v}$. Thus Equation (6.59) is shown.

We have $\widetilde{\mathbb{T}} \subset \mathbb{S}(\widetilde{W})$ and $\widetilde{Q}' \subset \mathbb{S}(U)$; therefore it follows from Equation (6.59) that

$$\widetilde{\mathbb{T}} \cap \widetilde{Q}' \subset \mathbb{S}(\widetilde{W} \cap U) = \mathbb{S}(\mathbb{R}z \oplus \mathbb{R}\widetilde{v}) \tag{6.60}$$

holds.

For (a). We suppose that $\mathfrak{a} \cap T_pQ'$ is not $\mathfrak{A}(Q',p)$-isotropic. Because this space is of dimension 1 and we have $v \in \mathfrak{a} \cap T_pQ'$, we then have $\varphi_{\mathfrak{A}(Q',p)}(v) \neq \frac{\pi}{4}$ and therefore by Equation (6.55) also $\varphi_{\mathfrak{A}'}(\widetilde{v}) \neq \frac{\pi}{4}$. We will now show

$$\widetilde{\mathbb{T}} \cap \widetilde{Q}' = \{\pm z\}\ ; \tag{6.61}$$

because of (6.56), $\mathbb{T} \cap Q' = \{p\}$ then follows immediately.

For the proof of (6.61): It is clear that $\pm z \in \widetilde{\mathbb{T}} \cap \widetilde{Q}'$ holds. Conversely, we suppose that $u \in \widetilde{\mathbb{T}} \cap \widetilde{Q}'$ is given. Because of (6.60) there then exist $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ so that $u = az + b\widetilde{v}$ holds. Because of $u \in \widetilde{Q}'$, we have

$$0 = \langle u, A'u \rangle_{\mathbb{C}} = a^2 \langle z, A'z \rangle_{\mathbb{C}} + 2ab \langle \widetilde{v}, A'z \rangle_{\mathbb{C}} + b^2 \langle \widetilde{v}, A'\widetilde{v} \rangle_{\mathbb{C}} \; .$$

We have $\langle z, A'z \rangle_{\mathbb{C}} = 0$ because of $z \in \widetilde{Q}'$ and $\langle \widetilde{v}, A'z \rangle_{\mathbb{C}} = 0$ because of $\widetilde{v} \in \overrightarrow{\mathcal{H}_z Q'} = (\mathrm{span}_{\mathfrak{A}'}\{z\})^{\perp,U}$, also we have $\langle \widetilde{v}, A'\widetilde{v} \rangle_{\mathbb{C}} \neq 0$ because of $\varphi_{\mathfrak{A}'}(\widetilde{v}) \neq \frac{\pi}{4}$. Thus we obtain $b = 0$, and hence $u = \pm z$. This completes the proof of Equation (6.61).

For (b). We now suppose that $\mathfrak{a} \cap T_pQ'$ is an $\mathfrak{A}(Q', p)$-isotropic subspace of T_pQ'; in particular v is $\mathfrak{A}(Q', p)$-isotropic, and therefore $\widetilde{v}$ is $\mathfrak{A}'$-isotropic. We now show

$$\widetilde{\mathbb{T}} \cap \widetilde{Q}' = \mathbb{S}(\mathbb{R}z \oplus \mathbb{R}\widetilde{v}) \; . \tag{6.62}$$

Indeed, the inclusion "$\subset$" of this equality has already been shown as (6.60). Conversely, we let $u \in \mathbb{S}(\mathbb{R}z \oplus \mathbb{R}\widetilde{v})$ be given, say $u = az + b\widetilde{v}$ with $a, b \in \mathbb{R}$ and $a^2 + b^2 = 1$. Then we have

$$\langle u, A'u \rangle_{\mathbb{C}} = a^2 \langle z, A'z \rangle_{\mathbb{C}} + 2ab \langle \widetilde{v}, A'z \rangle_{\mathbb{C}} + b^2 \langle \widetilde{v}, A'\widetilde{v} \rangle_{\mathbb{C}} \; .$$

As in the case (a), we have $\langle z, A'z \rangle_{\mathbb{C}} = \langle \widetilde{v}, A'z \rangle_{\mathbb{C}} = 0$, but now we also have $\langle \widetilde{v}, A'\widetilde{v} \rangle_{\mathbb{C}} = 0$ because $\widetilde{v}$ is $\mathfrak{A}'$-isotropic. Therefore we have $\langle u, A'u \rangle_{\mathbb{C}} = 0$ and hence $u \in \widetilde{Q}'$.

In particular we have $u \in \widetilde{Q}$, and therefore the vectors

$$x := \mathrm{Re}_A(u) = a\; \mathrm{Re}_A(z) + b\,\tfrac{\widetilde{v}_1}{\sqrt{2}} \in \widetilde{V}_1 \quad \text{and} \quad y := \mathrm{Im}_A(u) = a\; \mathrm{Im}_A(z) + b\,\tfrac{\widetilde{v}_2}{\sqrt{2}} \in \widetilde{V}_2$$

are of length $r = 1/\sqrt{2}$ by Proposition 2.23(b). Hence we have $u = x + Jy = \widetilde{f}(x, y) \in \widetilde{\mathbb{T}}$, completing the proof of Equation (6.62).

From Equation (6.62) we obtain

$$\mathbb{T} \cap Q' = \pi(\widetilde{\mathbb{T}}) \cap \pi(\widetilde{Q}') = \pi(\widetilde{\mathbb{T}} \cap \widetilde{Q}') = \pi(\mathbb{S}(\mathbb{R}z \oplus \mathbb{R}\widetilde{v})) \tag{6.63}$$

(note that $\widetilde{Q}'$ is saturated with respect to π). We now show

$$\pi(\mathbb{S}(\mathbb{R}z \oplus \mathbb{R}\widetilde{v})) = \gamma_v(\mathbb{R}) \; . \tag{6.64}$$

Indeed, by application of (6.15) to the complex t-subspace T_pQ' of T_pQ we have

$$\varphi_{\mathfrak{A}(Q,p)}(v) = \tfrac{\pi}{4} \; . \tag{6.65}$$

Proposition 5.15 thus shows

$$\forall s \in \mathbb{R} \; : \; \gamma_v(s) = \pi(\,\cos(s)\, z + \sin(s)\, \widetilde{v}\,) \; , \tag{6.66}$$

whence Equation (6.64) follows.

From Equations (6.63) and (6.64) we obtain $\mathbb{T} \cap Q' = \gamma_v(\mathbb{R})$. Also because of Equation (6.65), Proposition 5.18 shows that the unit speed geodesic γ_v is closed and of minimal period $\pi = 3.14\ldots$, and hence $\mathbb{T} \cap Q'$ is a circle of radius $\frac{1}{2}$. $\square$

Chapter 7

Families of congruent submanifolds

Among the totally geodesic submanifolds of an m-dimensional complex quadric Q (which we classified in Chapters 4 and 5), there are two series of families of congruent submanifolds which are of particular interest: the family of k-dimensional projective subspaces ($k \leq \frac{m}{2}$) contained in Q (corresponding to the type $(\mathrm{I}1, k)$) and the family of k-dimensional complex quadrics ($k < m$) which are totally geodesic in Q (corresponding to the type $(\mathrm{G}1, k)$).

The primary subject of the present chapter is to give these families the structure of a Riemannian manifold and to study them, in particular as submanifolds of the families of *all* k-dimensional projective subspaces resp. complex quadrics contained in the ambient projective space $\mathbb{P}^{m+1}$. A large part of these studies is focused on questions from the theory of reductive homogeneous spaces and of symmetric spaces, see Appendix A.1. In particular, we show in a general setting that a family of congruent manifolds can be seen as a naturally reductive homogeneous space. In the specific cases mentioned above, we investigate whether the reductive structure of the families is induced by a symmetric structure, and whether the families in Q are naturally reductive homogeneous subspaces of the corresponding families in $\mathbb{P}^{m+1}$.

In Section 7.1, fundamental facts on families of congruent homogeneous subspaces in Riemannian homogeneous spaces in general are presented. Section 7.2 is concerned with results on congruence families of projective subspaces and of quadrics in a projective space, and Section 7.3 finally discusses the corresponding families of projective subspaces and of quadrics which are contained in a fixed quadric Q.

The following notation should be kept in mind: If M_1, M_2, M are sets, $f : M_1 \times M_2 \to M$ is a map and $p_0 \in M_1$, $q_0 \in M_2$ holds, we consider the maps $f_{p_0} : M_2 \to M,\ q \mapsto f(p_0, q)$ and $f^{q_0} : M_1 \to M,\ p \mapsto f(p, q_0)$.

7.1 Families of submanifolds in general

Let M be a connected Riemannian homogeneous space and G be a Lie group which acts on M transitively and via isometries by the differentiable action $\varphi : G \times M \to M$. Moreover, let N_0 be a connected, closed homogeneous subspace of M, i.e. the group $K := \{\, g \in G \,|\, \varphi_g(N_0) = N_0 \,\}$ acts transitively on N_0. In this situation, K equals the intersection $\bigcap_{p \in N_0} (\varphi^p)^{-1}(N_0)$ and is therefore closed in G, hence a Lie subgroup of G (see [Var74], Theorem 2.12.6, p. 99). Because N_0 is closed in M, it follows therefrom that N_0 is a regular submanifold of M ([Var74], Theorem 2.9.7, p. 80).

In this situation we call the set

$$\mathfrak{F}^\varphi(N_0, M) := \{\, \varphi_g(N_0) \,|\, g \in G \,\}$$

of submanifolds of M the *φ-family of submanifolds induced by* N_0. In the case where G is the isometry group of M and φ its canonical action on M, we also speak of the *congruence family* $\mathfrak{F}(N_0, M)$ *of submanifolds induced by* N_0.

7.1 Proposition. *There is a unique differentiable structure on $\mathfrak{F}^\varphi(N_0, M)$ which is for any $N \in \mathfrak{F}^\varphi(N_0, M)$ characterized by the fact that $\psi^N : G \to \mathfrak{F}^\varphi(N_0, M)$ is a surjective submersion. With respect to this differentiable structure, also the transitive action $\psi : G \times \mathfrak{F}^\varphi(N_0, M) \to \mathfrak{F}^\varphi(N_0, M),\ (g, N) \mapsto \varphi_g(N)$ is differentiable.*

Proof. Because K is closed in G, the quotient G/K carries the structure of a differentiable manifold ([Var74], Theorem 2.9.4, p. 77), which we transfer onto $\mathfrak{F}^\varphi(N_0, M)$ by the G-equivariant bijection $G/K \to \mathfrak{F}^\varphi(N_0, M),\ g \cdot K \mapsto \psi^{N_0}(g)$. With respect to this differentiable structure, ψ is differentiable and $\psi^{N_0} : G \to \mathfrak{F}^\varphi(N_0, M)$ is a surjective submersion (see [Var74], Lemma 2.9.2, p. 76).

If now $N \in \mathfrak{F}(N_0, M)$ is given, there exists $g_0 \in G$ so that $N = \psi(g_0, N_0)$ holds; we then have $\psi^N = \psi^{N_0} \circ R_{g_0}$ with the diffeomorphism $R_{g_0} : G \to G,\ g \mapsto g \cdot g_0$, and therefore also $\psi^N : G \to \mathfrak{F}^\varphi(N_0, M)$ is a surjective submersion.

It is clear that there can be only one differentiable structure on $\mathfrak{F}^\varphi(N_0, M)$ so that $\psi^N : G \to \mathfrak{F}^\varphi(N_0, M)$ is a surjective submersion for some given $N \in \mathfrak{F}^\varphi(N_0, M)$. $\square$

Proposition 7.1 shows that $\mathfrak{F}^\varphi(N_0, M)$ is a homogeneous G-space. We now wish to construct the structure of a naturally reductive homogeneous space (see Section A.1) on $\mathfrak{F}^\varphi(N_0, M)$. For this purpose, let us denote the Lie algebras of G and K by $\mathfrak{g}$ and $\mathfrak{k}$, respectively.

We consider the more special situation that the Lie group homomorphism $\tau : G \to I(M),\ g \mapsto \varphi_g$ is a covering map onto an open subgroup of $I(M)$ (meaning that its linearization $\tau_L : \mathfrak{g} \to \mathfrak{i}(M)$ is an isomorphism of Lie algebras) and that M is a Riemannian symmetric G-space of compact type (see Section A.3 and Definition A.4), meaning in particular that the Killing form

$$\varkappa : \mathfrak{g} \times \mathfrak{g} \to \mathrm{I\!R},\ (X, Y) \mapsto \operatorname{tr}(\operatorname{ad}(X) \circ \operatorname{ad}(Y))$$

of $\mathfrak{g}$ is negative definite.

7.2 Proposition. *With* $\mathfrak{m} := \mathfrak{k}^{\perp,\varkappa} = \{\, X \in \mathfrak{g} \,|\, \forall Y \in \mathfrak{k} : \varkappa(X,Y) = 0 \,\}$, $(N_0, \mathfrak{m})$ *is a reductive datum for the homogeneous* G*-space* $\mathfrak{F}^{\varphi}(N_0, M)$ *(see Appendix A.1) and* $-\varkappa$ *induces a* G*-invariant Riemannian metric on* $\mathfrak{F}^{\varphi}(N_0, M)$ *. In this way* $\mathfrak{F}^{\varphi}(N_0, M)$ *becomes a naturally reductive homogeneous* G*-space.*

Proof. As $\varkappa$ is $\mathrm{Ad}(G)$-invariant and $\mathfrak{k}$ is $\mathrm{Ad}(K)$-invariant, $\mathfrak{m}$ is an $\mathrm{Ad}(K)$-invariant subspace of $\mathfrak{g}$ satisfying $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, therefore $(N_0, \mathfrak{m})$ is a reductive datum for $\mathfrak{F}^{\varphi}(N_0, M)$. The positive definite bilinear form $-\varkappa|(\mathfrak{m} \times \mathfrak{m})$ induces a G-invariant metric on $\mathfrak{F}^{\varphi}(N_0, M)$ because it is $\mathrm{Ad}(K)$-invariant; thereby $\mathfrak{F}^{\varphi}(N_0, M)$ becomes a Riemannian homogeneous G-space.

Because $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric with respect to $\varkappa$ for any $X \in \mathfrak{g}$, we have

$$\forall X, Y, Z \in \mathfrak{m} \;:\; \varkappa([Z,X],Y) + \varkappa(X,[Z,Y]) = 0 \;;$$

because $\mathfrak{k}$ and $\mathfrak{m}$ are orthogonal with respect to $\varkappa$,

$$\forall X, Y, Z \in \mathfrak{m} \;:\; \varkappa([Z,X]_{\mathfrak{m}},Y) + \varkappa(X,[Z,Y]_{\mathfrak{m}}) = 0$$

follows, where $X_{\mathfrak{m}}$ denotes the projection of $X \in \mathfrak{g}$ onto $\mathfrak{m}$ along $\mathfrak{k}$. The latter formula shows that $\mathfrak{F}^{\varphi}(N_0, M)$ is naturally reductive; see [KN69], Theorem X.3.3(2), p. 201. □

7.3 Remark. (a) In general, $\mathfrak{F}^{\varphi}(N_0, M)$ does not become a symmetric space in this way, as the examples of Theorems 7.5 and 7.11 will show.

(b) Put $G' := \tau(G)$, by hypothesis this is an open subgroup of $I(M)$ which still acts transitively on M, and denote by $\varphi' : G' \times M \to M$ the canonical action. Then the previous constructions can also be applied to (G', φ') in the place of (G, φ), giving rise to a naturally reductive space $\mathfrak{F}^{\varphi'}(N_0, M)$.

In this setting, $\mathfrak{F}^{\varphi}(N_0, M)$ and $\mathfrak{F}^{\varphi'}(N_0, M)$ coincide as Riemannian manifolds. Denoting this manifold by $\mathfrak{F}$, $(\mathrm{id}_{\mathfrak{F}}, \tau)$ is an almost-isomorphism of naturally reductive homogeneous spaces from the G-space $\mathfrak{F}^{\varphi}(N_0, M)$ onto the G'-space $\mathfrak{F}^{\varphi'}(N_0, M)$ (see Section A.1).

Proof. It is clear that $\mathfrak{F}^{\varphi}(N_0, M)$ and $\mathfrak{F}^{\varphi'}(N_0, M)$ coincide as sets, and that $(\mathrm{id}_{\mathfrak{F}}, \tau)$ is an almost-isomorphism of homogeneous spaces from the G-space $\mathfrak{F}^{\varphi}(N_0, M)$ onto the G'-space $\mathfrak{F}^{\varphi'}(N_0, M)$. Because of Proposition A.1(b), the latter fact shows in particular that $\mathfrak{F}^{\varphi}(N_0, M)$ and $\mathfrak{F}^{\varphi'}(N_0, M)$ coincide as differentiable manifolds. Denoting the objects belonging to (G', φ') by appending a prime (′) to the symbol for the corresponding object belonging to (G, φ), we now have $\tau^{-1}(K') = K$, therefore $\tau|K : K \to K'$ is a covering map of Lie groups, and hence

$$\tau_L(\mathfrak{k}) = \mathfrak{k}' \tag{7.1}$$

holds. Because the Lie algebra isomorphism $\tau_L : \mathfrak{g} \to \mathfrak{i}(M)$ satisfies

$$\forall X, Y \in \mathfrak{g} \;:\; \varkappa'(\tau_L(X), \tau_L(Y)) = \varkappa(X,Y) \;, \tag{7.2}$$

it follows from (7.1) that also $\tau_L(\mathfrak{m}) = \mathfrak{m}'$ holds, and therefore $(\mathrm{id}_{\mathfrak{F}}, \tau)$ is an almost-isomorphism of reductive homogeneous spaces. Now Equation (7.2) also shows that the Riemannian metrics on $\mathfrak{F}^{\varphi}(N_0, M)$ and on $\mathfrak{F}^{\varphi'}(N_0, M)$ are equal, and therefore $(\mathrm{id}_{\mathfrak{F}}, \tau)$ is an almost-isomorphism of naturally reductive homogeneous spaces. □

7.2 Congruence families in the complex projective space

From the point of view of algebraic geometry, the simplest submanifolds of the complex projective space are those defined by linear equations, namely the projective subspaces, and those defined by quadratic equations, namely the complex quadrics. In the present section we investigate the congruence families induced in a complex projective space by these submanifolds. It will turn out that the congruence families induced by complex projective spaces are isomorphic to complex Grassmannians and therefore not very interesting. However, the congruence families induced by complex quadrics provide more interesting examples.

We introduce some notations:

Let $\mathbb{V}$ be a unitary vector space of complex dimension $n \geq 2$. As usual, we denote the complex structure $v \mapsto i \cdot v$ of $\mathbb{V}$ by J, and the complex inner product of $\mathbb{V}$ by $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{C}}$. The latter induces the real inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{R}} := \mathrm{Re}(\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{C}})$ and thereby the norm $\|v\|$. We also consider the Hopf fibration $\pi : \mathbb{S}(\mathbb{V}) \to \mathbb{P}(\mathbb{V})$, this is a Hermitian submersion.

For every $k \in \{1, \dots, n-1\}$, let $G_k(\mathbb{V})$ denote the Grassmannian manifold of k-dimensional complex subspaces of $\mathbb{V}$, which is known to be a Hermitian symmetric space of type AIII (see [Hel78], p. 518). For every $V \in G_k(\mathbb{V})$, we put $[V] := \{\, \pi(v) \,|\, v \in \mathbb{S}(V)\,\}$, this being a $(k-1)$-dimensional projective subspace in $\mathbb{P}(\mathbb{V})$. If, on the other hand, Λ is a $(k-1)$-dimensional projective subspace of $\mathbb{P}(\mathbb{V})$, then $\widehat{\Lambda} := \{\, \lambda \cdot v \,|\, \lambda \in \mathbb{R},\ v \in \pi^{-1}(\Lambda)\,\}$ is an element of $G_k(\mathbb{V})$.

As before, we use the notation $\underline{B} \in I_h(\mathbb{P}(\mathbb{V}))$ for the holomorphic isometry corresponding to $B \in \mathrm{U}(\mathbb{V})$ by $\underline{B} \circ \pi = \pi \circ (B|\mathbb{S}(\mathbb{V}))$. Every holomorphic isometry of $\mathbb{P}(\mathbb{V})$ is obtained in this way. If G is a subgroup of $\mathrm{U}(\mathbb{V})$, we put $\underline{G} := \{\, \underline{B} \,|\, B \in G\,\}$; this is a subgroup of the group $I_h(\mathbb{P}(\mathbb{V}))$ of all holomorphic isometries of $\mathbb{P}(\mathbb{V})$.

We note that $\mathrm{SU}(\mathbb{V})$ acts transitively and via holomorphic isometries on $\mathbb{P}(\mathbb{V})$ by the action $\varphi : \mathrm{SU}(\mathbb{V}) \times \mathbb{P}(\mathbb{V}) \to \mathbb{P}(\mathbb{V}),\ (B, p) \mapsto \underline{B}(p)$; moreover $\mathbb{P}(\mathbb{V})$ is a Hermitian symmetric $\mathrm{SU}(\mathbb{V})$-space of compact type and $\tau : \mathrm{SU}(\mathbb{V}) \to I(\mathbb{P}(\mathbb{V})),\ B \mapsto \varphi_B$ is a covering map onto $I_h(\mathbb{P}(\mathbb{V})) = I(\mathbb{P}(\mathbb{V}))_0 = \underline{\mathrm{SU}(\mathbb{V})}$. It will turn out that already $\mathrm{SU}(\mathbb{V})$ acts transitively on the congruence families we consider in the sequel, and we can therefore consider these congruence families as naturally reductive homogeneous $\mathrm{SU}(\mathbb{V})$-spaces in the way described in Section 7.1.

Projective subspaces in $\mathbb{P}(\mathbb{V})$. We fix $k \in \{1, \dots, n-1\}$. The set of k-dimensional projective subspaces of $\mathbb{P}(\mathbb{V})$ forms a congruence family in $\mathbb{P}(\mathbb{V})$, which we denote by $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$. Because $\mathrm{SU}(\mathbb{V})$ acts transitively and by holomorphic isometries on $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$ via the action $\psi : (B, \Lambda) \mapsto \underline{B}(\Lambda)$, we consider $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$ as a naturally reductive homogeneous $\mathrm{SU}(\mathbb{V})$-space in the way explained above.

As the following theorem shows, the congruence family $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$ is isomorphic to the complex Grassmannian $G_{k+1}(\mathbb{V})$; of course this fact completely describes the geometry of this family.

7.4 Theorem. *The map*

$$\theta : G_{k+1}(\mathbb{V}) \to \mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V})),\ V \mapsto [V] \tag{7.3}$$

is an isomorphism of naturally reductive homogeneous $\mathrm{SU}(\mathbb{V})$*-spaces.*

Thus we see that by transferring the Hermitian symmetric structure of $G_{k+1}(\mathbb{V})$ *onto* $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$ *via* θ *,* $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$ *becomes a Hermitian symmetric* $\mathrm{SU}(\mathbb{V})$*-space (of type AIII as in [Hel78], p. 518 and of complex dimension* $(k+1)(n-k-1)$ *) whose symmetric structure is compatible with its original reductive structure.*

Proof. It is clear that θ is $\mathrm{SU}(\mathbb{V})$-equivariant, and therefore an isomorphism of homogeneous $\mathrm{SU}(\mathbb{V})$-spaces; in particular the isotropy groups of the $\mathrm{SU}(\mathbb{V})$-actions on $G_{k+1}(\mathbb{V})$ and on $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$ at corresponding points coincide. Because for both $G_{k+1}(\mathbb{V})$ and $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$, the reductive structure at some point is the orthocomplement of the Lie algebra of the isotropy group at that point with respect to the Killing form, it follows that θ is an isomorphism of reductive homogeneous $\mathrm{SU}(\mathbb{V})$-spaces. Because also the Riemannian metric of both spaces is the one induced by the Killing form, we see that θ is in fact an isomorphism of naturally reductive homogeneous $\mathrm{SU}(\mathbb{V})$ spaces. The remaining statements are obvious. □

Complex quadrics in $\mathbb{P}(\mathbb{V})$. We wish to study the set of k-dimensional (symmetric) complex quadrics (in the sense of Definition 6.1(a)) contained in $\mathbb{P}(\mathbb{V})$; it will turn out that this set is a congruence family in $\mathbb{P}(\mathbb{V})$.

7.5 Theorem. *Let* $k \in \{1, \dots, n-2\}$ *.*

(a) The set of k-dimensional complex quadrics in $\mathbb{P}(\mathbb{V})$ *is a congruence family, which we denote by* $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$ *. Already* $\mathrm{SU}(\mathbb{V})$ *acts transitively on* $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$ *.*

(b) In the way described in Section 7.1, $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$ *is a naturally reductive homogeneous* $\mathrm{SU}(\mathbb{V})$*-space. Its dimension is* $2(n-1)(k+2) - \frac{1}{2}(3k+4)(k+1)$ *.*

(c) (i) In the case $k < n-2$ *, the naturally reductive structure on* $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$ *is not induced by a symmetric structure.*

(ii) In the case $k = n-2$ *, we now regard* $\mathfrak{F}(Q^{n-2}, \mathbb{P}(\mathbb{V}))$ *as a naturally reductive homogeneous* $I_h(\mathbb{P}(\mathbb{V}))$*-space via the construction of Section 7.1. This naturally reductive homogeneous structure on* $\mathfrak{F}(Q^{n-2}, \mathbb{P}(\mathbb{V}))$ *is induced by a symmetric structure and in this way* $\mathfrak{F}(Q^{n-2}, \mathbb{P}(\mathbb{V}))$ *is an irreducible Riemannian symmetric space of type AI; this means that its universal cover is isomorphic to* $\mathrm{SU}(n)/\mathrm{SO}(n)$ *, see also [Hel78], p. 518.*

The naturally reductive homogeneous structures induced on $\mathfrak{F}(Q^{n-2}, \mathbb{P}(\mathbb{V}))$ *by* $I_h(\mathbb{P}(\mathbb{V}))$ *and by* $\mathrm{SU}(\mathbb{V})$ *are "isomorphic" in the way described in Remark 7.3(b).*

7.6 Remark. The Riemannian symmetric space $\mathfrak{F}(Q^{n-2}, \mathbb{P}(\mathbb{V}))$ of (c)(ii) cannot be equipped with a complex structure so that it becomes a Hermitian symmetric space, see [Hel78], p. 518.

For the proof of Theorem 7.5 we will use a more efficient way to describe k-dimensional complex quadrics in $\mathrm{I\!P}(\mathbb{V})$ than the one provided by Definition 6.1(a). For this purpose we introduce the concept of a *partial conjugation*:

7.7 Definition. *A* partial conjugation *on* $\mathbb{V}$ *is an anti-linear map* $A : \mathbb{V} \to \mathbb{V}$ *which is self-adjoint with respect to* $\langle \cdot, \cdot \rangle_{\mathrm{I\!R}}$ *and which satisfies* $A^3 = A$. *If* A *is a partial conjugation on* $\mathbb{V}$, *we put* $V(A) := \mathrm{Eig}(A, 1)$.

7.8 Proposition. *(a) Let* A *be a partial conjugation on* $\mathbb{V}$. *Then the real rank of* A *is necessarily even,* $A(\mathbb{V}) =: U$ *and* $\ker(A)$ *are complex,* A*-invariant subspaces of* $\mathbb{V}$, *we have* $\mathbb{V} = \ker(A) \oplus U$, *and* $A|U : U \to U$ *is a conjugation on* U *in the sense of Section 2.1.*

(b) An anti-linear map $A : \mathbb{V} \to \mathbb{V}$ *which is self-adjoint with respect to* $\langle \cdot, \cdot \rangle_{\mathrm{I\!R}}$ *is a partial conjugation on* $\mathbb{V}$ *if and only if* $A|A(\mathbb{V}) : A(\mathbb{V}) \to A(\mathbb{V})$ *is a conjugation on* $A(\mathbb{V})$ *in the sense of Section 2.1.*

In the sequel, we denote the set of partial conjugations on $\mathbb{V}$ which are of real rank $2k$ by $\mathrm{Con}_k(\mathbb{V})$; we have $\mathrm{Con}_n(\mathbb{V}) = \mathrm{Con}(\mathbb{V})$ by Proposition 7.8(b). When we wish to emphasize the difference between the more general concept of the partial conjugations of Definition 7.7 and the concept of the conjugations of Section 2.1, we call the latter *full conjugations.*

Proof of Proposition 7.8. For (a). Because A is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathrm{I\!R}}$, A is real diagonalizable and the eigenspaces of A are pairwise orthogonal; because of the equation $A^3 = A$ the only possible eigenvalues of A are $-1, 0, 1$. Therefore we have

$$\mathbb{V} = \mathrm{Eig}(A, 0) \oplus \mathrm{Eig}(A, 1) \oplus \mathrm{Eig}(A, -1) . \tag{7.4}$$

Moreover, because A is anti-linear, we have

$$\forall \lambda \in \mathrm{I\!R} \; : \; \mathrm{Eig}(A, -\lambda) = J(\mathrm{Eig}(A, \lambda)) . \tag{7.5}$$

It follows from Equation (7.5) that $\ker(A) = \mathrm{Eig}(A, 0)$ is a complex subspace of $\mathbb{V}$; clearly this space is A-invariant. From Equation (7.4) it follows that

$$U := A(\mathbb{V}) = \mathrm{Eig}(A, 1) \oplus \mathrm{Eig}(A, -1) \tag{7.6}$$

and $\mathbb{V} = \ker(A) \oplus U$ holds. We see from Equation (7.6) that U is A-invariant; U is complex because of Equation (7.5). Finally, (7.6) also shows that $A|U$ is an anti-linear isomorphism on U, hence the equation $A^3 = A$ implies $(A|U)^2 = \mathrm{id}_U$, whence it follows by application of Proposition 2.3(h) to $A|U$ that this map is a full conjugation on U.

For (b). Let $A : \mathbb{V} \to \mathbb{V}$ be an anti-linear map which is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathrm{I\!R}}$. It has already been shown in (a) that if A is a partial conjugation on $\mathbb{V}$, then $A|A(\mathbb{V})$ is a full conjugation on $U := A(\mathbb{V})$.

Conversely, we now suppose that $A|U$ is a full conjugation on U. Then we have to prove that A is a partial conjugation on $\mathbb{V}$, and for this it only remains to show that $A^3 = A$ holds. Because A is anti-linear and self-adjoint, $\ker(A)$ and U are complex, A-invariant subspaces of $\mathbb{V}$ and we have $\mathbb{V} = \ker(A) \oplus U$. For any given $v \in \mathbb{V}$ there thus exist unique $v_{\ker} \in \ker(A)$ and $v_U \in U$ so that $v = v_{\ker} + v_U$ holds, and we have

$$A^3 v = A^2(Av_{\ker} + Av_U) = A^2(\underbrace{Av_U}_{\in U}) \overset{(*)}{=} Av_U = Av_{\ker} + Av_U = Av ,$$

here the equals sign marked $(*)$ follows from the fact that $A|U : U \to U$ is a full conjugation. Thus we have shown $A^3 = A$. □

In generalization of the corresponding definitions from Sections 1.1 and 1.2 we define for any $A \in \mathrm{Con}_{k+2}(\mathbb{V})$ (where $k \geq 1$):

$$\widehat{Q}(A) := \{\, z \in A(\mathbb{V}) \setminus \{0\} \,|\, \langle z, Az \rangle_{\mathbb{C}} = 0 \,\}\ , \quad \widetilde{Q}(A) := \widehat{Q}(A) \cap \mathbb{S}(\mathbb{V}) \quad \text{and} \quad Q(A) := \pi(\widetilde{Q}(A))\ .$$

$Q(A)$ is a k-dimensional complex quadric of $\mathbb{P}(\mathbb{V})$ (in the sense of Definition 6.1(a)) and obviously, every k-dimensional complex quadric in $\mathbb{P}(\mathbb{V})$ is obtained in this way.

By generalization of Propositions 1.10 and 1.11 we have:

7.9 Proposition. *(a) Let $A \in \mathrm{Con}_{k+2}(\mathbb{V})$ and $B \in \mathrm{U}(\mathbb{V}) \dot{\cup} \overline{\mathrm{U}}(\mathbb{V})$ be given.*[18] *Then we have $A' := BAB^{-1} \in \mathrm{Con}_{k+2}(\mathbb{V})$, $\widehat{Q}(A') = B(\widehat{Q}(A))$, $\widetilde{Q}(A') = B(\widetilde{Q}(A))$ and $Q(A') = \underline{B}(Q(A))$. Moreover,*

$$\mathrm{Con}_{k+2}(\mathbb{V}) = \{\, BAB^{-1} \,|\, B \in \mathrm{U}(\mathbb{V}) \,\} \tag{7.7}$$

holds.

(b) For $A_1, A_2 \in \mathrm{Con}_{k+2}(\mathbb{V})$, we have

$$Q(A_1) = Q(A_2) \iff \exists\, \lambda \in \mathbb{S}^1 : A_2 = \lambda \cdot A_1\ .$$

Proof. For (a). Let $A \in \mathrm{Con}_{k+2}(\mathbb{V})$ and $B \in \mathrm{U}(\mathbb{V}) \dot{\cup} \overline{\mathrm{U}}(\mathbb{V})$ be given. Then $A' := BAB^{-1}$ is anti-linear, self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ and of real rank $2(k+2)$. Moreover, we have $(A')^3 = (BAB^{-1})^3 = B\,A^3\,B^{-1} = BAB^{-1} = A'$, and therefore $A' \in \mathrm{Con}_{k+2}(\mathbb{V})$. The statements on $\widehat{Q}(A')$, $\widetilde{Q}(A')$ and $Q(A')$ are now obvious.

For Equation (7.7): The inclusion "$\supset$" has already been shown. For the converse inclusion, let $A, A' \in \mathrm{Con}_{k+2}(\mathbb{V})$ be given. We choose orthonormal bases $(a_1, \dots, a_{k+2})$ of $V(A)$ and $(a'_1, \dots, a'_{k+2})$ of $V(A')$, as well as unitary bases $(a_{k+1}, \dots, a_n)$ of $\ker(A)$ and $(a'_{k+1}, \dots, a'_n)$ of $\ker(A')$. Then $\mathcal{B} := (a_1, \dots, a_n)$ and $\mathcal{B}' := (a'_1, \dots, a'_n)$ are unitary bases of $\mathbb{V}$, and if $B \in \mathrm{U}(\mathbb{V})$ denotes the unitary map which transforms $\mathcal{B}$ into $\mathcal{B}'$, we have $A' = BAB^{-1}$.

For (b). The implication "$\Longleftarrow$" is obvious. For the opposite implication, let $A_1, A_2 \in \mathrm{Con}_{k+2}(\mathbb{V})$ be given so that $Q(A_1) = Q(A_2)$ and therefore $\widehat{Q}(A_1) = \widehat{Q}(A_2)$ holds. By Proposition 2.27 there exist bases of $A_1(\mathbb{V})$ resp. of $A_2(\mathbb{V})$ which consist of elements of $\widehat{Q}(A_1)$ resp. of $\widehat{Q}(A_2)$, and therefore the hypothesis $\widehat{Q}(A_1) = \widehat{Q}(A_2)$ implies $A_1(\mathbb{V}) = A_2(\mathbb{V}) =: U$. $Q(A_1|U) = Q(A_2|U)$ is a (symmetric) complex quadric in $\mathbb{P}(U)$ in the sense of Chapter 1 and therefore Proposition 1.10 shows that there exists $\lambda \in \mathbb{S}^1$ so that $A_2|U = \lambda \cdot A_1|U$ holds. We also have $A_1|U^\perp = 0 = A_2|U^\perp$ and therefore $A_2 = \lambda \cdot A_1$ follows. □

Proof of Theorem 7.5. For (a). We denote the set of k-dimensional complex quadrics in $\mathbb{P}(\mathbb{V})$ by $\mathfrak{Q}_k$ and let $Q \in \mathfrak{Q}_k$ be given. Then there exists $A \in \mathrm{Con}_{k+2}(\mathbb{V})$ so that $Q = Q(A)$ holds. We will now show

$$\{\, f(Q) \,|\, f \in I(\mathbb{P}(\mathbb{V})) \,\} \subset \mathfrak{Q}_k \subset \{\, \underline{B}(Q) \,|\, B \in \mathrm{SU}(\mathbb{V}) \,\}\ . \tag{7.8}$$

Because of $\underline{\mathrm{SU}(\mathbb{V})} = I_h(\mathbb{P}(\mathbb{V})) \subset I(\mathbb{P}(\mathbb{V}))$ it then follows that both inclusions in (7.8) are in fact equalities. Thus we see that $I(\mathbb{P}(\mathbb{V}))$ acts transitively on $\mathfrak{Q}_k$, therefore $\mathfrak{Q}_k$ is a congruence family, and that already $\mathrm{SU}(\mathbb{V})$ acts (via $B \mapsto \underline{B}$) transitively on this family.

For the proof of the first inclusion in (7.8), we let $f \in I(\mathbb{P}(\mathbb{V}))$ be given. f is either holomorphic or anti-holomorphic, moreover there exists a unitary or anti-unitary transformation $B : \mathbb{V} \to \mathbb{V}$

[18] Remember that $\overline{\mathrm{U}}(\mathbb{V})$ denotes the set of anti-unitary transformations of $\mathbb{V}$.

so that $f = \underline{B}$ holds. Then we have $BAB^{-1} \in \mathrm{Con}_{k+2}(\mathbb{V})$ by Proposition 7.9(a) and therefore $f(Q) = \underline{B}(Q) = Q(BAB^{-1}) \in \mathfrak{Q}_k$.

For the proof of the second inclusion in (7.8), we let $Q' \in \mathfrak{Q}_k$ be given. Then there exists another partial conjugation $A' \in \mathrm{Con}_{k+2}(\mathbb{V})$ so that $Q' = Q(A')$ holds. By Equation (7.7) in Proposition 7.9 it follows that there exists $B \in \mathrm{U}(\mathbb{V})$ with $A' = BAB^{-1}$. If we choose $\lambda \in \mathbb{S}^1$ so that $\det(\lambda B) = 1$ and therefore $\lambda B \in \mathrm{SU}(\mathbb{V})$ holds, we have $\underline{\lambda B}(Q) = \underline{B}(Q) = Q(BAB^{-1}) = Q(A') = Q'$, and therefore Q' is a member of the right-hand set in (7.8).

Henceforth, we will denote the congruence family $\mathfrak{Q}_k$ by $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$.

For (b). It follows from the introductory remarks of the present section and the fact that already $\mathrm{SU}(\mathbb{V})$ acts transitively on $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$ (see (a)) that $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$ can be equipped with the structure of a naturally reductive homogeneous $\mathrm{SU}(\mathbb{V})$-space in the way described in Section 7.1.

Again we fix $Q \in \mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$, say $Q = Q(A)$ with $A \in \mathrm{Con}_{k+2}(\mathbb{V})$. Then the isotropy group K of the action of $\mathrm{SU}(\mathbb{V})$ on $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$ is in block matrix notation with respect to the decomposition $\mathbb{V} = U \oplus \ker(A)$ with $U := A(\mathbb{V})$ given by

$$K = \left\{ \begin{pmatrix} \lambda B_1^{\mathbb{C}} & 0 \\ 0 & B_2 \end{pmatrix} \middle| \begin{array}{c} B_1 \in \mathrm{O}(V(A)),\ B_2 \in \mathrm{U}(\ker(A)),\ \lambda \in \mathbb{S}^1 \\ \lambda^{k+2} \det(B_1) \det(B_2) = 1 \end{array} \right\} . \tag{7.9}$$

Indeed, suppose that $B \in K$ is given. Then we have $B \in \mathrm{SU}(\mathbb{V})$ and $Q(BAB^{-1}) = \underline{B}(Q) = Q = Q(A)$; by Proposition 7.9(b) it follows that there exists $\lambda \in \mathbb{S}^1$ so that $BAB^{-1} = \lambda^2 A$ holds. We then have

$$B(V(A)) = V(BAB^{-1}) = V(\lambda^2 A) = \lambda V(A) ,$$

whence it follows that $B_1 := (\lambda^{-1} B)|V(A) \in \mathrm{O}(V(A))$ holds. Moreover, it follows that B leaves U and therefore also $U^\perp = \ker(A)$ invariant; thus we have $B_2 := B|\ker(A) \in \mathrm{U}(\ker(A))$ and clearly $B = \left(\begin{smallmatrix} \lambda B_1^{\mathbb{C}} & 0 \\ 0 & B_2 \end{smallmatrix}\right)$ holds. Finally, because of $B \in \mathrm{SU}(\mathbb{V})$ we have

$$1 = \det(B) = \lambda^{k+2} \cdot \det(B_1^{\mathbb{C}}) \cdot \det(B_2) = \lambda^{k+2} \cdot \det(B_1) \cdot \det(B_2) .$$

Thus, the inclusion "$\subset$" of Equation (7.9) is shown; the inclusion "$\supset$" is obvious.

It should be noted that in the right-hand side of Equation (7.9), $\lambda \in \mathbb{S}^1$ can attain only discrete values for each pair (B_1, B_2). Therefore we have

$$\dim K = \dim \mathrm{O}(V(A)) + \dim \mathrm{U}(\ker(A)) = \tfrac{1}{2}(k+2)(k+1) + (n-k-2)^2$$

and hence

$$\begin{aligned} \dim \mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V})) = \dim \mathrm{SU}(\mathbb{V}) - \dim K &= (n^2-1) - \left(\tfrac{1}{2}(k+2)(k+1) + (n-k-2)^2\right) \\ &= 2(n-1)(k+2) - \tfrac{1}{2}(3k+4)(k+1) . \end{aligned}$$

For (c)(i). We first show that the Lie algebra $\mathfrak{k}$ of K is (again in block matrix notation with respect to the decomposition $\mathbb{V} = U \oplus \ker(A)$) given by

$$\mathfrak{k} = \left\{ \begin{pmatrix} X^{\mathbb{C}} + \alpha(Y)\, J|U & 0 \\ 0 & Y \end{pmatrix} \middle| X \in \mathfrak{o}(V(A)),\ Y \in \mathfrak{u}(\ker(A)) \right\} \tag{7.10}$$

with the $\mathrm{I\!R}$-linear form $\alpha : \mathfrak{u}(\ker(A)) \to \mathrm{I\!R},\ Y \mapsto \frac{i}{k+2}\operatorname{tr}_{\mathbb{C}} Y$ (note that $Y \in \mathfrak{u}(\ker(A))$ is anti-Hermitian and therefore we have $\operatorname{tr}_{\mathbb{C}} Y \in i\mathrm{I\!R}$).

For the proof of Equation (7.10) we first note that the right-hand side $\widetilde{\mathfrak{k}}$ of Equation (7.10) is a linear subspace of $\mathfrak{su}(\mathbb{V})$ whose dimension equals $\dim(K)$; therefore it suffices to show $\widetilde{\mathfrak{k}} \subset \mathfrak{k}$, and for this it is in turn sufficient to prove $\exp(\widetilde{\mathfrak{k}}) \subset K$, where $\exp$ is the usual exponential map. For this we let $Z \in \widetilde{\mathfrak{k}}$ be given, say $Z = \left(\begin{smallmatrix} X^{\mathbb{C}} + \alpha(Y)\, J|U & 0 \\ 0 & Y \end{smallmatrix}\right)$ with $X \in \mathfrak{o}(V(A))$ and $Y \in \mathfrak{u}(\ker(A))$, and put $B := \exp(Z)$. Along with Z, B leaves U and $\ker(A)$ invariant. We have

$$\begin{aligned}
\exp(\alpha(Y)\, J|U) &= \sum_{k=0}^{\infty} \tfrac{1}{k!}\, \alpha(Y)^k\, (J|U)^k \\
&= \sum_{k=0}^{\infty} \tfrac{(-1)^k}{(2k)!}\, \alpha(Y)^{2k}\, \mathrm{id}_U \;+\; \sum_{k=0}^{\infty} \tfrac{(-1)^k}{(2k+1)!}\, \alpha(Y)^{2k+1}\, (J|U) \\
&= \cos(\alpha(Y))\, \mathrm{id}_U + \sin(\alpha(Y))\, (J|U) = e^{i\alpha(Y)}\, \mathrm{id}_U
\end{aligned}$$

and therefore

$$B|U = \exp(Z|U) = \exp(X^{\mathbb{C}} + \alpha(Y) J|U) \overset{(*)}{=} \exp(X^{\mathbb{C}}) \cdot \exp(\alpha(Y) J|U) = e^{i\alpha(Y)} \cdot \exp(X)^{\mathbb{C}} = \lambda \cdot B_1^{\mathbb{C}}$$

with $\lambda := e^{i\alpha(Y)} \in \mathbb{S}^1$ and $B_1 := \exp(X) \in \mathrm{O}(V(A))$; here the equals sign marked $(*)$ follows from the fact that the endomorphisms $X^{\mathbb{C}}$ and $\alpha(Y)J|U$ of U commute. Moreover, we have

$$B|(\ker A) = \exp(Z|\ker A) = \exp(Y) =: B_2 \in \mathrm{U}(\ker A)$$

and

$$\lambda^{k+2}\, \det(B_1)\, \det(B_2) = e^{(k+2)i\alpha(Y)}\, e^{\operatorname{tr}(X)}\, e^{\operatorname{tr}(Y)} = 1$$

by the definition of the linear form α and the fact that we have $\operatorname{tr}(X) = 0$ because of $X \in \mathfrak{o}(V(A))$. It follows that $\exp(Z) \in K$ holds, compare Equation (7.9).

Let now $\mathfrak{m} = \mathfrak{k}^{\perp,\varkappa}$ be the reductive structure of $\mathfrak{F}(Q^k, \mathrm{I\!P}(\mathbb{V}))$ at the point Q as described in Proposition 7.2(a). Then we have $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ with

$$\mathfrak{m}_1 := \left\{ \begin{pmatrix} 0 & -Z^* \\ Z & 0 \end{pmatrix} \,\middle|\, Z : U \to \ker A \text{ complex-linear} \right\} \tag{7.11}$$

$$\text{and} \quad \mathfrak{m}_2 := \left\{ \begin{pmatrix} J \circ X^{\mathbb{C}} & 0 \\ 0 & 0 \end{pmatrix} \,\middle|\, X \in \mathrm{End}_+(V(A)), \operatorname{tr} X = 0 \right\}.$$

For the proof of this statement, we first note that $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \{0\}$, $\mathfrak{m}_1 \oplus \mathfrak{m}_2 \subset \mathfrak{su}(\mathbb{V})$ and $\dim(\mathfrak{k}) + \dim(\mathfrak{m}_1) + \dim(\mathfrak{m}_2) = \dim(\mathfrak{su}(\mathbb{V}))$ holds; therefore it suffices to show that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are orthogonal to $\mathfrak{k}$ with respect to the Killing form $\varkappa$ of $\mathfrak{su}(\mathbb{V})$. For this purpose we choose an orthonormal basis $(a_1, \dots, a_{k+2})$ of $V(A)$. Then $(a_1, \dots, a_{k+2})$ also is a unitary basis of U, which we expand to a unitary basis $(a_1, \dots, a_n)$ of $\mathbb{V}$. We have (see [IT91], p. 60)

$$\forall Z_1, Z_2 \in \mathfrak{su}(\mathbb{V}) \;:\; \varkappa(Z_1, Z_2) = (-2n) \cdot \sum_{\nu=1}^{n} \langle Z_1 a_\nu, Z_2 a_\nu \rangle_{\mathrm{I\!R}} \tag{7.12}$$

Because the elements of $\mathfrak{k}$ leave the perpendicular, complementary spaces U and $\ker A$ invariant, whereas the elements of $\mathfrak{m}_1$ interchange these spaces, it immediately follows from Equation (7.12) that $\mathfrak{m}_1$ is $\varkappa$-orthogonal to $\mathfrak{k}$. Now let $Z_1 = \left(\begin{smallmatrix} X^{\mathbb{C}}+\alpha(Y)\,J|U & 0 \\ 0 & Y \end{smallmatrix}\right) \in \mathfrak{k}$ (with $X \in \mathfrak{o}(V(A))$ and $Y \in \mathfrak{u}(\ker(A))$) and $Z_2 = \left(\begin{smallmatrix} J\circ\widetilde{X}^{\mathbb{C}} & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{m}_2$ (with $\widetilde{X} \in \mathrm{End}_+(V(A))$ and $\operatorname{tr}\widetilde{X} = 0$) be given. Then we have by Equation (7.12):

$$\begin{aligned} -\tfrac{1}{2n}\cdot\varkappa(Z_1,Z_2) &= \sum_{\nu=1}^{n}\langle Z_1 a_\nu, Z_2 a_\nu\rangle_{\mathbb{R}} \\ &= \sum_{\nu=1}^{k+2}\big(\langle \underbrace{Xa_\nu}_{\in V(A)}, \underbrace{J\widetilde{X}a_\nu}_{\in JV(A)}\rangle_{\mathbb{R}} + \langle \alpha(Y)\,Ja_\nu, J\widetilde{X}a_\nu\rangle_{\mathbb{R}}\big) \;=\; \alpha(Y)\cdot\operatorname{tr}\widetilde{X} \;=\; 0\,. \end{aligned}$$

This completes the proof that $\mathfrak{m}_2$ is $\varkappa$-orthogonal to $\mathfrak{k}$.

We now consider the endomorphisms $D, E : \mathbb{V}\to\mathbb{V}$ given by

$$Da_1 = Ja_1,\; Da_2 = -Ja_2,\; Da_j = 0 \text{ for } j\geq 3$$

and $E = a_1\wedge a_{k+3}$, i.e.

$$Ea_1 = -a_{k+3},\; Ea_{k+3} = a_1,\; Ea_j = 0 \text{ for } j\in\{2,\dots,n\}\setminus\{k+3\}\,,$$

respectively. We have $D\in\mathfrak{m}_2\subset\mathfrak{m}$ and $E\in\mathfrak{m}_1\subset\mathfrak{m}$. However, a simple calculation shows $[D,E]a_1 = Ja_{k+3}$ and therefore $[D,E]\notin\mathfrak{k}$. Thus, we have $[\mathfrak{m},\mathfrak{m}]\not\subset\mathfrak{k}$, showing that the reductive structure of $\mathfrak{F}(Q^k,\mathbb{P}(\mathbb{V}))$ can not be induced by a symmetric structure on $\mathfrak{F}(Q^k,\mathbb{P}(\mathbb{V}))$.

For (c)(ii). In the case $k = n-2$, we have $\ker A = \{0\}$ and hence A is a full conjugation on $\mathbb{V}$. From Equation (7.9) and Proposition 2.17(a) we see that

$$K = \{\,\lambda B\in\mathrm{SU}(\mathbb{V}) \mid B\in\mathrm{Aut}_s(\mathfrak{A}),\; \lambda\in\mathbb{S}^1,\; \lambda^n = \det(B)\,\} \tag{7.13}$$

holds, where $\mathfrak{A} := \mathbb{S}^1\cdot A$ is the $\mathbb{C}$Q-structure induced by A; therefrom it follows that we have $K_0 = \mathrm{Aut}_s(\mathfrak{A})_0$.

We now regard $\mathfrak{F}(Q^{n-2},\mathbb{P}(\mathbb{V}))$ as an $I_h(\mathbb{P}(\mathbb{V}))$-space. To justify this claim, we note that because of $I_h(\mathbb{P}(\mathbb{V})) = \underline{\mathrm{SU}(\mathbb{V})}$, $I_h(\mathbb{P}(\mathbb{V}))$ acts transitively on $\mathfrak{F}(Q^{n-2},\mathbb{P}(\mathbb{V}))$; the isotropy group of this action at the point Q is $\underline{K} = \underline{\mathrm{Aut}_s(\mathfrak{A})}$. Moreover $\tau : \mathrm{SU}(\mathbb{V})\to I_h(\mathbb{P}(\mathbb{V})) = I(\mathbb{P}(\mathbb{V}))_0,\; B\mapsto\underline{B}$ is a covering map of Lie groups, and therefore the actions of the Lie groups $\mathrm{SU}(\mathbb{V})$ and of $I_h(\mathbb{P}(\mathbb{V}))$ on $\mathfrak{F}(Q^{n-2},\mathbb{P}(\mathbb{V}))$ induce "isomorphic" naturally reductive structures on $\mathfrak{F}(Q^{n-2},\mathbb{P}(\mathbb{V}))$ as was explained in Remark 7.3(b) ($K' = \underline{K}$).

Let us now consider the anti-holomorphic isometry $\underline{A} : \mathbb{P}(\mathbb{V})\to\mathbb{P}(\mathbb{V})$ and the involutive Lie group automorphism

$$\sigma : I_h(\mathbb{P}(\mathbb{V}))\to I_h(\mathbb{P}(\mathbb{V})),\; f\mapsto \underline{A}\circ f\circ\underline{A}^{-1}\,. \tag{7.14}$$

It is easily seen that $\mathrm{Fix}(\sigma) = \underline{\mathrm{Aut}_s(\mathfrak{A})} = \underline{K}$ holds, and therefore σ gives rise to a symmetric space structure on the $I_h(\mathbb{P}(\mathbb{V}))$-space $\mathfrak{F}(Q^{n-2},\mathbb{P}(\mathbb{V}))$. The reductive structure

$\underline{\mathfrak{m}}^\sigma = \mathrm{Eig}(\sigma_L, -1)$ induced by this symmetric structure is orthogonal to the Lie algebra $\underline{\mathfrak{k}} = \mathrm{Eig}(\sigma_L, 1)$ of $\underline{K}$ with respect to the Killing form $\underline{\varkappa}$ of $I_h(\mathbb{P}(\mathbb{V})) = \underline{\mathrm{SU}(\mathbb{V})}$ (indeed, σ_L is a Lie algebra automorphism, and therefore we have for every $X \in \underline{\mathfrak{m}}^\sigma$ and $Y \in \underline{\mathfrak{k}}$: $\underline{\varkappa}(X,Y) = \underline{\varkappa}(\sigma_L(X), \sigma_L(Y)) = \underline{\varkappa}(-X,Y) = -\underline{\varkappa}(X,Y)$ and hence $\underline{\varkappa}(X,Y) = 0$), whence $\underline{\mathfrak{m}}^\sigma = \underline{\mathfrak{k}}^{\perp,\underline{\varkappa}}$ follows. Therefore the symmetric structure on $\mathfrak{F}(Q^{n-2}, \mathbb{P}(\mathbb{V}))$ defined by σ induces the original reductive structure on this space.

It remains to show the statement on the universal cover of $\mathfrak{F}(Q^{n-2}, \mathbb{P}(\mathbb{V}))$. For this, we consider the homogeneous $\mathrm{SU}(\mathbb{V})$-space $\mathrm{SU}(\mathbb{V})/K_0$ and the involutive Lie group automorphism

$$\widetilde{\sigma} : \mathrm{SU}(\mathbb{V}) \to \mathrm{SU}(\mathbb{V}),\ B \mapsto A \circ B \circ A^{-1}\ ; \tag{7.15}$$

we have $\mathrm{Fix}(\widetilde{\sigma}) = \mathrm{Aut}_s(\mathfrak{A})_0 = K_0$, and therefore $\widetilde{\sigma}$ induces a symmetric structure on $\mathrm{SU}(\mathbb{V})/K_0$.

Because $\mathrm{SU}(\mathbb{V})$ is simply connected and K_0 is connected, $\mathrm{SU}(\mathbb{V})/K_0$ is simply connected, as can be read off the exact homotopy sequence for the fibre bundle $\mathrm{SU}(\mathbb{V}) \to \mathrm{SU}(\mathbb{V})/K_0$. Moreover $K_0 = \mathrm{Aut}_s(\mathfrak{A})_0$ is isomorphic to $\mathrm{SO}(n)$ (see Proposition 2.17(a)), and therefore $\mathrm{SU}(\mathbb{V})/K_0$ is isomorphic to $\mathrm{SU}(n)/\mathrm{SO}(n)$.

We now consider the covering map $\psi : \mathrm{SU}(\mathbb{V})/K_0 \to \mathrm{SU}(\mathbb{V})/K$ (whose number of leaves equals the number of connected components of K) and note that the group covering map $\tau : \mathrm{SU}(\mathbb{V}) \to I_h(\mathbb{P}(\mathbb{V}))$ gives rise to a map $\underline{\tau} : \mathrm{SU}(\mathbb{V})/K \to I_h(\mathbb{P}(\mathbb{V}))/\underline{K}$ so that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{SU}(\mathbb{V}) \arrow[r, "\mathrm{id}_{\mathrm{SU}(\mathbb{V})}"] \arrow[d] & \mathrm{SU}(\mathbb{V}) \arrow[r, "\tau"] \arrow[d] & I_h(\mathbb{P}(\mathbb{V})) \arrow[d] \\
\mathrm{SU}(\mathbb{V})/K_0 \arrow[r, "\psi"'] & \mathrm{SU}(\mathbb{V})/K \arrow[r, "\underline{\tau}"'] & I_h(\mathbb{P}(\mathbb{V}))/\underline{K}\ ;
\end{tikzcd}$$

here the vertical arrows represent the canonical projections. Both $\mathrm{SU}(\mathbb{V})/K$ and $I_h(\mathbb{P}(\mathbb{V}))/\underline{K}$ are isomorphic to $\mathfrak{F}(Q^{n-2}, \mathbb{P}(\mathbb{V}))$, and Remark 7.3(b) shows that $\underline{\tau}$ corresponds to the identity map on $\mathfrak{F}(Q^{n-2}, \mathbb{P}(\mathbb{V}))$ under these isomorphisms. In particular, $\underline{\tau}$ is a diffeomorphism.

From the commutativity of the diagram it follows that $(\underline{\tau} \circ \psi, \tau)$ is a homomorphism of homogeneous spaces; moreover from (7.14) and (7.15) it is seen that $\tau \circ \widetilde{\sigma} = \sigma \circ \tau$ holds, and therefore $(\underline{\tau} \circ \psi, \tau)$ is in fact a homomorphism of symmetric spaces from the $\mathrm{SU}(\mathbb{V})$-space $\mathrm{SU}(\mathbb{V})/K_0$ onto the $I_h(\mathbb{P}(\mathbb{V}))$-space $I_h(\mathbb{P}(\mathbb{V}))/\underline{K}$; moreover $\underline{\tau} \circ \psi$ is a covering map.

Because the $I_h(\mathbb{P}(\mathbb{V}))$-spaces $I_h(\mathbb{P}(\mathbb{V}))/\underline{K}$ and $\mathfrak{F}(Q^{n-2}, \mathbb{P}(\mathbb{V}))$ are isomorphic as symmetric spaces, we therefore see that $\underline{\tau} \circ \psi$ gives rise to a covering map of symmetric spaces from $\mathrm{SU}(\mathbb{V})/K_0$ onto $\mathfrak{F}(Q^{n-2}, \mathbb{P}(\mathbb{V}))$; remember that $\mathrm{SU}(\mathbb{V})/K_0$ is isomorphic to $\mathrm{SU}(n)/\mathrm{SO}(n)$. □

7.3 Congruence families in the complex quadric

Continuing to use the notations of Section 7.2, we now suppose that $(\mathbb{V}, \mathfrak{A})$ is a $\mathbb{C}$Q-space and consider the complex quadric $Q := Q(\mathfrak{A})$ in $\mathbb{P}(\mathbb{V})$. Q now plays the role of the ambient space, in which we will study congruence families induced by projective subspaces or complex quadrics contained in Q. These families will turn out to be nice submanifolds of $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$ and $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$, respectively. We put $m := \dim_{\mathbb{C}} Q = n-2$ and suppose $m \geq 3$ (so that every isometry of Q is either holomorphic or anti-holomorphic, see Theorem 3.23(c)).

We note that $\mathrm{Aut}_s(\mathfrak{A})_0$ acts transitively and via holomorphic isometries on Q by the action $\varphi : \mathrm{Aut}_s(\mathfrak{A})_0 \times Q \to Q,\ (B,p) \mapsto \underline{B}(p)$ (see Proposition 3.9); moreover Q is a Hermitian symmetric $\mathrm{Aut}_s(\mathfrak{A})_0$-space of compact type and $\tau : \mathrm{Aut}_s(\mathfrak{A})_0 \to I(Q),\ B \mapsto \varphi_B$ is a covering map onto $I_h(Q)_0 = I(Q)_0$, see Theorem 3.23(a). It will turn out that $\mathrm{Aut}_s(\mathfrak{A})_0$ acts transitively on the connected components of the congruence families we consider, and thus we will regard them as naturally reductive $\mathrm{Aut}_s(\mathfrak{A})_0$-spaces in the way described in Section 7.1.

Totally geodesic complex subquadrics in Q. We fix $k \in \{1, \dots, m-1\}$. As we saw in Chapter 6, the set of k-dimensional complex subquadrics contained in Q does not form a single congruence family, but rather an infinite multitude of such families parametrized by the characteristic angle $t \in [0, \frac{\pi}{4}]$, see Theorem 6.6 and Corollary 6.17. Here we study the congruence family given by $t = 0$, i.e. the congruence family of those k-dimensional complex subquadrics of Q which are totally geodesic submanifolds of Q (of type $(\mathrm{G1}, k)$). We denote this congruence family by $\mathfrak{F}(Q^k_{tg}, Q)$.

7.10 Theorem. *(a) Already $\mathrm{Aut}_s(\mathfrak{A})_0$ acts transitively on $\mathfrak{F}(Q^k_{tg}, Q)$ via τ, and therefore we consider $\mathfrak{F}(Q^k_{tg}, Q)$ as a naturally reductive homogeneous $\mathrm{Aut}_s(\mathfrak{A})_0$-space as described in Section 7.1. As such, $\mathfrak{F}(Q^k_{tg}, Q)$ is isomorphic to the real Grassmannian $G_{k+2}(V(A))$ (where $A \in \mathfrak{A}$). In particular, the reductive structure of $\mathfrak{F}(Q^k_{tg}, Q)$ is induced by a symmetric structure. We have $\dim_{\mathbb{R}} \mathfrak{F}(Q^k_{tg}, Q) = (k+2)(m-k)$.*

(b) $\mathfrak{F}(Q^k_{tg}, Q)$ is a compact, connected submanifold of $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$. As a reductive homogeneous space, it is a subspace of $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$, and therefore, it is a totally geodesic submanifold of $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$; moreover the Riemannian metric of $\mathfrak{F}(Q^k_{tg}, Q)$ is the $\frac{m}{2(m+2)}$-fold of the Riemannian metric induced by $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$.

Proof. In the following, we fix $A \in \mathfrak{A}$. *For (a).* For every $V' \in G_{k+2}(V(A))$, $V' \oplus JV'$ is a $(k+2)$-dimensional $\mathbb{C}$Q-subspace of $\mathbb{V}$ and therefore we have $Q \cap [V' \oplus JV'] \in \mathfrak{F}(Q^k_{tg}, Q)$ by Lemma 5.8. We now consider the map

$$f : G_{k+2}(V(A)) \to \mathfrak{F}(Q^k_{tg}, Q),\ V' \mapsto Q \cap [V' \oplus JV'] .$$

f is surjective because of Proposition 5.10. f is also injective: Suppose that $V_1', V_2' \in G_{k+2}(V(A))$ are given with $f(V_1') = f(V_2')$. For $\ell \in \{1,2\}$ we consider the partial conjugation $A_\ell \in \mathrm{Con}_{k+2}(\mathbb{V})$ characterized by $A_\ell|(V_\ell' \oplus JV_\ell') = A|(V_\ell' \oplus JV_\ell')$ and $A_\ell|(V_\ell' \oplus JV_\ell')^\perp = 0$; then we have $A_\ell(\mathbb{V}) = V_\ell' \oplus JV_\ell'$ and $Q(A_\ell) = f(V_\ell')$. By hypothesis we have $Q(A_1) = Q(A_2)$, hence there exists $\lambda \in \mathbb{S}^1$ with $A_1 = \lambda A_2$ by Proposition 7.9(b). Therefrom $A_1(\mathbb{V}) = A_2(\mathbb{V})$ follows. We now obtain

$$V_1' = (V_1' \oplus JV_1') \cap V(A) = A_1(\mathbb{V}) \cap V(A) = A_2(\mathbb{V}) \cap V(A) = (V_2' \oplus JV_2') \cap V(A) = V_2' ,$$

completing the proof of the injectivity of f.

The Lie group $\mathrm{SO}(V(A))$ acts transitively on $G_{k+2}(V(A))$, and the Lie group $\mathrm{Aut}_s(\mathfrak{A})_0$ acts transitively on $\mathfrak{F}(Q^k_{tg}, Q)$ via τ (see Corollary 6.17(b)). Moreover, with the isomorphism of Lie groups $\mathrm{SO}(V(A)) \to \mathrm{Aut}_s(\mathfrak{A})_0,\ L \mapsto L^{\mathbb{C}}$ (see Proposition 2.17(a)), (f, F) is an isomorphism of homogeneous spaces from the $\mathrm{SO}(V(A))$-space $G_{k+2}(V(A))$ onto the $\mathrm{Aut}_s(\mathfrak{A})_0$-space $\mathfrak{F}(Q^k_{tg}, Q)$.

Because on both $G_{k+2}(V(A))$ and on $\mathfrak{F}(Q^k_{tg}, Q)$, the reductive structure is given by $\mathfrak{m} = \mathfrak{k}^{\perp,\varkappa}$ via the Lie algebra $\mathfrak{k}$ of the respective isotropy group and the Killing form $\varkappa$ of the acting group, and because on both spaces the Riemannian metric is induced by the Killing form, (f, F) is in fact an isomorphism of naturally reductive homogeneous spaces. In particular, we have $\dim \mathfrak{F}(Q^k_{tg}, Q) = \dim G_{k+2}(V(A))$, which gives the formula for the dimension.

For (b). The Lie group $G := \mathrm{SU}(\mathbb{V})$ acts (transitively) on $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$, hence the Lie subgroup $G' := \mathrm{Aut}_s(\mathfrak{A})_0$ of G also acts on $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$, and the set $\mathfrak{F}(Q^k_{tg}, Q)$ is an orbit of the latter action. Considering $\mathfrak{F}(Q^k_{tg}, Q)$ in this way, it is a differentiable submanifold of $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$; its differentiable structure is characterized by the fact that for fixed $Q' \in \mathfrak{F}(Q^k_{tg}, Q)$, the map $G' \to \mathfrak{F}(Q^k_{tg}, Q),\ B \mapsto \underline{B}(Q')$ is a surjective submersion. Therefore this differentiable structure coincides with the original differentiable structure on the family $\mathfrak{F}(Q^k_{tg}, Q)$ defined in Proposition 7.1. Because G' is compact and connected, $\mathfrak{F}(Q^k_{tg}, Q)$ is a compact, and hence regular, connected submanifold of $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$, also see [Var74], Theorem 2.9.7, p. 80. The inclusion map $\mathfrak{F}(Q^k_{tg}, Q) \hookrightarrow \mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$ is equivariant with respect to the action of G', and hence it follows that the homogeneous G'-space $\mathfrak{F}(Q^k_{tg}, Q)$ is a homogeneous subspace of the homogeneous G-space $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$.

The reductive structure of the $\mathrm{SO}(V(A))$-space $G_{k+2}(V(A))$ at $V' \in G_{k+2}(V(A))$ is in block matrix notation with respect to the decomposition $V(A) = V' \oplus (V')^{\perp,V(A)}$ given by

$$\left\{ \begin{pmatrix} 0 & -Z^* \\ Z & 0 \end{pmatrix} \middle| Z : V' \to (V')^{\perp,V(A)} \text{ linear} \right\} .$$

Under the isomorphism (f, F) of reductive homogeneous spaces, this space is transformed into the reductive structure of the G'-space $\mathfrak{F}(Q^k_{tg}, Q)$ at the "point" $Q' := f(V') \in \mathfrak{F}(Q^k_{tg}, Q)$, which is therefore given by

$$\mathfrak{m}' = \left\{ \begin{pmatrix} 0 & -(Z^{\mathbb{C}})^* \\ Z^{\mathbb{C}} & 0 \end{pmatrix} \middle| Z : V(A') \to (\ker A' \cap V(A)) \text{ linear} \right\} ,$$

where $A' \in \mathrm{Con}_{k+2}(\mathbb{V})$ is the partial conjugation characterized by $A'|(V' \oplus JV') = A|(V' \oplus JV')$ and $A'|(V' \oplus JV')^{\perp} = 0$, and where the block matrix is to be read with respect to the decomposition $\mathbb{V} = A'(\mathbb{V}) \oplus \ker A'$; note that $V(A') = V'$ holds. If we denote the reductive structure of the G-space $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$ at $Q' \in \mathfrak{F}(Q^k_{tg}, Q)$ by $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ as in the proof of Theorem 7.5(c)(i), we now see by comparison with Equation (7.11) that $\mathfrak{m}' \subset \mathfrak{m}_1 \subset \mathfrak{m}$ holds. Thus $\mathfrak{F}(Q^k_{tg}, Q)$ is a reductive homogeneous subspace of $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$.

Moreover, if we denote the usual scalar product of endomorphisms by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, the Killing forms of $\mathfrak{g}'$ and of $\mathfrak{g}$ are given by $(X, Y) \mapsto -m \cdot \langle\!\langle X, Y \rangle\!\rangle$ and $(X, Y) \mapsto -2(m+2) \cdot \langle\!\langle X, Y \rangle\!\rangle$, respectively (see Proposition 2.17(a) and [IT91], p. 60). It follows from this fact that the Riemannian metric of $\mathfrak{F}(Q^k_{tg}, Q)$ is the $\frac{m}{2(m+2)}$-fold of the Riemannian metric induced by $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$. □

Projective subspaces in Q. We now suppose $k \leq \frac{m}{2}$. As we saw in Section 5.5, there exist k-dimensional complex-projective subspaces of $\mathbb{P}(\mathbb{V})$ which are contained in Q; they are exactly the totally geodesic submanifolds of Q of type $(\mathrm{I}1, k)$.

We now characterize the position of these subspaces in the following way: Put $\widetilde{Q} := \widetilde{Q}(\mathfrak{A})$ and denote by

$$G_{k+1}(\mathbb{V}, \widetilde{Q}) = \{\, U \in G_{k+1}(\mathbb{V}) \,|\, \mathbb{S}(U) \subset \widetilde{Q} \,\}$$

the set of complex-$(k+1)$-dimensional, isotropic subspaces of $\mathbb{V}$. Note that $G_1(\mathbb{V}, \widetilde{Q}) = Q$ holds. Denoting for any $\Lambda \in \mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$ by $\widehat{\Lambda} \in G_{k+1}(\mathbb{V})$ the linear space characterized by $[\widehat{\Lambda}] = \Lambda$, we therefore have

$$\forall \Lambda \in \mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V})) \ : \big(\Lambda \subset Q \iff \widehat{\Lambda} \in G_{k+1}(\mathbb{V}, \widetilde{Q})\big) \,. \tag{7.16}$$

For each $U \in G_{k+1}(\mathbb{V}, \widetilde{Q})$ and any $A \in \mathfrak{A}$ there exists a *partial complex structure* $j : V(A) \to V(A)$ (i.e. a skew-adjoint endomorphism with $j^3 = -j$) of rank $2(k+1)$ so that

$$U = \{\, x + Jjx \,|\, x \in j(V(A)) \,\} \tag{7.17}$$

holds; note that $j|j(V(A))$ is an orthogonal complex structure on $j(V(A))$. Conversely, for every partial complex structure j on $V(A)$, the corresponding space U defined by Equation (7.17) is a member of $G_{k+1}(\mathbb{V}, \widetilde{Q})$.

Proof of the last statements. Let $U \in G_{k+1}(\mathbb{V}, \widetilde{Q})$ be given. By Proposition 2.20(e),(f) there exist a $2(k+1)$-dimensional subspace $Y \subset V(A)$ and an orthogonal complex structure $\tau : Y \to Y$ so that $U = \{\, x + J\tau x \,|\, x \in Y \,\}$ holds. Let $j : V(A) \to V(A)$ be the linear map characterized by $j|Y = \tau$ and $j|Y^\perp = 0$. Then j is a partial complex structure of rank $2(k+1)$ so that Equation (7.17) holds.

Conversely, if a partial complex structure $j : V(A) \to V(A)$ is given, we have $\langle x + Jjx, A(x + Jjx)\rangle_{\mathbb{C}} = \langle x + Jjx, x - Jjx\rangle_{\mathbb{C}} = \langle x, x\rangle_{\mathbb{C}} - \langle jx, jx\rangle_{\mathbb{C}} = 0$ for every $x \in j(V(A))$ and therefore the space U corresponding to j is indeed isotropic. □

7.11 Theorem. *(a) $I(Q)$ acts transitively on the set of k-dimensional projective subspaces contained in Q, and therefore this set forms a congruence family, which we denote by $\mathfrak{F}(\mathbb{P}^k, Q)$.*[19] *In fact, already $\mathrm{Aut}_s(\mathfrak{A})$ acts transitively on $\mathfrak{F}(\mathbb{P}^k, Q)$ via the two-fold covering map $\mathrm{Aut}_s(\mathfrak{A}) \to I_h(Q),\ B \mapsto \underline{B}|Q$.*

(b) $\mathfrak{F}(\mathbb{P}^k, Q)$ is a complex, compact submanifold of $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$. It is of complex dimension $\frac{1}{2} \cdot (k+1)(2m-3k)$ and thus of complex codimension $\frac{1}{2} \cdot (k+1)(k+2)$ in $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$.

(c) If $2k < m$, the manifold $\mathfrak{F}(\mathbb{P}^k, Q)$ is connected; if $2k = m$, it consists of exactly two connected components. In either case, $\mathrm{Aut}_s(\mathfrak{A})_0$ acts transitively on the connected components of $\mathfrak{F}(\mathbb{P}^k, Q)$, and they will therefore be considered as naturally reductive homogeneous $\mathrm{Aut}_s(\mathfrak{A})_0$-spaces in the way described in Section 7.1. Furthermore:

[19] $\mathfrak{F}(\mathbb{P}^k, Q)$ is isomorphic to the typical fibre of a twistor bundle, see [Raw84] Proposition 2.1 (p. 88) and p. 102.

(i) If $2k = m$, the reductive structures of the connected components of $\mathfrak{F}(\mathbb{P}^k, Q)$ are induced by symmetric structures, and in this regard they are isomorphic to the irreducible Hermitian symmetric space of compact type $\mathrm{SO}(\,2(k+1)\,)/\mathrm{U}(k+1)$ of type DIII, see [Hel78], p. 518. Moreover, they are symmetric subspaces and therefore totally geodesic submanifolds of the symmetric space $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$; their Riemannian metric is the $\frac{m}{2(m+2)}$-fold of the Riemannian metric induced by $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$.

(ii) If $1 \leq 2k < m$, the reductive structure of $\mathfrak{F}(\mathbb{P}^k, Q)$ is not induced by a symmetric structure. Therefore, $\mathfrak{F}(\mathbb{P}^k, Q)$ (equipped with the reductive structure mentioned above) is also not a reductive homogeneous subspace of $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$ (although it is a homogeneous subspace). Moreover, $\mathfrak{F}(\mathbb{P}^k, Q)$ is not a totally geodesic submanifold of $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$.

Proof. Throughout the proof we fix $A \in \mathfrak{A}$. *For (a).* It is easily seen from the representation of the members of $G_{k+1}(\mathbb{V}, \widetilde{Q})$ via partial complex structures as in Equation (7.17) that the action of $\mathrm{Aut}(\mathfrak{A})\dot{\cup}\overline{\mathrm{Aut}}(\mathfrak{A})$ leaves $G_{k+1}(\mathbb{V}, \widetilde{Q})$ invariant, and that already $\mathrm{Aut}_s(\mathfrak{A})$ acts transitively on this space. Because the $\mathrm{Aut}(\mathfrak{A})\dot{\cup}\overline{\mathrm{Aut}}(\mathfrak{A})$-equivariant map $\theta : G_{k+1}(\mathbb{V}) \to \mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$, $U \mapsto [U]$ maps $G_{k+1}(\mathbb{V}, \widetilde{Q})$ bijectively onto the set $\mathfrak{P}_k$ of k-dimensional projective subspaces of $\mathbb{P}(\mathbb{V})$ which are contained in Q (see (7.16)), it follows that $\underline{\mathrm{Aut}(\mathfrak{A})\dot{\cup}\overline{\mathrm{Aut}}(\mathfrak{A})} = I_h(Q)\dot{\cup}I_{ah}(Q) = I(Q)$ leaves $\mathfrak{P}_k$ invariant, and that already $\underline{\mathrm{Aut}_s(\mathfrak{A})} = I_h(Q)$ acts transitively on $\mathfrak{P}_k$. Therefore $\mathfrak{P}_k$ is indeed a congruence family, which is in the sequel denoted by $\mathfrak{F}(\mathbb{P}^k, Q)$.

For (b). The Lie group $\underline{\mathrm{SU}(\mathbb{V})}$ acts (transitively) on $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$, hence the Lie subgroup $\underline{\mathrm{Aut}_s(\mathfrak{A})}$ of $\underline{\mathrm{SU}(\mathbb{V})}$ also acts on $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$, and the set $\mathfrak{F}(\mathbb{P}^k, Q)$ is an orbit of the latter action. Considering $\mathfrak{F}(\mathbb{P}^k, Q)$ in this way, it is a differentiable submanifold of $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$; its differentiable structure is characterized by the fact that for fixed $\Lambda \in \mathfrak{F}(\mathbb{P}^k, Q)$, the map $\underline{\mathrm{Aut}_s(\mathfrak{A})} \to \mathfrak{F}(\mathbb{P}^k, Q)$, $f \mapsto f(\Lambda)$ is a surjective submersion. Therefore this differentiable structure coincides with the original differentiable structure on the family $\mathfrak{F}(\mathbb{P}^k, Q)$ defined in Proposition 7.1. Because $\underline{\mathrm{Aut}_s(\mathfrak{A})}$ is compact, $\mathfrak{F}(\mathbb{P}^k, Q)$ is a compact, and hence regular submanifold of $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$, also see [Var74], Theorem 2.9.7, p. 80.

However, a more explicit proof is needed to show that $\mathfrak{F}(\mathbb{P}^k, Q)$ is a complex submanifold of $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$.

Because of (7.16), the biholomorphic map $\theta^{-1} : \mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V})) \to G_{k+1}(\mathbb{V})$, $\Lambda \mapsto \widehat{\Lambda}$ (see Theorem 7.4) maps $\mathfrak{F}(\mathbb{P}^k, Q)$ onto $G_{k+1}(\mathbb{V}, \widetilde{Q})$. Therefore it is sufficient to show that $G_{k+1}(\mathbb{V}, \widetilde{Q})$ is a complex submanifold of $G_{k+1}(\mathbb{V})$. We abbreviate $r := k+1$.

We now consider the Stiefel manifold $\widehat{\mathrm{St}}_r(\mathbb{V}) := \{\, u \in L(\mathbb{C}^r, \mathbb{V}) \mid u \text{ is injective} \,\}$; this is a complex manifold, and the canonical projection $\widehat{\varrho} : \widehat{\mathrm{St}}_r(\mathbb{V}) \to G_r(\mathbb{V})$, $u \mapsto u(\mathbb{C}^r)$ is a holomorphic submersion. Moreover, we fix $A \in \mathfrak{A}$ and consider the non-degenerate, symmetric $\mathbb{C}$-bilinear form $\beta : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$, $(v, w) \mapsto \langle v, Aw \rangle_{\mathbb{C}}$ and the holomorphic map

$$g : \widehat{\mathrm{St}}_r(\mathbb{V}) \to \mathbb{C}^{r(r+1)/2},\ u \mapsto \big(\,\beta(u_\mu, u_\nu)\,\big)_{1 \leq \mu \leq \nu \leq r} ;$$

here we put $u_\mu := u(e_\mu)$ for $u \in L(\mathbb{C}^r, \mathbb{V}) \supset \widehat{\mathrm{St}}_r(\mathbb{V})$ and $\mu \in \{1, \ldots, r\}$, where $(e_1, \ldots, e_r)$ is the canonical basis of $\mathbb{C}^r$. Note that because of Proposition 2.20(a), $g^{-1}(\{0\}) = \widehat{\varrho}^{-1}(G_r(\mathbb{V},\widetilde{Q}))$ holds.

Immediately, we will show that g is a submersion; it then follows that $g^{-1}(\{0\}) = \widehat{\varrho}^{-1}(G_r(\mathbb{V},\widetilde{Q}))$ is a regular, complex submanifold of $\widehat{\mathrm{St}}_r(\mathbb{V})$ (see [Nar68], Corollary 2.5.5, p. 81). Thus we may then conclude that $G_r(\mathbb{V},\widetilde{Q})$ is a complex submanifold of $G_r(\mathbb{V})$. (Local trivializations of $\widehat{\varrho}$ give rise to local parameterizations of $G_r(\mathbb{V},\widetilde{Q})$.) Moreover, we see that the complex codimension of $\widehat{\varrho}^{-1}(G_r(\mathbb{V},\widetilde{Q}))$ in $\mathrm{St}_r(\mathbb{V})$ is equal to $\frac{1}{2}r(r+1) = \frac{1}{2}(k+1)(k+2)$, and therefore the complex codimension of $G_r(\mathbb{V},\widetilde{Q})$ in $G_r(\mathbb{V})$ also is $\frac{1}{2}(k+1)(k+2)$. Therefrom follow the formulas for the codimension and (together with Theorem 7.4) the dimension of $\mathfrak{F}(\mathbb{P}^k,Q)$.

For the proof of the submersivity of g let $u \in \widehat{\mathrm{St}}_r(\mathbb{V})$ be given. Then we have $\overrightarrow{T_u\widehat{\mathrm{St}}_r(\mathbb{V})} = L(\mathbb{C}^r,\mathbb{V})$ and

$$\forall \xi \in T_u\widehat{\mathrm{St}}_r(\mathbb{V}) \;:\; \overrightarrow{T_u g(\xi)} = \big(\beta((\overrightarrow{\xi})_\mu, u_\nu) + \beta(u_\mu, (\overrightarrow{\xi})_\nu)\big)_{1\le\mu\le\nu\le r} \;\in\; \mathbb{C}^{r(r+1)/2} .$$

To show that $T_u g : T_u\widehat{\mathrm{St}}_r(\mathbb{V}) \to T_u\mathbb{C}^{r(r+1)/2}$ is surjective, it is therefore sufficient to prove that the linear forms $(\lambda_{\mu\nu})_{\mu\le\nu}$ with

$$\lambda_{\mu\nu} : L(\mathbb{C}^r,\mathbb{V}) \to \mathbb{C},\; a \mapsto \beta(a_\mu, u_\nu) + \beta(u_\mu, a_\nu)$$

are linear independent. For this we first note that because of the non-degeneracy of β we have

$$\forall z \in \mathbb{C}^r \;\exists\, w \in \mathbb{V} \;:\; \big(\beta(w,u_\nu)\big)_{1\le\nu\le r} = z\,. \tag{7.18}$$

Now let $(\alpha_{\mu\nu})_{\mu\le\nu} \in \mathbb{C}^{r(r+1)/2}$ be given so that $\sum_{\mu\le\nu} \alpha_{\mu\nu}\,\lambda_{\mu\nu} = 0$ holds. Further, let $\mu_0 \le \nu_0$ be given and put $\ell := 1$ in the case $\mu_0 < \nu_0$, $\ell := 2$ in the case $\mu_0 = \nu_0$. By (7.18) there exists $a \in L(\mathbb{C}^r,\mathbb{V})$ so that

$$\forall \mu,\nu \in \{1,\dots,r\} \;:\; \beta(a_\mu,u_\nu) = \tfrac{1}{\ell}\cdot\delta_{\mu,\mu_0}\cdot\delta_{\nu,\nu_0}$$

holds. Then we have for every $\mu \le \nu$: $\lambda_{\mu\nu}(a) = \delta_{\mu,\mu_0}\cdot\delta_{\nu,\nu_0}$ and therefore

$$0 = \sum_{\mu\le\nu} \alpha_{\mu\nu}\,\lambda_{\mu\nu}(a) = \alpha_{\mu_0\,\nu_0}\,.$$

This shows the linear independence of $(\lambda_{\mu\nu})$.

For (c). We fix an arbitrary subspace $\Lambda_0 \in \mathfrak{F}(\mathbb{P}^k,Q)$. Then we have $W_0 := \widehat{\Lambda_0} \in G_r(\mathbb{V},\widetilde{Q})$ by (7.16); therefore, there exists a partial complex structure j_0 on $V(A)$ of rank $2(k+1)$ such that $W_0 = \{\, x + Jj_0x \,|\, x \in j_0(V(A))\,\}$ holds.

Because the group $\mathrm{Aut}_s(\mathfrak{A}) \cong \mathrm{O}(V(A))$, which acts transitively on $\mathfrak{F}(\mathbb{P}^k,Q)$, has exactly two connected components (Proposition 2.17(a)), $\mathfrak{F}(\mathbb{P}^k,Q)$ has at most two connected components, and $G := \mathrm{Aut}_s(\mathfrak{A})_0$ acts transitively on each of them, so they are homogeneous G-spaces, and become naturally reductive homogeneous G-spaces by the construction described in Section 7.1. Moreover, G is a subgroup of $\mathrm{SU}(\mathbb{V})$, so the connected components of $\mathfrak{F}(\mathbb{P}^k,Q)$ are homogeneous subspaces of the homogeneous $\mathrm{SU}(\mathbb{V})$-space $\mathfrak{F}(\mathbb{P}^k,\mathbb{P}(\mathbb{V}))$.

To investigate whether $\mathfrak{F}(\mathbb{P}^k, Q)$ is connected, we consider the isotropy group K of the action of $\mathrm{Aut}_s(\mathfrak{A})$ on $\mathfrak{F}(\mathbb{P}^k, Q)$ at Λ_0; K is also given by $\{\, B \in \mathrm{Aut}_s(\mathfrak{A}) \,|\, B j_0^{\mathbb{C}} = j_0^{\mathbb{C}} B \,\}$. As we will now see, the connectedness of $\mathfrak{F}(\mathbb{P}^k, Q)$ depends on whether K is contained in the neutral component G of $\mathrm{Aut}_s(\mathfrak{A})$.

In the case $2k < m$ there exists a 1-codimensional subspace of $V(A)$ which contains $j_0(V(A))$, and if $L \in \mathrm{O}(V(A))$ is the reflection in such a subspace, we have $L^{\mathbb{C}} \in K$ and $\det L = -1$. Because of $\mathrm{Aut}_s(\mathfrak{A}) = G \,\dot\cup\, \{\, B \circ L^{\mathbb{C}} \,|\, B \in G \,\}$, we see that in this case already the connected group G acts transitively on $\mathfrak{F}(\mathbb{P}^k, Q)$, and therefore $\mathfrak{F}(\mathbb{P}^k, Q)$ is connected.

On the other hand, in the case $2k = m$ the unitary group $\mathrm{U}(V(A), j_0)$ is (via complexification with respect to the complex structure of $\mathbb{V}$) isomorphic to K, hence K is connected and therefore contained in G. It follows that $\{\, \underline{B}(\Lambda_0) \,|\, B \in G \,\}$ and $\{\, \underline{B}(\Lambda_0) \,|\, B \in \mathrm{Aut}_s(\mathfrak{A}) \setminus G \,\}$ are disjoint, open subsets of $\mathfrak{F}(\mathbb{P}^k, Q)$ which together constitute all of $\mathfrak{F}(\mathbb{P}^k, Q)$. Therefore these two sets are the two connected components of $\mathfrak{F}(\mathbb{P}^k, Q)$ in this case.

For (c)(i). We suppose $2k = m$ and denote the connected component of $\mathfrak{F}(\mathbb{P}^k, Q)$ which contains Λ_0 by $\mathfrak{F}(\mathbb{P}^k, Q)'$. As we saw above, G acts transitively on $\mathfrak{F}(\mathbb{P}^k, Q)'$, and therefore $\mathfrak{F}(\mathbb{P}^k, Q)'$ is a $(\varphi|(G \times Q))$-family of submanifolds in the sense of Section 7.1, and we regard this family as a naturally reductive space in the way described there.

We next describe how the reductive structure of $\mathfrak{F}(\mathbb{P}^k, Q)'$ is induced by a symmetric structure. For this we consider the reflection $S : \mathbb{V} \to \mathbb{V}$ in W_0 and the involutive Lie group automorphism

$$\widetilde{\sigma} : \mathrm{SU}(\mathbb{V}) \to \mathrm{SU}(\mathbb{V}),\ B \mapsto SBS^{-1} \,.$$

We have $j_0^{\mathbb{C}} = -J \circ S$ (this equation is easily checked on W_0 and on $W_0^{\perp} = \{\, x - J j_0 x \,|\, x \in V(A) \,\}$) and consequently

$$\forall B \in G \,:\, \widetilde{\sigma}(B) = j_0^{\mathbb{C}}\, B\, (j_0^{\mathbb{C}})^{-1} \,\in\, G \,. \tag{7.19}$$

Thus we see that $\sigma := \widetilde{\sigma}|G$ is an involutive Lie group automorphism on G. It also follows from (7.19) that $\mathrm{Fix}(\sigma) = K$ holds (remember that we have $K \subset G$ in the present situation), and therefore σ gives rise to a symmetric structure on $\mathfrak{F}(\mathbb{P}^k, Q)'$. The reductive structure $\mathfrak{m}^{\sigma} = \mathrm{Eig}(\sigma_L, -1)$ induced by this symmetric structure is orthogonal to $\mathfrak{k} = \mathrm{Eig}(\sigma_L, 1)$ with respect to the Killing form $\varkappa$ of $\mathfrak{g}$ (by the same argument that was already used in the proof of Theorem 7.5(c)(ii)), whence it follows that the symmetric structure on $\mathfrak{F}(\mathbb{P}^k, Q)'$ defined by σ induces the original reductive structure on this space. Note that $\mathfrak{F}(\mathbb{P}^k, Q)'$ is isomorphic to $G/K \cong \mathrm{SO}(\,2(k+1)\,)/\mathrm{U}(k+1)$.

Now we show that equipped with this symmetric structure, $\mathfrak{F}(\mathbb{P}^k, Q)'$ is a symmetric subspace of $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$. For this purpose we note that the isotropy group of the action of $\mathrm{SU}(\mathbb{V})$ on $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$ is $\widetilde{K} := \{\, B \in \mathrm{SU}(\mathbb{V}) \,|\, B(W_0) = W_0 \,\}$, that $\mathrm{Fix}(\widetilde{\sigma}) = \widetilde{K}$ holds, that therefore $\widetilde{\sigma}$ gives rise to a symmetric structure on $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$, and that this symmetric structure is the one described in Theorem 7.4 which induces the original reductive structure on $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$. Because σ is the restriction of $\widetilde{\sigma}$ to G, it follows that $\mathfrak{F}(\mathbb{P}^k, Q)'$ is a symmetric subspace, and therefore a totally geodesic submanifold, of $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$.

By the same argument as in the proof of Theorem 7.10(b) we see that the Riemannian metric of $\mathfrak{F}(\mathbb{P}^k, Q)'$ is the $\frac{m}{2(m+2)}$-fold of the Riemannian metric induced by $\mathfrak{F}(Q^k, \mathbb{P}(\mathbb{V}))$.

For (c)(ii). We now suppose $2k < m$ and once again abbreviate $r := k+1$, then we have $4 \leq 2r < m+2$. In this setting G acts transitively on $\mathfrak{F}(\mathbb{P}^k, Q)$, which therefore is a $(\varphi|(G \times Q))$-family of submanifolds in the sense of Section 7.1; we regard this family as a naturally reductive space in the way described there.

We derive an explicit description of the reductive structure $\mathfrak{m} := \mathfrak{m}_{\Lambda_0}$ of $\mathfrak{F}(\mathbb{P}^k, Q)$ at $\Lambda_0 \in \mathfrak{F}(\mathbb{P}^k, Q)$: Now, the isotropy group of G at Λ_0 is $K \cap G = \{\, B \in G \,|\, B j_0^{\mathbb{C}} = j_0^{\mathbb{C}} B \,\}$; hence its Lie algebra is $\mathfrak{k} := \{\, X \in \mathfrak{g} \,|\, X j_0^{\mathbb{C}} = j_0^{\mathbb{C}} X \,\} = \ker \mathrm{ad}(j_0^{\mathbb{C}})$, note $j_0^{\mathbb{C}} \in \mathfrak{g} = \mathfrak{aut}_s(\mathfrak{A})$ (see Proposition 2.17(a)). Because $\mathrm{ad}(j_0^{\mathbb{C}})$ is skew-adjoint with respect to the Killing form of $\mathfrak{g}$, it follows that the reductive structure of $\mathfrak{F}(\mathbb{P}^k, Q)$ is given by $\mathfrak{m} = \mathrm{ad}(j_0^{\mathbb{C}})(\mathfrak{g})$.

There exists an orthonormal basis $(a_1, \ldots, a_{m+2})$ of $V(A)$ so that

$$\forall \nu \in \{1, \ldots, m+2\} \;:\; j_0 a_\nu = \begin{cases} a_{\nu+r} & \text{for } 1 \leq \nu \leq r \\ -a_{\nu-r} & \text{for } r+1 \leq \nu \leq 2r \\ 0 & \text{for } 2r+1 \leq \nu \leq m+2 \end{cases}$$

holds. We consider the endomorphisms $X := a_1 \wedge a_r \in \mathfrak{g}$ and $Y := a_{2r} \wedge a_{2r+1} \in \mathfrak{g}$, i.e.

$$\begin{aligned} X a_1 = -a_r,\; X a_r = a_1,\; X a_\nu = 0 \text{ otherwise} \\ \text{and} \quad Y a_{2r} = -a_{2r+1},\; Y a_{2r+1} = a_{2r},\; Y a_\nu = 0 \text{ otherwise.} \end{aligned}$$

Then we have $X' := \mathrm{ad}(j_0^{\mathbb{C}})X$, $Y' := \mathrm{ad}(j_0^{\mathbb{C}})Y \in \mathfrak{m}$. We further put $Z := [X', Y']$. Then a simple calculation shows $(\mathrm{ad}(j_0)Z)a_1 = -a_{2r+1}$, and therefore $Z \notin \ker \mathrm{ad}(j_0) = \mathfrak{k}$. Thus we have $[\mathfrak{m}, \mathfrak{m}] \not\subset \mathfrak{k}$, and therefore the reductive structure of $\mathfrak{F}(\mathbb{P}^k, Q)$ cannot come from a symmetric structure. Furthermore, because $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$ is a symmetric space, the submanifold $\mathfrak{F}(\mathbb{P}^k, Q)$ cannot be a reductive homogeneous subspace (because it would then be a totally geodesic submanifold and hence a symmetric subspace).

Finally, assume that $\mathfrak{F}(\mathbb{P}^k, Q)$ were a totally geodesic submanifold of $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$. Because $\mathfrak{F}(\mathbb{P}^k, Q)$ is connected and complete (as a Riemannian naturally reductive homogeneous space), $\mathfrak{F}(\mathbb{P}^k, Q)$ would then be a Riemannian symmetric G_0'-subspace of the $\mathrm{SU}(\mathbb{V})$-space $\mathfrak{F}(\mathbb{P}^k, \mathbb{P}(\mathbb{V}))$, where

$$G' := \{\, B \in \mathrm{SU}(\mathbb{V}) \,|\, \underline{B}(\mathfrak{F}(\mathbb{P}^k, Q)) = \mathfrak{F}(\mathbb{P}^k, Q) \,\}$$

([KN69], Theorem XI.4.2, p. 235). We will show that $G_0' = G$ holds. Then $\mathfrak{F}(\mathbb{P}^k, Q)$ would be a Riemannian symmetric G-space; its symmetric structure would induce the original naturally reductive structure of $\mathfrak{F}(\mathbb{P}^k, Q)$ because of the same argument as in the proof of Theorem 7.5(c)(ii), but this is impossible by our previous result.

To prove $G_0' = G$, we first show

$$G' = \{\, \mu B_0 \in \mathrm{SU}(\mathbb{V}) \,|\, B_0 \in \mathrm{Aut}_s(\mathfrak{A}),\; \mu \in \mathbb{S}^1,\; \mu^{m+2} = \det(B_0) \,\}\,. \tag{7.20}$$

If B is taken from the right-hand side of Equation (7.20), then we have $B \in \mathrm{Aut}(\mathfrak{A})$ and therefore $\underline{B}(Q) = Q$. It follows that we have $\underline{B}(\mathfrak{F}(\mathbb{P}^k, Q)) = \mathfrak{F}(\mathbb{P}^k, Q)$ and thus $B \in G'$. Conversely, let $B \in G'$ be given. We have

$$\underline{B}(Q) = Q\,. \tag{7.21}$$

In fact, let $p \in Q$ be given. Then there exists $\Lambda \in \mathfrak{F}(\mathbb{P}^k, Q)$ with $p \in \Lambda$. Because of $B \in G'$ we then also have $\underline{B}(\Lambda) \in \mathfrak{F}(\mathbb{P}^k, Q)$, in particular $\underline{B}(p) \in Q$. Thus we have shown $\underline{B}(Q) \subset Q$. Analogously we obtain $\underline{B}^{-1}(Q) \subset Q$ and therefore $Q \subset \underline{B}(Q)$. Therefrom Equation (7.21) follows.

Now let us fix $A \in \mathfrak{A}$. From Equation (7.21) we see $Q(BAB^{-1}) = Q(A)$; by Proposition 1.10 it follows that there exists $\lambda \in \mathbb{S}^1$ so that $BAB^{-1} = \lambda A$ holds. If we choose $\mu \in \mathbb{S}^1$ with $\mu^2 = \lambda$, we therefore have $(\overline{\mu}B)A(\overline{\mu}B)^{-1} = A$ and hence $B_0 := \overline{\mu}B \in \mathrm{Aut}_s(\mathfrak{A})$. We have

$$1 = \det(B) = \det(\mu B_0) = \mu^{m+2}\underbrace{\det(B_0)}_{\in\{\pm 1\}}$$

and therefore $\det(B_0) = \mu^{m+2}$. This shows that $B = \mu B_0$ is a member of the right-hand side of Equation (7.20).

We now show $G'_0 = G$. Because $G = \mathrm{Aut}_s(\mathfrak{A})_0$ is a connected group which is contained in G' by Equation (7.20), we have $G \subset G'_0$. For the converse direction, we fix $A \in \mathfrak{A}$ and $v_0 \in \mathbb{S}(V(A))$. For every $B \in G'$, say $B = \mu B_0$ with

$$B_0 \in \mathrm{Aut}_s(\mathfrak{A}) \quad \text{and} \quad \mu \in \mathbb{S}^1,\ \mu^{m+2} = \det(B_0) \in \{\pm 1\}\,,$$

we have

$$\langle Bv_0, ABv_0\rangle_{\mathbb{C}} = \mu^2 \cdot \langle B_0v_0, A(\underbrace{B_0v_0}_{\in V(A)})\rangle_{\mathbb{C}} = \mu^2 \cdot \|B_0v_0\|^2 = \mu^2\,.$$

Because we have $\mu^{m+2} = 1$, we thus see that the continuous map

$$f : G' \to \mathbb{S}^1,\ B \mapsto \langle Bv_0, ABv_0\rangle_{\mathbb{C}}$$

attains only discrete values and is therefore on G'_0 identically equal to $f(\mathrm{id}_{\mathbb{V}}) = 1$. Consequently, we have

$$\begin{aligned} G'_0 \subset f^{-1}(\{1\}) &= \{\,\mu B_0 \mid B_0 \in \mathrm{Aut}_s(\mathfrak{A}),\ \mu^2 = 1,\ \mu^{m+2} = \det(B_0)\,\} \\ &\subset \{\,\mu B_0 \mid B_0 \in \mathrm{Aut}_s(\mathfrak{A}),\ \mu \in \{\pm 1\}\,\} = \mathrm{Aut}_s(\mathfrak{A}) \end{aligned}$$

and therefore $G'_0 \subset \mathrm{Aut}_s(\mathfrak{A})_0 = G$. $\square$

Chapter 8

The geometry of Q^1, Q^3, Q^4 and Q^6

As was first noted by È. Cartan in [Car14], there are some "intersection points" between the seven infinite series of classical irreducible Riemannian symmetric spaces. To determine which they are, one can use the well-known fact that two simply connected Riemannian symmetric spaces of compact type are isomorphic if and only if their Lie triple systems have the same "extended Dynkin diagram", i.e. if they have isomorphic configurations of simple roots and if the multiplicities of corresponding simple roots are equal ([Loo69], Theorem VII.3.9(a), p. 145).

As an inspection of the table of Dynkin diagrams of irreducible Lie triple systems (see [Loo69], Table 4 on p. 119 and Table 8 on p. 146) shows, the following isomorphies and no other exist between complex quadrics of specific dimension and members of other Riemannian symmetric spaces:

$$Q^1 \cong \mathbb{S}^2 \cong \mathbb{P}^1, \quad Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1, \quad Q^3 \cong \mathrm{Sp}(2)/\mathrm{U}(2),$$
$$Q^4 \cong G_2(\mathbb{C}^4) \quad \text{and} \quad Q^6 \cong \mathrm{SO}(8)/\mathrm{U}(4).$$

The subject of the present chapter is the explicit construction of these isomorphisms (except for $Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$, which has already been described in Section 3.4).

It should be noted that the concept of an isomorphism between the mentioned spaces can be understood with respect to several categories: First, the isomorphisms can be understood as isomorphisms of complex manifolds, as one would do in complex analysis.

But the viewpoint most natural in the present situation is that of the theory of symmetric spaces, the spaces involved being irreducible symmetric spaces. Then, an isomorphism is an isomorphism of symmetric spaces as defined in Sections A.2, A.3. (Are we speaking of isomorphisms of affine symmetric spaces, of Riemannian symmetric spaces or of Hermitian symmetric spaces? This distinction does not play an important role here: Q^m is irreducible for $m \neq 2$, and therefore an isomorphism of affine symmetric spaces from Q^m ($m \in \{1,3,4,6\}$) to another Riemannian (Hermitian) symmetric space already is an isomorphism of Riemannian (Hermitian) symmetric spaces.)

One might also take the viewpoint of Riemannian geometry and ask for isometries. However, then one first has to answer the question which Riemannian metric one should use on the spaces

$\mathrm{Sp}(2)/\mathrm{U}(2)$, $G_2(\mathbb{C}^4)$ and $\mathrm{SO}(8)/\mathrm{U}(4)$ occurring in the isomorphies for Q^3, Q^4 resp. Q^6. They are symmetric spaces, so one will naturally use such a metric that the acting group acts via isometries. Because the spaces are irreducible, two such metrics differ only by a positive, real factor. However, no member of this $\mathbb{R}_+$-family of metrics is singled out in a geometric way.[20] Thus we see that the geometrically relevant concept here is not that of an isometry, but that of a homothety.

Finally, the complex Grassmannian $G_2(\mathbb{C}^4)$ can be equipped with the structure of a quaternionic Kähler manifold, see Section 8.3, therefore the isomorphy to Q^4 shows that Q^4 can also be equipped with such a structure, such that the isomorphy holds also as an isomorphy of quaternionic-Kähler manifolds. In that situation we will find a relation between the quaternionic Kähler structure and the $\mathbb{C}$Q-structure on Q^4.

In the construction of the specific isomorphisms, we will employ the following strategies:

(a) $Q^1 \cong \mathbb{P}^1$. ($\mathrm{SO}(3)/\mathrm{SO}(2) \cong \mathrm{SU}(2)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1))$.) In Section 8.1 we will describe an isomorphism between Q^1 and $\mathbb{S}^2_{r=1/\sqrt{2}}$ (note that both these spaces are represented by the quotient $\mathrm{SO}(3)/\mathrm{SO}(2)$); it is well-known that the oriented euclidean sphere $\mathbb{S}^2_{r=1/\sqrt{2}}$ is isomorphic to $\mathbb{P}^1$.

(b) $Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$. ($\mathrm{SO}(4)/(\mathrm{SO}(2) \times \mathrm{SO}(2)) \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/(\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)) \times \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)))$.) This isomorphism has already been constructed in Section 3.4; it is based on the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to Q^2 \subset \mathbb{P}^3$.

(c) $Q^4 \cong G_2(\mathbb{C}^4)$. ($\mathrm{SO}(6)/(\mathrm{SO}(2) \times \mathrm{SO}(4)) \cong \mathrm{SU}(4)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2))$.) This isomorphism will be constructed in Section 8.2 via the Plücker embedding. In Section 8.3, we will see in what way $G_2(\mathbb{C}^n)$ carries the structure of a quaternionic Kähler manifold; the isomorphy $Q^4 \cong G_2(\mathbb{C}^4)$ therefore shows that Q^4 (unlike the complex quadrics of every other dimension) also carries the structure of a quaternionic Kähler manifold. Moreover, we will find a relation between this quaternionic Kähler structure and the $\mathbb{C}$Q-structure on Q^4.

(d) $Q^3 \cong \mathrm{Sp}(2)/\mathrm{U}(2)$. We will construct this isomorphism by realizing $\mathrm{Sp}(2)/\mathrm{U}(2)$ as a $\mathrm{Sp}(2)$-orbit in $G_2(\mathbb{C}^4)$ and then restricting the isomorphism described in (c) to this orbit.

(e) $Q^6 \cong \mathrm{SO}(8)/\mathrm{U}(4)$. We will use the theory of spinors and the Principle of Triality (which are described in Appendix B) to construct an isomorphism between Q^6 and a connected component of the congruence family $\mathfrak{F}(\mathbb{P}^3, Q^6)$. The latter is isomorphic to $\mathrm{SO}(8)/\mathrm{U}(4)$ by Theorem 7.11(c)(i).

During the construction of the isomorphisms, we will also obtain the following isomorphisms of Lie groups:

[20] Actually, this is not quite true for $G_2(\mathbb{C}^4)$. Here the construction of the quaternionic Kähler structure provides a "canonical" Riemannian metric, see Section 8.3.

(a) $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$ (in connection with $Q^3 \cong \mathrm{Sp}(2)/\mathrm{U}(2)$)

(b) $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$ (in connection with $Q^4 \cong G_2(\mathbb{C}^4)$).

8.1 Q^1 is isomorphic to $\mathbb{S}^2_{r=1/\sqrt{2}}$

Let $(\mathbb{V}, \mathfrak{A})$ be a 3-dimensional $\mathbb{C}$Q-space. Then $Q := Q(\mathfrak{A})$ is a 1-dimensional complex quadric, and we also consider its pre-image under the Hopf fibration $\widetilde{Q} := \widetilde{Q}(\mathfrak{A})$. Moreover, we fix $A \in \mathfrak{A}$. Then the 2-sphere $\mathbb{S} := \mathbb{S}_r(V(A))$ of radius $r := 1/\sqrt{2}$ is a Riemannian symmetric $\mathrm{SO}(V(A))$-space in the usual way.

We fix an orientation on $V(A)$. Then there exists one and only one skew-symmetric bilinear map $\times : V(A) \times V(A) \to V(A),\ (x,y) \mapsto x \times y$ so that for every orthonormal system (x,y) in $V(A)$, $(x, y, x \times y)$ is a positively oriented orthonormal basis of $V(A)$. $\times$ is called the *cross product* on $V(A)$.

We equip $\mathbb{S}$ with the complex structure $J^{\mathbb{S}} : T\mathbb{S} \to T\mathbb{S}$ given by

$$\forall q \in \mathbb{S},\ v \in T_q\mathbb{S} \ : \ \overrightarrow{J^{\mathbb{S}}_q(v)} = (\sqrt{2}\, q) \times \overrightarrow{v} \ . \tag{8.1}$$

In this way, $\mathbb{S}$ becomes a Hermitian symmetric space.

We also consider the Hopf fibration $\pi : \mathbb{S}(\mathbb{V}) \to \mathbb{P}(\mathbb{V})$.

8.1 Proposition. *There is one and only one map* $f_1 : Q \to \mathbb{S}$ *so that*

$$\forall z \in \widetilde{Q} \ : \ f_1(\pi(z)) = \sqrt{2}\, \mathrm{Re}_A(z) \times \mathrm{Im}_A(z)$$

holds and f_1 *is a holomorphic isometry. Moreover, with the isomorphism of Lie groups*

$$F_1 : \mathrm{Aut}_s(\mathfrak{A})_0 \to \mathrm{SO}(V(A)),\ B \mapsto B|V(A)$$

(f_1, F_1) *is an isomorphism of Hermitian symmetric spaces from the* $\mathrm{Aut}_s(\mathfrak{A})_0$*-space* Q *to the* $\mathrm{SO}(V(A))$*-space* $\mathbb{S}$. *Thus, we have shown the following isomorphy in the category of Hermitian symmetric spaces:*

$$\boxed{Q^1 \cong \mathbb{S}^2_r} \ .$$

Proof. For $z \in \widetilde{Q}$, $(\sqrt{2}\, \mathrm{Re}_A(z), \sqrt{2}\, \mathrm{Im}_A(z))$ is an orthonormal system in $V(A)$ by Proposition 2.23(b), and hence

$$(\ \sqrt{2}\, \mathrm{Re}_A(z),\ \sqrt{2}\, \mathrm{Im}_A(z),\ 2\,(\mathrm{Re}_A(z) \times \mathrm{Im}_A(z))\ )$$

is a positively oriented orthonormal basis of $V(A)$. It follows that $\|\,\mathrm{Re}_A(z) \times \mathrm{Im}_A(z)\| = \frac{1}{2}$ holds, and therefore the map

$$\widetilde{f_1} : \widetilde{Q} \to \mathbb{S},\ z \mapsto \sqrt{2} \cdot (\mathrm{Re}_A(z) \times \mathrm{Im}_A(z)) \tag{8.2}$$

in fact maps into $\mathbb{S}$. We also see that

$$\begin{aligned} & (\mathrm{Re}_A(z) \times \mathrm{Im}_A(z)) \times \mathrm{Im}_A(z) = -\tfrac{1}{2} \cdot \mathrm{Re}_A(z) \\ \text{and} \quad & \mathrm{Re}_A(z) \times (\mathrm{Re}_A(z) \times \mathrm{Im}_A(z)) = -\tfrac{1}{2} \cdot \mathrm{Im}_A(z) \end{aligned} \tag{8.3}$$

holds.

For $z, z' \in \widetilde{Q}$ we have

$$\begin{aligned} \widetilde{f}_1(z) = \widetilde{f}_1(z') &\iff \mathrm{Re}_A(z) \times \mathrm{Im}_A(z) = \mathrm{Re}_A(z') \times \mathrm{Im}_A(z') \\ &\iff \begin{array}{c} (\mathrm{Re}_A(z), \mathrm{Im}_A(z)) \text{ and } (\mathrm{Re}_A(z'), \mathrm{Im}_A(z')) \text{ are orthonormal bases} \\ \text{of the same 2-dimensional subspace of } V(A) \text{ with the same orientation.} \end{array} \\ &\iff \exists\, t \in \mathbb{R} \;:\; \begin{cases} \mathrm{Re}_A(z) = \cos(t)\,\mathrm{Re}_A(z') - \sin(t)\,\mathrm{Im}_A(z') \\ \mathrm{Im}_A(z) = \sin(t)\,\mathrm{Re}_A(z') + \cos(t)\,\mathrm{Im}_A(z') \end{cases} \\ &\iff \exists\, t \in \mathbb{R} \;:\; z = e^{it} \cdot z' \\ &\iff \pi(z) = \pi(z') \,. \end{aligned}$$

This equivalence shows the existence and injectivity of f_1. f_1 is also surjective: If $q \in \mathbb{S}$ is given, choose $x, y \in V(A)$ with $\|x\| = \|y\| = \frac{1}{\sqrt{2}}$ so that $(\sqrt{2}\,x, \sqrt{2}\,y, \sqrt{2}\,q)$ is a positively oriented orthonormal basis of $V(A)$. Then we have $p := \pi(x + Jy) \in Q$ and $f_1(p) = q$.

$\widetilde{f}_1$ is differentiable; because π is a surjective submersion, it follows that f_1 is also differentiable.

We next show that f_1 is a holomorphic isometry. It suffices to show that for any given $z \in \widetilde{Q}$

$$T_z\widetilde{f}_1|\mathcal{H}_zQ : \mathcal{H}_zQ \to T_{\widetilde{f}_1(z)}\mathbb{S}$$

is a $\mathbb{C}$-linear isometry.

We have by Theorem 2.26:

$$\overrightarrow{\mathcal{H}_zQ} = \mathbb{C} \cdot (\mathrm{Re}_A(z) \times \mathrm{Im}_A(z)) \,.$$

Thus any given $v \in \mathcal{H}_zQ$ can be represented as $\overrightarrow{v} = c \cdot (\mathrm{Re}_A(z) \times \mathrm{Im}_A(z))$ with a suitable $c = a + ib \in \mathbb{C}$. Then we have

$$\overrightarrow{T_z\widetilde{f}_1(v)} = \sqrt{2}\cdot(\,\mathrm{Re}_A(\overrightarrow{v})\times\mathrm{Im}_A(z) \;+\; \mathrm{Re}_A(z)\times\mathrm{Im}_A(\overrightarrow{v})\,) \overset{(8.3)}{=} -\tfrac{1}{\sqrt{2}}\cdot(\,a\,\mathrm{Re}_A(z) + b\,\mathrm{Im}_A(z)\,) \tag{8.4}$$

and therefore

$$\begin{aligned} \|T_z\widetilde{f}_1(v)\|^2 &= \tfrac{1}{2} \cdot (a^2\,\|\mathrm{Re}_A(z)\|^2 + b^2\,\|\mathrm{Im}_A(z)\|^2) = \tfrac{1}{4} \cdot (a^2 + b^2) \\ &= (a^2 + b^2) \cdot \|\mathrm{Re}_A(z) \times \mathrm{Im}_A(z)\|^2 = \|v\|^2 \,. \end{aligned}$$

Hence $T_z\widetilde{f}_1|\mathcal{H}_zQ$ is an $\mathbb{R}$-linear isometry. Moreover, we have $\overrightarrow{Jv} = (-b+ia)\cdot(\mathrm{Re}_A(z)\times\mathrm{Im}_A(z))$ and therefore

$$\begin{aligned} \overrightarrow{J^{\mathbb{S}}T_z\widetilde{f}_1(v)} &\overset{(8.1)}{=} (\sqrt{2}\,\widetilde{f}_1(z)) \;\times\; \overrightarrow{T_z\widetilde{f}_1(v)} \\ &\overset{(8.2)}{\underset{(8.4)}{=}} (2\;\mathrm{Re}_A(z) \times \mathrm{Im}_A(z)) \;\times\; (-\tfrac{1}{\sqrt{2}}\,(a\,\mathrm{Re}_A(z) + b\,\mathrm{Im}_A(z))) \\ &\overset{(8.3)}{=} -\tfrac{1}{\sqrt{2}}\,(-b\,\mathrm{Re}_A(z) + a\,\mathrm{Im}_A(z)) \overset{(8.4)}{=} \overrightarrow{T_z\widetilde{f}_1(Jv)} \,. \end{aligned}$$

Thus $T_z\widetilde{f}_1|\mathcal{H}_zQ$ is a $\mathbb{C}$-linear isometry.

F_1 is an isomorphism of Lie groups by Proposition 2.17(a). For any $B \in \mathrm{Aut}_s(\mathfrak{A})_0$ and $z \in \widetilde{Q}$ we have $B|V(A) \in \mathrm{SO}(V(A))$, and therefore

$$\begin{aligned} F_1(B)\widetilde{f}_1(z) &= B(\sqrt{2}\,(\mathrm{Re}_A\,z \times \mathrm{Im}_A\,z)) = \sqrt{2}\,(B(\mathrm{Re}_A\,z) \times B(\mathrm{Im}_A\,z)) \\ &= \sqrt{2}\,(\mathrm{Re}_A(Bz) \times \mathrm{Im}_A(Bz)) = \widetilde{f}_1(Bz) \end{aligned}$$

holds. Hence (f_1, F_1) is an isomorphism of homogeneous spaces. Because the Lie groups $\mathrm{Aut}_s(\mathfrak{A})_0$ and $\mathrm{SO}(V(A))$ acting on the symmetric space Q resp. $\mathbb{S}$ are of compact type, (f_1, F_1) is in fact an isomorphism of affine symmetric spaces by Proposition A.5. Because f_1 is a holomorphic isometry, it follows that (f_1, F_1) is an isomorphism of Hermitian symmetric spaces. □

8.2 Q^4 is isomorphic to $G_2(\mathbb{C}^4)$

In the present section, we first describe a specific embedding $\mathcal{P} : G_2(\mathbb{C}^n) \to \mathbb{P}(\bigwedge^2\mathbb{C}^n)$ from the complex Grassmannian $G_2(\mathbb{C}^n)$, called the *Plücker embedding.* Then we show how $G_2(\mathbb{C}^n)$ can be equipped with the structure of a Hermitian symmetric space. In the case $n = 4$, it will turn out that the image Q of the Plücker embedding is a 4-dimensional symmetric complex quadric in $\mathbb{P}(\bigwedge^2\mathbb{C}^4)$, and that $\mathcal{P}$ gives rise to an isomorphism of Hermitian symmetric spaces from $G_2(\mathbb{C}^4)$ to Q.

At first, we let W be a complex linear space of arbitrary dimension n. To W we associate the linear space $\bigwedge^2 W$ of bivectors of W (see Section B.1), the complex projective space $\mathbb{P}(\bigwedge^2 W)$ and the holomorphic fibre bundle $\widehat{\pi} : \bigwedge^2 W \setminus \{0\} \to \mathbb{P}(\bigwedge^2 W),\ \xi \mapsto \mathbb{C}\,\xi$. We also consider the Stiefel manifold

$$\widehat{\mathrm{St}}_2(W) := \{\, u \in L(\mathbb{C}^2, W) \mid u \text{ is injective} \,\}$$

of 2-frames in W, which is an open subset and hence a complex submanifold of the $\mathbb{C}$-linear space $L(\mathbb{C}^2, W)$, the Grassmannian $G_2(W)$ and the projection $\widehat{\theta} : \widehat{\mathrm{St}}_2(W) \to G_2(W),\ u \mapsto u(\mathbb{C}^2)$. As is well-known, there is exactly one way to equip $G_2(W)$ with the structure of a complex submanifold so that $\widehat{\theta}$ becomes a holomorphic submersion.[21]

The following fact is well-known:

8.2 Proposition. *$\xi \in \bigwedge^2 W$ is decomposable (i.e. $\xi = v \wedge w$) if and only if $\xi \wedge \xi = 0$ holds.*

Proof. See [Car51], p. 11. □

In the sequel, we put $u_\mu := u(e_\mu)$ for $u \in L(\mathbb{C}^2, W)$ and $\mu \in \{1, 2\}$; here (e_1, e_2) is the canonical basis of $\mathbb{C}^2$.

[21] The construction of the manifold structure of the real Grassmannian $G_k(\mathbb{R}^n)$, to which the construction of the complex manifold structure of the complex Grassmannian $G_k(\mathbb{C}^n)$ is entirely analogous, is for example described in [Boo86], p. 63f.

8.3 Proposition. *There exists a holomorphic embedding* $\mathcal{P} : G_2(W) \to \mathbb{P}(\bigwedge^2 W)$ *characterized by*

$$\forall u \in \widehat{\mathrm{St}}_2(W) \; : \; \mathcal{P}(u(\mathbb{C}^2)) = \widehat{\pi}(u_1 \wedge u_2) \; . \tag{8.5}$$

Its image in $\mathbb{P}(\bigwedge^2 W)$ *is the complex submanifold*

$$\{\, \widehat{\pi}(\xi) \,|\, \xi \in \textstyle\bigwedge^2 W \setminus \{0\},\; \xi \wedge \xi = 0 \,\} \; .$$

$\mathcal{P}$ *is called the* Plücker embedding*.*

Proof. Let us consider the holomorphic map $\widehat{\mathcal{P}} : \widehat{\mathrm{St}}_2(W) \to \bigwedge^2 W,\; u \mapsto u_1 \wedge u_2$. One sees that

$$\forall\, u, u' \in \widehat{\mathrm{St}}_2(W) \; : \big(\, \widehat{\pi}(\widehat{\mathcal{P}}(u)) = \widehat{\pi}(\widehat{\mathcal{P}}(u')) \iff \widehat{\theta}(u) = \widehat{\theta}(u') \,\big)$$

holds. Therefore, there exists exactly one map $\mathcal{P} : G_2(W) \to \mathbb{P}(\bigwedge^2 W)$ so that Equation (8.5) holds, and it is injective. Moreover, $\mathcal{P}$ is holomorphic along with $\widehat{\mathcal{P}}$ because $\widehat{\theta}$ is a holomorphic surjective submersion. The image of $\mathcal{P}$ is as stated in the proposition because of Proposition 8.2. $\mathrm{GL}(W)$ acts transitively on $G_2(W)$ via $(B, U) \mapsto B(U)$ and it acts transitively on $\mathcal{P}(G_2(W))$ via[22] $(B, \xi) \mapsto B^{(2)}(\xi)$. $\mathcal{P}$ is equivariant with respect to these actions and therefore Proposition A.1(b) shows $\mathcal{P} : G_2(W) \to \mathcal{P}(G_2(W))$ to be a diffeomorphism. Because $G_2(W)$ is compact, it follows that $\mathcal{P} : G_2(W) \to \mathbb{P}(\bigwedge^2 W)$ is an embedding. □

We now suppose that W is a unitary space. Then the inner product on W gives rise to the inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ on $\bigwedge^2 W$ characterized by

$$\forall v_1, v_2, w_1, w_2 \in W \; : \; \langle v_1 \wedge v_2, w_1 \wedge w_2 \rangle_{\mathbb{C}} = \langle v_1, w_1 \rangle_{\mathbb{C}} \cdot \langle v_2, w_2 \rangle_{\mathbb{C}} - \langle v_1, w_2 \rangle_{\mathbb{C}} \cdot \langle v_2, w_1 \rangle_{\mathbb{C}} \tag{8.6}$$

(also see Appendix B.2), and the latter inner product in turn induces the Fubini/Study metric on the projective space $\mathbb{P}(\bigwedge^2 W)$. In this way $\mathbb{P}(\bigwedge^2 W)$ becomes a Kähler manifold and the Hopf fibration $\pi : \mathbb{S}(\bigwedge^2 W) \to \mathbb{P}(\bigwedge^2 W)$ becomes a Riemannian submersion.

We will equip the complex manifold $G_2(W)$ with a Riemannian metric so that it becomes a Kähler manifold and the Plücker embedding $\mathcal{P} : G_2(W) \to \mathbb{P}(\bigwedge^2 W)$ becomes an isometric embedding. It should be noted that the construction of this Riemannian metric is entirely analogous to the construction of the Fubini/Study metric on a complex projective space via the Hopf fibration.

The restriction $\widehat{R} := \widetilde{R}|(\widehat{\mathrm{St}}_2(W) \times \mathrm{GL}(\mathbb{C}^2)) : \widehat{\mathrm{St}}_2(W) \times \mathrm{GL}(\mathbb{C}^2) \to \widehat{\mathrm{St}}_2(W)$ of the map

$$\widetilde{R} : L(\mathbb{C}^2, W) \times \mathrm{End}(\mathbb{C}^2) \to L(\mathbb{C}^2, W),\; (u, A) \mapsto u \circ A \tag{8.7}$$

is a Lie group action of $\mathrm{GL}(\mathbb{C}^2)$ on $\widehat{\mathrm{St}}_2(W)$ from the right. The orbits of this action are the fibres of $\widehat{\theta}$, and on them, the action is simply transitive. Therefore $\widehat{\theta}$ becomes a principal fibre bundle with structure group $\mathrm{GL}(\mathbb{C}^2)$ via $\widehat{R}$.

We will now first construct a Hermitian metric on $\widehat{\mathrm{St}}_2(W)$, then reduce the principal fibre bundle $\widehat{\theta}$ with structure group $\mathrm{GL}(\mathbb{C}^2)$ to a principal bundle $\theta : \mathrm{St}_2(W) \to G_2(W)$ with

[22] For the meaning of $B^{(2)}$ see Section B.1.

structure group $\mathrm{U}(2)$ (where $\mathrm{St}_2(W)$ is a Riemannian submanifold of $\widehat{\mathrm{St}}_2(W)$) and use θ to project the Riemannian metric of $\mathrm{St}_2(W)$ onto $G_2(W)$, thereby obtaining the desired metric on $G_2(W)$.

The complex inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ on W gives rise to the complex inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}}^{L}$ on $L(\mathbb{C}^2,W)$ given by

$$\forall u,u' \in L(\mathbb{C}^2,W) \;:\; \langle u,u'\rangle_{\mathbb{C}}^{L} := \langle u_1,u_1'\rangle_{\mathbb{C}} + \langle u_2,u_2'\rangle_{\mathbb{C}} \,. \tag{8.8}$$

As an open subset of $L(\mathbb{C}^2,W)$, the Stiefel manifold $\widehat{\mathrm{St}}_2(W)$ becomes a Hermitian manifold with the Hermitian metric induced by this inner product. Besides the vertical bundle $\mathcal{V}^{\widehat{\theta}} := \ker(T\widehat{\theta})$ of $\widehat{\theta}$, we thus also have the horizontal bundle $\mathcal{H}^{\widehat{\theta}} := (\mathcal{V}^{\widehat{\theta}})^{\perp,T\widehat{\mathrm{St}}_2(W)}$. The following proposition gives an explicit description of these subbundles of $T\widehat{\mathrm{St}}_2(W)$:

8.4 Proposition. *Let $u \in \widehat{\mathrm{St}}_2(W)$ be given. Then we have:*

(a) $\overrightarrow{\mathcal{V}_u^{\widehat{\theta}}} = \{\, u \circ X \,|\, X \in \mathrm{End}(\mathbb{C}^2)\,\} = \{\, v \in L(\mathbb{C}^2,W) \,|\, v(\mathbb{C}^2) \subset \widehat{\theta}(u)\,\}$.

(b) $\overrightarrow{\mathcal{H}_u^{\widehat{\theta}}} = \{\, v \in L(\mathbb{C}^2,W) \,|\, v(\mathbb{C}^2) \perp \widehat{\theta}(u)\,\}$.

It follows from this proposition that the linear subspaces $\overrightarrow{\mathcal{V}_u^{\widehat{\theta}}}$ and $\overrightarrow{\mathcal{H}_u^{\widehat{\theta}}}$ of $L(\mathbb{C}^2,W)$ do not depend on the choice of u within any given fibre of $\widehat{\theta}$. We also note that part (b) of the proposition shows that every space $\mathcal{H}_u^{\widehat{\theta}}$, and therefore also the Grassmann manifold $G_2(W)$, is of complex dimension $2\cdot(n-2)$.

Proof of Proposition 8.4. For (a). Because the fibres of $\widehat{\theta}$ are equal to the orbits of the $\mathrm{GL}(\mathbb{C}^2)$-action $\widehat{R}$, we have

$$\mathcal{V}_u^{\widehat{\theta}} = \ker T_u\widehat{\theta} = T_u(\widehat{\theta}^{-1}(\{\widehat{\theta}(u)\})) = T_u\{\, \widehat{R}(u,A) \,|\, A \in \mathrm{GL}(\mathbb{C}^2)\,\} \,.$$

Because $\widehat{R}(u,\cdot) : \mathrm{GL}(\mathbb{C}^2) \to \widehat{\mathrm{St}}_2(W)$ is the restriction of the linear map $\widetilde{R}(u,\cdot) : \mathrm{End}(\mathbb{C}^2) \to L(\mathbb{C}^2,W)$ to the open subset $\mathrm{GL}(\mathbb{C}^2)$ of $\mathrm{End}(\mathbb{C}^2)$, the previous equation implies

$$\overrightarrow{\mathcal{V}_u^{\widehat{\theta}}} = \{\, \widetilde{R}(u,X) \,|\, X \in \mathrm{End}(\mathbb{C}^2)\,\} \,,$$

whence the first equals sign in (a) follows. The second equals sign is a consequence of the fact that $u : \mathbb{C}^2 \to \widehat{\theta}(u)$ is an isomorphism of linear spaces.

For (b). Let us abbreviate $U := \widehat{\theta}(u)$. Then the orthogonal decomposition $W = U \oplus U^{\perp,W}$ induces an orthogonal decomposition

$$L(\mathbb{C}^2,W) = L(\mathbb{C}^2,U) \oplus L(\mathbb{C}^2,U^{\perp,W}) \tag{8.9}$$

(where we interpret linear maps into U or into $U^{\perp,W}$ also as linear maps into W, and therefore $L(\mathbb{C}^2,U)$ and $L(\mathbb{C}^2,U^{\perp,W})$ as linear subspaces of $L(\mathbb{C}^2,W)$). Indeed, it follows from Equation (8.8) that the sum on the right-hand side of (8.9) is orthogonally direct, and the inclusion "$\supset$" in (8.9) is obvious. For the opposite inclusion, we denote by

$$P_U : W \to U \quad\text{and}\quad Q_U : W \to U^{\perp,W}$$

the orthogonal projection onto U resp. onto $U^{\perp,W}$; these maps are $\mathbb{C}$-linear and $P_U + Q_U = \mathrm{id}_W$ holds. Now let $v \in L(\mathbb{C}^2, W)$ be given, then we have $v^U := P_U \circ v \in L(\mathbb{C}^2, U)$ and $v^{U^\perp} := Q_U \circ v \in L(\mathbb{C}^2, U^{\perp,W})$, and $v^U + v^{U^\perp} = (P_U + Q_U) \circ v = v$ holds.

The statement of (b) follows from Equation (8.9) because we have $\overrightarrow{\mathcal{V}^{\widehat{\theta}}_u} = L(\mathbb{C}^2, U)$ by (a). $\square$

We now reduce the structure group of the principal fibre bundle $\widehat{\theta}$ to $\mathrm{U}(2)$. Thereby we obtain the principal fibre bundle $\theta := \widehat{\theta}|\mathrm{St}_2(W) : \mathrm{St}_2(W) \to G_2(W)$ with structure group $\mathrm{U}(2)$, where we have

$$\mathrm{St}_2(W) := \{\, u \in \widehat{\mathrm{St}}_2(W) \mid u : \mathbb{C}^2 \to W \text{ is a (linear) isometric immersion} \,\}$$

and $\mathrm{U}(2)$ acts on $\mathrm{St}_2(W)$ from the right by the action $R := \widehat{R}|(\mathrm{St}_2(W) \times \mathrm{U}(2))$. In the sequel, we regard the (non-complex) submanifold $\mathrm{St}_2(W)$ of $\widehat{\mathrm{St}}_2(W)$ as a Riemannian submanifold, and in this regard, the elements of $\mathrm{U}(2)$ act on $\mathrm{St}_2(W)$ by isometries. It should also be noted that $\mathrm{St}_2(W)$ is contained in the sphere $\mathbb{S}_{\sqrt{2}}(L(\mathbb{C}^2, W))$.

Let us denote by $\mathcal{V}^\theta := \ker(T\theta)$ and $\mathcal{H}^\theta := (\mathcal{V}^\theta)^{\perp, T\mathrm{St}_2(W)}$ the vertical resp. horizontal subbundle of $T\mathrm{St}_2(W)$ induced by the fibre bundle θ. Then we have for any $u \in \mathrm{St}_2(W)$

$$\mathcal{V}^\theta_u = \mathcal{V}^{\widehat{\theta}}_u \cap T_u\mathrm{St}_2(W) \quad \text{and} \quad \mathcal{H}^\theta_u = \mathcal{H}^{\widehat{\theta}}_u \,; \tag{8.10}$$

the latter equality holds because θ and $\widehat{\theta}$ are fibre bundles over the same manifold $G_2(W)$. We note that because the horizontal structure $\mathcal{H}^\theta$ is $\mathrm{U}(2)$-invariant, it is in fact a connection in the sense of EHRESMANN (meaning that curves in $G_2(W)$ have a global $\mathcal{H}^\theta$-horizontal lift).

For any $u \in \mathrm{St}_2(W)$, the map $\theta_*|\mathcal{H}^\theta_u : \mathcal{H}^\theta_u \to T_{\theta(u)}G_2(W)$ is an $\mathbb{R}$-linear isomorphism, and therefore there exists one and only one real inner product on the linear space $T_{\theta(u)}G_2(W)$ so that that map becomes a linear isometry. Because $\mathrm{U}(2)$ acts via R transitively and via isometries on the fibres of θ, the real inner product on $T_{\theta(u)}G_2(W)$ obtained in this way does not depend on the choice of u within any given fibre of θ. It follows that these inner products constitute a Riemannian metric on $G_2(W)$ (that it is indeed differentiable follows from the usual argument involving local sections of θ) which is characterized by the fact that $\theta : \mathrm{St}_2(W) \to G_2(W)$ becomes a Riemannian submersion.

Moreover, for any $u \in \mathrm{St}_2(W)$, $\mathcal{H}^\theta_u = \mathcal{H}^{\widehat{\theta}}_u$ is a complex linear subspace of $T_uL(\mathbb{C}^2, W)$, and because $\widehat{\theta}$ is a holomorphic submersion, the complex structure $J_{\theta(u)}$ of $G_2(W)$ at the point $\theta(u)$ is conjugate under the linear isomorphism $(\widehat{\theta}_*|\mathcal{H}^{\widehat{\theta}}_u) = (\theta_*|\mathcal{H}^\theta_u) : \mathcal{H}^\theta_u \to T_{\theta(u)}G_2(W)$ to the multiplication with i. Therefore $J_{\theta(u)}$ is orthogonal and skew-adjoint with respect to the inner product on $T_{\theta(u)}G_2(W)$. It follows that $G_2(W)$ becomes a Hermitian manifold with its original complex structure and the Riemannian metric just introduced. We regard it as such from now on.

8.5 Proposition. *$G_2(W)$ is a Kähler manifold.*

Proof. The proposition follows from the fact that $G_2(W)$ is a Hermitian symmetric space (which we will show below), but can also be shown directly using the fact that $\theta_*|\mathcal{H}^\theta : \mathcal{H}^\theta \to TG_2(W)$

behaves essentially like the differential of an affine map, as is exemplified by the O'Neill equations ([O'N83], Lemma 7.45, p. 212).

To state the proof in detail, let us denote by ∇^G, ∇^{St} and ∇^L the Levi-Civita covariant derivatives of $G_2(W)$, $\mathrm{St}_2(W)$ resp. $L(\mathbb{C}^2, W)$, and by J^G and J^L the complex structures of $G_2(W)$ resp. $L(\mathbb{C}^2, W)$.

Let a curve $c : I \to G_2(W)$ and a vector field $X \in \mathfrak{X}_c(G_2(W))$ be given. Then we have to show $\nabla^G_\partial J^G X = J^G \nabla^G_\partial X$. Because $\mathcal{H}^\theta$ is a connection in the sense of EHRESMANN, there exists a global $\mathcal{H}^\theta$-horizontal lift $\widetilde{c} : I \to \mathrm{St}_2(W)$ of c. We let $\widetilde{X} \in \mathfrak{X}_{\widetilde{c}}(\mathrm{St}_2(W))$ be the $\mathcal{H}^\theta$-horizontal lift of X along $\widetilde{c}$.

Because θ is the restriction of the holomorphic map $\widehat{\theta}$ and we have $\mathcal{H}^\theta_u = \mathcal{H}^{\widehat{\theta}}_u$ for every $u \in \mathrm{St}_2(W)$, we see that

$$\mathcal{H}^\theta \text{ is } J^L\text{-invariant and } J^L|\mathcal{H}^\theta \text{ is conjugate to } J^G \text{ under } \theta_*|\mathcal{H}^\theta . \tag{8.11}$$

It follows that $J^L \widetilde{X}$ is the $\mathcal{H}^\theta$-horizontal lift of $J^G X$.

Because of the $\mathcal{H}^\theta$-horizontality of $\widetilde{c}_*\partial$, the O'Neill equation ([O'N83], Lemma 7.45, p. 212) shows that the $\mathcal{H}^\theta$-component of $\nabla^{\mathrm{St}}_\partial \widetilde{X}$ equals the $\mathcal{H}^\theta$-horizontal lift of $\nabla^G_\partial X$ along $\widetilde{c}$, and therefore we have

$$\theta_* \nabla^{\mathrm{St}}_\partial \widetilde{X} = \nabla^G_\partial X . \tag{8.12}$$

Analogously, we have

$$\theta_* \nabla^{\mathrm{St}}_\partial J^L \widetilde{X} = \nabla^G_\partial J^G X . \tag{8.13}$$

Furthermore, let us denote by h the second fundamental form of $\mathrm{St}_2(W) \hookrightarrow L(\mathbb{C}^2, W)$. Because of the Gauss equation of first order and the fact that J^L is ∇^L-parallel, we then have

$$\begin{aligned} \nabla^{\mathrm{St}}_\partial J^L \widetilde{X} &= \nabla^L_\partial J^L \widetilde{X} - h(J^L \widetilde{X}, \dot{\widetilde{c}}) \\ &= J^L\, \nabla^L_\partial \widetilde{X} - h(J^L \widetilde{X}, \dot{\widetilde{c}}) \\ &= J^L\, \nabla^{\mathrm{St}}_\partial \widetilde{X} + \underbrace{J^L\, h(\widetilde{X}, \dot{\widetilde{c}})}_{(*)} - \underbrace{h(J^L \widetilde{X}, \dot{\widetilde{c}})}_{(\dagger)} . \end{aligned} \tag{8.14}$$

$h(\widetilde{X}, \dot{\widetilde{c}})$ is orthogonal to $T\mathrm{St}_2(W)$, in particular to $\mathcal{H}^\theta$; because $\mathcal{H}^\theta$ is J^L-invariant (see (8.11)), it follows that the term marked $(*)$ above is orthogonal to $\mathcal{H}^\theta$. Also, the vector field $J^L \widetilde{X}$ is $\mathcal{H}^\theta$-valued, hence tangent to $\mathrm{St}_2(W)$, and therefore the term marked $(\dagger)$ above is also orthogonal to $\mathcal{H}^\theta$. Thus we obtain from (8.14):

$$\theta_* \nabla^{\mathrm{St}}_\partial J^L \widetilde{X} = \theta_* J^L\, \nabla^{\mathrm{St}}_\partial \widetilde{X} . \tag{8.15}$$

Putting these results together, we get

$$\nabla^G_\partial J^G X \overset{(8.13)}{=} \theta_* \nabla^{\mathrm{St}}_\partial J^L \widetilde{X} \overset{(8.15)}{=} \theta_* J^L\, \nabla^{\mathrm{St}}_\partial \widetilde{X} \overset{(8.11)}{=} J^G \theta_* \nabla^{\mathrm{St}}_\partial \widetilde{X} \overset{(8.12)}{=} J^G \nabla^G_\partial X . \qquad \square$$ [23]

[23] It is also possible to state the proof using global vector fields, rather than vector fields along curves. The approach involving curves was chosen here because it can be applied analogously to prove that the quaternionic structure we will construct on $G_2(W)$ in Section 8.3 is parallel.

8.6 Proposition. *The Plücker embedding $\mathcal{P} : G_2(W) \to \mathbb{P}(\bigwedge^2 W)$ is an isometric embedding.*

Proof. Because $\mathcal{P}$ is an embedding by Proposition 8.3, it suffices to show that $\mathcal{P}$ is an isometric immersion. Because the map $\widetilde{\mathcal{P}} : \mathrm{St}_2(W) \to \bigwedge^2 W,\ u \mapsto u_1 \wedge u_2$ satisfies

$$\mathcal{P} \circ \theta = \pi \circ \widetilde{\mathcal{P}} \,,$$

and θ and π are Riemannian submersions, it suffices to show that for every given $u \in \mathrm{St}_2(W)$, we have

$$\widetilde{\mathcal{P}}_* \mathcal{H}^\theta_u \subset \mathcal{H}^\pi_{\widetilde{\mathcal{P}}(u)} \quad \text{and} \quad \forall \xi, \eta \in \mathcal{H}^\theta_u \,:\, \langle \widetilde{\mathcal{P}}_* \xi, \widetilde{\mathcal{P}}_* \eta \rangle_{\mathbb{C}} = \langle \xi, \eta \rangle_{\mathbb{C}} \,. \tag{8.16}$$

Here, $\mathcal{H}^\theta_u$ and $\mathcal{H}^\pi_{\widetilde{\mathcal{P}}(u)}$ denote the horizontal spaces of the Riemannian submersions θ and π at the points u and $\widetilde{\mathcal{P}}(u)$, respectively; by Equation (8.10), Proposition 8.4(b), and Equation (1.6), we have

$$\mathcal{H}^\theta_u = \{\, \xi \in T_u L(\mathbb{C}^2, W) \mid \overrightarrow{\xi}(\mathbb{C}^2) \perp u(\mathbb{C}^2) \,\} \,, \tag{8.17}$$

$$\mathcal{H}^\pi_{\widetilde{\mathcal{P}}(u)} = \{\, \zeta \in T_{\widetilde{\mathcal{P}}(u)} {\textstyle\bigwedge}^2 W \mid \langle \overrightarrow{\zeta}, \widetilde{\mathcal{P}}(u) \rangle_{\mathbb{C}} = 0 \,\} \,. \tag{8.18}$$

We also note that

$$\forall \xi \in T_u \mathrm{St}_2(W) \,:\, \overrightarrow{\widetilde{\mathcal{P}}_* \xi} = (\overrightarrow{\xi})_1 \wedge u_2 + u_1 \wedge (\overrightarrow{\xi})_2 \tag{8.19}$$

holds. For the proof of the first part of (8.16), let $\xi \in \mathcal{H}^\theta_u$ be given. Then we have

$$\begin{aligned}
\langle \overrightarrow{\widetilde{\mathcal{P}}_* \xi}, \mathcal{P}(u) \rangle_{\mathbb{C}} &\overset{(8.19)}{=} \langle (\overrightarrow{\xi})_1 \wedge u_2 + u_1 \wedge (\overrightarrow{\xi})_2, u_1 \wedge u_2 \rangle_{\mathbb{C}} \\
&\overset{(8.6)}{=} \langle (\overrightarrow{\xi})_1, u_1 \rangle_{\mathbb{C}} \cdot \langle u_2, u_2 \rangle_{\mathbb{C}} - \langle (\overrightarrow{\xi})_1, u_2 \rangle_{\mathbb{C}} \cdot \langle u_2, u_1 \rangle_{\mathbb{C}} \\
&\qquad + \langle u_1, u_1 \rangle_{\mathbb{C}} \cdot \langle (\overrightarrow{\xi})_2, u_2 \rangle_{\mathbb{C}} - \langle u_1, u_2 \rangle_{\mathbb{C}} \cdot \langle (\overrightarrow{\xi})_2, u_1 \rangle_{\mathbb{C}} \\
&\overset{(8.17)}{=} 0 \,,
\end{aligned}$$

whence $\widetilde{\mathcal{P}}_* \xi \in \mathcal{H}^\pi_{\widetilde{\mathcal{P}}(u)}$ follows by Equation (8.18).

For the proof of the second part of (8.16), we let $\xi, \eta \in \mathcal{H}^\theta_u$ be given. Then we have

$$\begin{aligned}
\langle \widetilde{\mathcal{P}}_* \xi, \widetilde{\mathcal{P}}_* \eta \rangle_{\mathbb{C}} &\overset{(8.19)}{=} \langle (\overrightarrow{\xi})_1 \wedge u_2 + u_1 \wedge (\overrightarrow{\xi})_2, (\overrightarrow{\eta})_1 \wedge u_2 + u_1 \wedge (\overrightarrow{\eta})_2 \rangle_{\mathbb{C}} \\
&= \underbrace{\langle (\overrightarrow{\xi})_1 \wedge u_2, (\overrightarrow{\eta})_1 \wedge u_2 \rangle_{\mathbb{C}}}_{\underset{(*)}{=} \langle (\overrightarrow{\xi})_1, (\overrightarrow{\eta})_1 \rangle_{\mathbb{C}}} + \underbrace{\langle (\overrightarrow{\xi})_1 \wedge u_2, u_1 \wedge (\overrightarrow{\eta})_2 \rangle_{\mathbb{C}}}_{\underset{(*)}{=} 0} \\
&\qquad + \underbrace{\langle u_1 \wedge (\overrightarrow{\xi})_2, (\overrightarrow{\eta})_1 \wedge u_2 \rangle_{\mathbb{C}}}_{\underset{(*)}{=} 0} + \underbrace{\langle u_1 \wedge (\overrightarrow{\xi})_2, u_1 \wedge (\overrightarrow{\eta})_2 \rangle_{\mathbb{C}}}_{\underset{(*)}{=} \langle (\overrightarrow{\xi})_2, (\overrightarrow{\eta})_2 \rangle_{\mathbb{C}}} \\
&= \langle (\overrightarrow{\xi})_1, (\overrightarrow{\eta})_1 \rangle_{\mathbb{C}} + \langle (\overrightarrow{\xi})_2, (\overrightarrow{\eta})_2 \rangle_{\mathbb{C}} = \langle \overrightarrow{\xi}, \overrightarrow{\eta} \rangle^L_{\mathbb{C}} \,;
\end{aligned}$$

here the equals signs marked $(*)$ follow by a straightforward calculation from Equation (8.6), the fact that (u_1, u_2) is a unitary system in W and the fact that we have $\langle (\overrightarrow{\xi})_\mu, u_\nu \rangle_{\mathbb{C}} = \langle (\overrightarrow{\eta})_\mu, u_\nu \rangle_{\mathbb{C}} = 0$ for $\mu, \nu = 1, 2$ (which is a consequence of Equation (8.17)). $\square$

$G_2(W)$ is a homogeneous $\mathrm{SU}(W)$-space via the transitive Lie group action

$$\varphi : \mathrm{SU}(W) \times G_2(W) \to G_2(W),\ (B,U) \mapsto B(U)\ .$$

This homogeneous space can be regarded as a Hermitian symmetric space in the following way: We fix an origin point $U_0 \in G_2(W)$ and consider the linear involution $S : W \to W$ with $S|U_0 = -\mathrm{id}_{U_0}$, $S|U_0^\perp = \mathrm{id}_{U_0^\perp}$. Then $\sigma_{G_2(W)} : \mathrm{SU}(W) \to \mathrm{SU}(W),\ B \mapsto SBS^{-1}$ is an involutive Lie group automorphism, whose fixed point group $\mathrm{Fix}(\sigma_{G_2(W)}) = \{\, B \in \mathrm{SU}(W) \,|\, B(U_0) = U_0 \,\}$ coincides with the isotropy group of φ at U_0. Thus, $(\mathrm{SU}(W), \varphi, U_0, \sigma_{G_2(W)})$ is an affine symmetric $G_2(W)$-space, which turns out to be irreducible, of compact type, and Hermitian symmetric with respect to the Hermitian metric described above.

We now specialize to the situation where W is a 4-dimensional oriented[24] unitary space. Then the restriction of the Hodge operator of $\bigwedge W$ (see Section B.2) to $\bigwedge^2 W$ is an anti-linear map $* : \bigwedge^2 W \to \bigwedge^2 W$. In fact, Proposition B.2(c),(e) shows $*$ to be a conjugation on $\bigwedge^2 W$, so $\bigwedge^2 W$ becomes a $\mathbb{C}$Q-space via the $\mathbb{C}$Q-structure $\mathfrak{A} := \{\, \lambda \cdot * \,|\, \lambda \in \mathbb{S}^1 \,\}$.

8.7 Theorem. *(a) The image of the Plücker embedding $\mathcal{P} : G_2(W) \to \mathbb{P}(\bigwedge^2 W)$ is the 4-dimensional complex quadric $Q(*)$ and $f_4 := (\mathcal{P} : G_2(W) \to Q(*))$ is a biholomorphic isometry. $Q(*)$ will be called the* Plücker quadric.

(b) For any $B \in \mathrm{SU}(W)$ we have $B^{(2)} \in \mathrm{Aut}_s(\mathfrak{A})_0$ and

$$\Phi : \mathrm{SU}(W) \to \mathrm{Aut}_s(\mathfrak{A})_0,\ B \mapsto B^{(2)}$$

is a two-fold covering map of Lie groups with kernel $\{\pm\mathrm{id}_W\}$. Consequently we have the following isomorphy of Lie groups:

$$\boxed{\mathrm{SU}(4) \cong \mathrm{Spin}(6)}\ . \tag{8.20}$$

(c) (f_4, Φ) is an almost-isomorphism of Hermitian symmetric spaces (as defined in Section A.3) from the $\mathrm{SU}(W)$-space $G_2(W)$ to the $\mathrm{Aut}_s(\mathfrak{A})_0$-space $Q(*)$. Thus we have shown the following almost-isomorphy of Hermitian symmetric spaces:

$$\boxed{Q^4 \cong G_2(\mathbb{C}^4)}\ .$$

Proof. For (a). Let us denote by $\omega \in \bigwedge^4 W$ the positive unit 4-vector of W (see Section B.2). Then we have for any $\xi \in \bigwedge^2 W \setminus \{0\}$ by Proposition 8.3:

$$\begin{aligned}\widehat{\pi}(\xi) \in \mathcal{P}(G_2(W)) &\iff \xi \wedge \xi = 0 \iff \xi \wedge (**\xi) = 0 \\ &\iff \langle \xi, *\xi \rangle_{\mathbb{C}} \cdot \omega = 0 \iff \langle \xi, *\xi \rangle_{\mathbb{C}} = 0 \iff \xi \in \widehat{Q}(*)\end{aligned}$$

(also see Proposition B.2(a)). This shows that $\mathcal{P}(G_2(W)) = Q(*)$ holds. It is also a consequence of Propositions 8.3 and 8.6 that f_4 is a biholomorphic isometry.

[24] As was explained in the Introduction, we apply the concept of an orientation also to complex linear spaces.

For (b). Let $B \in \mathrm{SU}(W)$ be given, then we have $B^{(2)} \in \mathrm{U}(\bigwedge^2 W)$. Let $(w_1, \dots, w_4)$ be any positively oriented unitary basis of W, then $(Bw_1, \dots, Bw_4)$ is another positively oriented unitary basis of W and with help of Proposition B.2(b) we get

$$B^{(2)}(*(w_1 \wedge w_2)) = B^{(2)}(w_3 \wedge w_4) = Bw_3 \wedge Bw_4 = *(Bw_1 \wedge Bw_2) = *B^{(2)}(w_1 \wedge w_2)\,.$$

Because $\bigwedge^2 W$ possesses a basis (ξ_j) where each ξ_j is of the form $w_1 \wedge w_2$ with a unitary 2-frame (w_1, w_2), and any such 2-frame can be extended to a positively oriented unitary basis $(w_1, \dots, w_4)$ of W, it follows that $B^{(2)} \circ * = * \circ B^{(2)}$ and thus $B^{(2)} \in \mathrm{Aut}_s(\mathfrak{A})$ holds.

Therefore, we can define the Lie group homomorphism Φ as in the proposition as a map into $\mathrm{Aut}_s(\mathfrak{A})$. Because $\mathrm{SU}(W)$ is connected Φ in fact maps into $\mathrm{Aut}_s(\mathfrak{A})_0$. The theorem of Beez (Proposition B.1) shows that the kernel of Φ is $\{\pm \mathrm{id}_W\}$, and therefore Φ is a two-fold covering map of Lie groups onto its image. $\mathrm{Aut}_s(\mathfrak{A})_0$ is isomorphic to $\mathrm{SO}(V(*)) \cong \mathrm{SO}(6)$, and therefore we have $\dim \mathrm{Aut}_s(\mathfrak{A})_0 = 15 = \dim \mathrm{SU}(W)$, whence it follows that the image of Φ is $\mathrm{Aut}_s(\mathfrak{A})_0$. It now also follows that the two-fold covering map Φ induces an isomorphism $\mathrm{SU}(W) \to \mathrm{Spin}(6)$ "over" Φ.

For (c). From the definitions of f_4 and Φ one sees immediately that (f_4, Φ) is an almost-isomorphism of homogeneous spaces from the $\mathrm{SU}(W)$-space $G_2(W)$ onto the $\mathrm{Aut}_s(\mathfrak{A})_0$-space $Q(*)$. Because the Lie groups $\mathrm{SU}(W)$ and $\mathrm{Aut}_s(\mathfrak{A})_0$ are of compact type, Proposition A.5 shows that (f_4, Φ) is an almost-isomorphisms of affine symmetric spaces; because f_4 is also a holomorphic isometry, (f_4, Φ) is in fact an almost-isomorphism of Hermitian symmetric spaces. □

8.3 $G_2(\mathbb{C}^n)$ and Q^4 as quaternionic Kähler manifolds

The complex 2-Grassmannians $G_2(\mathbb{C}^n)$ can be equipped with a quaternionic Kähler structure which is compatible with the complex structure of these spaces. A construction of this quaternionic Kähler structure has been given by J. Berndt in [Ber97]. In the present section, I present this construction in a simplified way, which is based on a discussion with Prof. H. Reckziegel.

As we saw in Section 8.2, the complex quadric Q^4 is as a Hermitian symmetric space isomorphic to $G_2(\mathbb{C}^4)$, and therefore it follows that also Q^4 can be equipped with a quaternionic Kähler structure compatible with its complex structure. It will be proven that this quaternionic Kähler structure on Q^4 is also compatible with its $\mathbb{C}$Q-structure.

8.8 Definition. *(a) Let V be a euclidean space. Then a 3-dimensional linear subspace $\mathfrak{J} \subset \mathrm{End}_-(V)$ is called a* quaternionic structure *on V if there is a basis (J_1, J_2, J_3) of $\mathfrak{J}$ which satisfies*

$$\forall k \in \{1,2,3\} \,:\, \big(\, J_k \circ J_k = -\mathrm{id}_V \quad \textit{and} \quad J_k \circ J_{k+1} = -J_{k+1} \circ J_k = J_{k+2} \,\big) \tag{8.21}$$

(where the indices are to be read modulo 3). Any such basis of $\mathfrak{J}$ is called a canonical basis *of $\mathfrak{J}$. If V is equipped with a fixed quaternionic structure, we call $(V, \mathfrak{J})$ or simply V a* quaternionic euclidean space.

(b) *Let M be a Riemannian manifold. Then a* quaternionic Kähler structure *on M is a rank 3 subbundle $\mathfrak{J}$ of the bundle* $\mathrm{End}_-(TM)$ *of skew-adjoint endomorphisms with the following properties:*

(i) *For every $p \in M$, $\mathfrak{J}_p$ is a quaternionic structure on the euclidean space T_pM.*

(ii) *$\mathfrak{J}$ is parallel with respect to the Levi-Civita covariant derivative ∇ of M, i.e.*

$$\forall J \in \Gamma(\mathfrak{J}),\ X \in \mathfrak{X}(M)\ :\ \nabla_X J \in \Gamma(\mathfrak{J})\ .$$

If M is equipped with a fixed quaternionic Kähler structure $\mathfrak{J}$, we call $(M, \mathfrak{J})$ (or shortly M) a quaternionic Kähler manifold.

(c) *We call a Kähler manifold M a* $\mathbb{C}$&$\mathbb{H}$-Kähler manifold *if it is additionally equipped with a quaternionic Kähler structure $\mathfrak{J}$ in such a way that the transformations in $\mathfrak{J}_p$ are $\mathbb{C}$-linear for every $p \in M$.*

8.9 Remarks. (a) Suppose that $(V, \mathfrak{J})$ is a quaternionic euclidean space, let us fix a canonical basis (J_1, J_2, J_3) of $\mathfrak{J}$, and equip the $\mathbb{R}$-linear space $\mathfrak{J}$ with the inner product and the orientation so that (J_1, J_2, J_3) becomes a positively oriented, orthonormal basis of $\mathfrak{J}$. Then it can be shown that another basis (J_1', J_2', J_3') of $\mathfrak{J}$ is a canonical basis if and only if it is a positively oriented, orthonormal basis. In particular, the inner product and the orientation with which we equipped $\mathfrak{J}$ does not depend on the choice of the canonical basis (J_1, J_2, J_3).

(b) Quaternionic euclidean spaces $(V, \mathfrak{J})$ can be interpreted as left-modules over the skew-field of quaternions in the following way: We regard the 3-dimensional, $\mathbb{R}$-linear space $\mathfrak{J}$ as an oriented, euclidean space as in (a), and denote by $\mathfrak{J} \times \mathfrak{J} \to \mathfrak{J},\ (J_1, J_2) \mapsto J_1 \times J_2$ the *cross product map* thereby induced; this map is characterized by being bilinear and skew-symmetric and having the following property: for any orthonormal system (J_1, J_2) in $\mathfrak{J}$, $(J_1, J_2, J_1 \times J_2)$ is a positively oriented, orthonormal basis of $\mathfrak{J}$, and hence a canonical basis by (a).

Now consider the 4-dimensional linear space $\widetilde{\mathfrak{J}} := \mathbb{R}\,\mathrm{id}_V \oplus \mathfrak{J}$. Then the composition of elements of $\widetilde{\mathfrak{J}}$ is explicitly described by the equation

$$\forall c, c' \in \mathbb{R},\ J, J' \in \mathfrak{J}\ :\ (c\,\mathrm{id}_V + J) \circ (c'\,\mathrm{id}_V + J') = (c\,c' - \langle J, J' \rangle_{\mathfrak{J}}) \cdot \mathrm{id}_V + c\,J' + c'\,J + J \times J'\ ;$$

this equation is shown by representing J and J' in components with respect to a canonical basis of $\mathfrak{J}$.

It follows that $(\widetilde{\mathfrak{J}}, +, \circ)$ is a skew-field which is isomorphic to the skew-field of quaternions, and V becomes a $\widetilde{\mathfrak{J}}$-left-module by the definition

$$\forall \widetilde{J} \in \widetilde{\mathfrak{J}},\ v \in V\ :\ \widetilde{J} \cdot v := \widetilde{J}(v)\ .$$

It is a consequence of this interpretation of V that we necessarily have $\dim_{\mathbb{R}} V = 4r$, $r \in \mathbb{N}$. The number r is called the *quaternionic dimension* of V.

We will now construct a quaternionic Kähler structure on $G_2(\mathbb{C}^n)$ so that $G_2(\mathbb{C}^n)$ becomes a $\mathbb{C}$&$\mathbb{H}$-Kähler manifold. For this purpose, we return to the general situation of the previous section, where W is a unitary space of arbitrary dimension n.

We denote by $\mathbb{H}$ the model of the skew-field of quaternions as a real subalgebra of $\mathrm{End}(\mathbb{C}^2)$. Specifically, we have

$$\mathbb{H} = \left\{ \left(\begin{smallmatrix} a & b \\ -\overline{b} & \overline{a} \end{smallmatrix} \right) \;\middle|\; a, b \in \mathbb{C} \right\} , \tag{8.22}$$

where we identify complex (2×2)-matrices with the endomorphisms on $\mathbb{C}^2$ they describe with respect to the canonical basis of $\mathbb{C}^2$. The addition on $\mathbb{H}$ is given by the addition of $\mathrm{End}(\mathbb{C}^2)$ and the multiplication on $\mathbb{H}$ is given by composition of linear maps (see [BtD85], p. 5f.). Moreover, the real part map $\mathrm{Re} : \mathbb{H} \to \mathbb{R},\ q \mapsto \mathrm{Re}(q)$ and the conjugation map $\mathbb{H} \to \mathbb{H},\ q \mapsto \overline{q}$ are given by

$$\forall q \in \mathbb{H} \;:\; \left(q = \left(\begin{smallmatrix} a & b \\ -\overline{b} & \overline{a} \end{smallmatrix} \right) \implies \mathrm{Re}(q) = \mathrm{Re}(a)\ ,\ \overline{q} = \left(\begin{smallmatrix} \overline{a} & -b \\ \overline{b} & a \end{smallmatrix} \right) \right) . \tag{8.23}$$

Note that $\overline{q}$ is the adjoint matrix of q; for this reason we have in particular

$$\forall q, q' \in \mathbb{H} \;:\; \overline{q \cdot q'} = \overline{q'} \cdot \overline{q} . \tag{8.24}$$

The subfield $\{ q \in \mathbb{H} \,|\, q = \overline{q} \}$ of $\mathbb{H}$ is isomorphic to $\mathbb{R}$, and we regard $\mathbb{R}$ as a subfield of $\mathbb{H}$ in this way. $\mathrm{Im}(\mathbb{H}) := \{ q \in \mathbb{H} \,|\, \mathrm{Re}(q) = 0 \}$ is a 3-dimensional linear subspace of $\mathbb{H}$ complementary to $\mathbb{R} \subset \mathbb{H}$. The members of $\mathbb{R} \subset \mathbb{H}$ and $\mathrm{Im}(\mathbb{H})$ are called *real* resp. *imaginary quaternions.* We call a basis (i, j, k) of $\mathrm{Im}(\mathbb{H})$ a *canonical basis of* $\mathrm{Im}(\mathbb{H})$ if

$$i^2 = j^2 = k^2 = ijk = -1 \tag{8.25}$$

holds; then we also have

$$ij = -ji = k\ , \quad jk = -kj = i \quad \text{and} \quad ki = -ik = j\ . \tag{8.26}$$

For any orthonormal system (i, j) in $\mathrm{Im}(\mathbb{H})$, (i, j, ij) is a canonical basis of $\mathrm{Im}(\mathbb{H})$.

It should also be noted that $\mathbb{H}$ becomes a euclidean space via the inner product given by

$$\forall q, q' \in \mathbb{H} \;:\; \langle q, q' \rangle := \mathrm{Re}(q \cdot \overline{q'}) .$$

Then $\mathbb{S}(\mathbb{H}) = \mathrm{SU}(2)$ holds.

In what follows, the action

$$\chi : \mathrm{U}(2) \times \mathrm{End}(\mathbb{C}^2) \to \mathrm{End}(\mathbb{C}^2),\ (A, X) \mapsto A \circ X \circ A^{-1}$$

will be of importance. For $A \in \mathrm{SU}(2) = \mathbb{S}(\mathbb{H})$, $\chi(A, \cdot)$ leaves $\mathbb{H}$ invariant (remember that the multiplication of $\mathbb{H}$ is given by the composition of endomorphisms of $\mathbb{C}^2$); moreover, as it is well-known, $\chi(A, \cdot)|\mathbb{H}$ is an orthogonal transformation on $\mathbb{H}$ which leaves $\mathrm{Im}(\mathbb{H})$ invariant, and the map $\mathrm{SU}(2) \to \mathrm{SO}(\mathrm{Im}(\mathbb{H})),\ A \mapsto \chi(A, \cdot)|\,\mathrm{Im}(\mathbb{H})$ is "the" universal covering of $\mathrm{SO}(\mathrm{Im}(\mathbb{H})) \cong \mathrm{SO}(3)$. Furthermore, we have for any $A \in \mathrm{SU}(2) = \mathbb{S}(\mathbb{H})$ $A^{-1} = \overline{A}$ and therefore for any $q \in \mathbb{H}$

$$\overline{\chi(A, q)} = \overline{A \circ q \circ A^{-1}} = \overline{A \circ q \circ \overline{A}} \overset{(8.24)}{=} A \circ \overline{q} \circ A^{-1} = \chi(A, \overline{q}) .$$

We also have

$$\forall A \in \mathrm{U}(2),\ \lambda \in \mathbb{S}^1\ :\ \chi(\lambda\, A, \cdot) = \chi(A, \cdot)\ . \tag{8.27}$$

For any given $A \in \mathrm{U}(2)$ we have $\lambda\, A \in \mathrm{SU}(2)$, where $\lambda \in \mathbb{S}^1$ is chosen such that $\lambda^{-2} = \det(A)$ holds. By application of the preceding results to $\lambda\, A$ and use of Equation (8.27), it therefore follows that even for every $A \in \mathrm{U}(2)$, $\chi(A, \cdot)|\mathbb{H}$ is an orthogonal transformation on $\mathbb{H}$ leaving $\mathrm{Im}(\mathbb{H})$ invariant and that

$$\forall A \in \mathrm{U}(2),\ q \in \mathbb{H}\ :\ \overline{\chi(A, q)} = \chi(A, \overline{q})\ . \tag{8.28}$$

holds.

We now turn $L(\mathbb{C}^2, W)$ into a left-$\mathbb{H}$-linear space by the definition

$$\forall q \in \mathbb{H},\ u \in L(\mathbb{C}^2, W)\ :\ q \cdot u := \widetilde{R}(u, \overline{q}) = u \circ \overline{q}\ , \tag{8.29}$$

where $\widetilde{R}$ is defined in (8.7). Thereby every $q \in \mathbb{H}$ gives rise to a complex vector bundle endomorphism $J_q^L : TL(\mathbb{C}^2, W) \to TL(\mathbb{C}^2, W)$ characterized by

$$\forall \xi \in TL(\mathbb{C}^2, W)\ :\ \overrightarrow{J_q^L(\xi)} = q \cdot \overrightarrow{\xi}\ . \tag{8.30}$$

In the case $u \in \mathrm{St}_2(W)$, J_q^L leaves the horizontal space $\mathcal{H}_u^\theta$ of the fibre bundle $\theta : \mathrm{St}_2(W) \to G_2(W),\ u \mapsto u(\mathbb{C}^2)$ (see Proposition 8.4(b) and Equation (8.10)) invariant, and therefore there exists one and only one $\mathbb{C}$-linear transformation $J_{q,u} : T_{\theta(u)}G_2(W) \to T_{\theta(u)}G_2(W)$ such that

$$(\theta_* | \mathcal{H}_u^\theta) \circ (J_q^L | \mathcal{H}_u^\theta) = J_{q,u} \circ (\theta_* | \mathcal{H}_u^\theta) \tag{8.31}$$

holds; moreover

$$\mathfrak{J}_u := \{\, J_{q,u} \,|\, q \in \mathrm{Im}(\mathbb{H})\,\}$$

is a quaternionic structure on the euclidean space $T_{\theta(u)}G_2(W)$. To show that the quaternionic structures defined in this way give rise uniquely to a rank-3-subbundle of the bundle $\mathrm{End}_-(TG_2(W))$, we need to show that $\mathfrak{J}_u$ does not depend on the representant u chosen within a fibre of θ. For this purpose, we describe the transformation behaviour of the maps $J_{q,u}$:

8.10 Proposition. *Let $u, u' \in \mathrm{St}_2(W)$ with $\theta(u) = \theta(u')$ and $q \in \mathrm{Im}(\mathbb{H})$ be given. Then there exists a unique $q' \in \mathrm{Im}(\mathbb{H})$ so that*

$$J_{q,u'} = J_{q',u}$$

holds; this q' is given by

$$q' = A \circ q \circ A^{-1}\ ,$$

if $u' = u \circ A$ holds with some $A \in U(2)$.

For the proof of this proposition see below.

Proposition 8.10 shows that $\mathfrak{J}_u$ does not depend on the choice of u within any fibre of θ. Therefore there exists a family $\mathfrak{J} := (\mathfrak{J}_U)_{U \in G_2(W)}$ of 3-dimensional linear subspaces so that $\mathfrak{J}_{\theta(u)} = \mathfrak{J}_u$

holds for every $u \in \mathrm{St}_2(W)$. To show that $\mathfrak{J}$ is in fact a differentiable vector subbundle of $\mathrm{End}_-(TG_2(W))$ we let a local section $\sigma : D \to \mathrm{End}_-(TG_2(W))$ (where $D \subset G_2(W)$ is an open subset) be given, fix $q \in \mathrm{Im}(\mathbb{H})$ and consider the map $S : D \to \mathrm{End}_-(TG_2(W))$, $U \mapsto S_U$ characterized by

$$\forall U \in D \,:\, \big(\, S_U \in \mathrm{End}_-(T_U G_2(W)) \quad \text{and} \quad (\theta_* \circ J_q^L)|\mathcal{H}^\theta_{\sigma(U)} = (S_U \circ \theta_*)|\mathcal{H}^\theta_{\sigma(U)}\,\big)\,. \tag{8.32}$$

It is clear that S maps into $\mathfrak{J}_{|D}$, and we will show below that S is differentiable and therefore a local section in the bundle $\mathrm{End}_-(TG_2(W)) \to G_2(W)$. By letting q run through a basis of $\mathrm{Im}(\mathbb{H})$, one then obtains a local basis field of $\mathfrak{J}$ over D, and this shows that $\mathfrak{J}$ is indeed a differentiable vector subbundle of $\mathrm{End}_-(TG_2(W))$.

To show that S is differentiable, we let a vector field $X \in \mathfrak{X}(D)$ be given, and let $\widetilde{X}$ be the $\mathcal{H}^\theta$-horizontal lift of X with respect to θ, i.e. the vector field $\widetilde{X} \in \Gamma(\mathcal{H}^\theta_{|\theta^{-1}(D)})$ characterized by

$$X \circ (\theta|\theta^{-1}(D)) = \theta_* \circ \widetilde{X}\,. \tag{8.33}$$

Then we have for every $U \in D$

$$S_U(X_U) = S_U(X_{\theta(\sigma(U))}) \overset{(8.33)}{=} (S_U \circ \theta_*)(\widetilde{X}_{\sigma(U)}) \overset{(8.32)}{=} (\theta_* \circ J_q^L)\widetilde{X}_{\sigma(U)}\,.$$

We see from this calculation that $D \to TG_2(W)$, $U \mapsto S_U(X_U)$ is a differentiable vector field, and it follows that S is differentiable.

8.11 Remark. Proposition 8.10 also shows that the bundle $\mathfrak{J}$ is associated with the principal fibre bundle $\theta : \mathrm{St}_2(W) \to G_2(W)$ with structure group $\mathrm{U}(2)$ via the association map

$$\rho : \mathrm{St}_2(W) \times \mathrm{Im}(\mathbb{H}) \to \mathfrak{J},\ (u,q) \mapsto J_{q,u}\,; \tag{8.34}$$

its typical fibre is $\mathrm{Im}(\mathbb{H})$, which we consider with the action $\chi|(\mathrm{U}(2) \times \mathrm{Im}(\mathbb{H}))$ from the left (see [Bou67], Section 6.5.1).

Proof of Proposition 8.10. Because $\mathrm{U}(2)$ acts via R simply transitively on the fibres of θ, there exists a unique $A \in \mathrm{U}(2)$ so that $u' = R^A(u) = u \circ A$ holds, and it follows from the preceding consideration that $q' := \chi(A,q) \in \mathrm{Im}(\mathbb{H})$ holds.

Now, we have $(R^A)_* \mathcal{H}^\theta_u = \mathcal{H}^\theta_{u'}$ and for every $\xi \in \mathcal{H}^\theta_u$

$$\begin{aligned}\overrightarrow{(T_u R^A|\mathcal{H}^\theta_u)^{-1} \circ J_q^L \circ T_u R^A(\xi)} &= \overrightarrow{J_q^L \circ T_u R^A(\xi)} \circ A^{-1} = \overrightarrow{T_u R^A(\xi)} \circ \overline{q} \circ A^{-1} = \overrightarrow{\xi} \circ A \circ \overline{q} \circ A^{-1} \\ &= \overrightarrow{\xi} \circ \chi(A,\overline{q}) \overset{(8.28)}{=} \overrightarrow{\xi} \circ \overline{\chi(A,q)} = \overrightarrow{\xi} \circ \overline{q'} = \overrightarrow{J^L_{q'}(\xi)}\,,\end{aligned}$$

whence

$$(T_u R^A|\mathcal{H}^\theta_u)^{-1} \circ (J_q^L|\mathcal{H}^\theta_{u'}) \circ (T_u R^A|\mathcal{H}^\theta_u) = (J^L_{q'}|\mathcal{H}^\theta_u) \tag{8.35}$$

follows.

By linearization of the equation $\theta = \theta \circ (R^A)^{-1}$, we obtain

$$(T_{u'}\theta|\mathcal{H}^\theta_{u'}) = (T_u\theta|\mathcal{H}^\theta_u) \circ (T_u R^A|\mathcal{H}^\theta_u)^{-1}\,,$$

and the definition of $J_{q,u'}$ together with this equation yields

$$\begin{aligned} J_{q,u'} &= (T_{u'}\theta|\mathcal{H}^\theta_{u'}) \circ (J^L_q|\mathcal{H}^\theta_{u'}) \circ (T_{u'}\theta|\mathcal{H}^\theta_{u'})^{-1} \\ &= (T_u\theta|\mathcal{H}^\theta_u) \circ (T_uR^A|\mathcal{H}^\theta_u)^{-1} \circ (J^L_q|\mathcal{H}^\theta_{u'}) \circ (T_uR^A|\mathcal{H}^\theta_u) \circ (T_u\theta|\mathcal{H}^\theta_u)^{-1} . \end{aligned} \tag{8.36}$$

The result is now obtained by plugging Equation (8.35) into Equation (8.36). □

8.12 Proposition. *$\mathfrak{J}$ is a quaternionic Kähler structure, and by equipping $G_2(W)$ with the Kähler structure from Section 8.2 and the mentioned quaternionic Kähler structure, it becomes a $\mathbb{C}$&$\mathbb{H}$-Kähler manifold.*

Proof. It only remains to show that the subbundle $\mathfrak{J}$ of $\mathrm{End}_-(TG_2(W))$ is parallel with respect to the Levi-Civita covariant derivative ∇^G of $G_2(W)$. For this, it suffices to show that for any curve $c : I \to G_2(W)$, any $\mathcal{H}^\theta$-horizontal lift $\widetilde{c} : I \to \mathrm{St}_2(W)$ of c and any $q \in \mathrm{Im}(\mathbb{H})$, the endomorphism field $J(t) := J_{q,\widetilde{c}(t)}$ is ∇^G-parallel, and this is done analogously as in the proof of Proposition 8.5. □

8.13 Remark. There are two ways to construct a covariant derivative on the vector bundle $\mathfrak{J}$: First, the U(2)-invariant connection $(\mathcal{H}^\theta)$ of the principal fibre bundle θ induces a connection $(\mathcal{H}^{\mathfrak{J}})$ on $\mathfrak{J}$ via the association map ρ of Equation (8.34) (see Remark 8.11). $(\mathcal{H}^{\mathfrak{J}})$ is induced by a covariant derivative on $\mathfrak{J}$; this is a consequence of the fact that $\forall c \in \mathbb{R},\ J \in \mathfrak{J} : \mathcal{H}^{\mathfrak{J}}_{cJ} = Th_c(\mathcal{H}^{\mathfrak{J}}_J)$ holds with $h_c : \mathfrak{J} \to \mathfrak{J},\ J \mapsto c\,J$ (see [Poo81], Definition 2.26, p. 54 and Theorem 2.52, p. 74). Second, one can consider the Levi-Civita covariant derivative on $G_2(W)$, which induces a covariant derivative ∇^{End} on the endomorphism bundle $\mathrm{End}(TG_2(W)) \to G_2(W)$; $\mathfrak{J}$ is a ∇^{End}-parallel subbundle of the latter bundle by Proposition 8.12, and therefore the restriction of ∇^{End} gives a covariant derivative on $\mathfrak{J}$.

The proof of Proposition 8.12 shows that these two constructions in fact give rise to the same covariant derivative on $\mathfrak{J}$.

8.14 Remark. As has been shown by Berndt (see [Ber97], Section 10, Theorem 2), the curvature tensor $R_{G_2(W)}$ of $G_2(W)$ can be described via the Riemannian metric $\langle\cdot,\cdot\rangle$, the complex structure J and the quaternionic structure $\mathfrak{J}$ of this Grassmannian: For any $U \in G_2(W)$ and any canonical basis (J_1, J_2, J_3) of $\mathfrak{J}_U$, we have

$$\begin{aligned} \forall u,v,w \in T_pG_2(W) \ : \ R_{G_2(W)}(u,v)w =& \langle v,w\rangle u - \langle u,w\rangle v \\ &+ \langle Jv,w\rangle Ju - \langle Ju,w\rangle Jv - 2\langle Ju,v\rangle Jw \\ &+ \sum_{\mu=1}^{3}\Big(\langle J_\mu v,w\rangle J_\mu u - \langle J_\mu u,w\rangle J_\mu v - 2\langle J_\mu u,v\rangle J_\mu w\Big) \\ &+ \sum_{\mu=1}^{3}\Big(\langle J_\mu Jv,w\rangle J_\mu Ju - \langle J_\mu Ju,w\rangle J_\mu Jv\Big) . \end{aligned}$$

As we did in the previous section, we now specialize to the situation where $\dim W = 4$ holds, and W is additionally equipped with a complex orientation. Then we have the Hodge operator $* : \bigwedge^2 W \to \bigwedge^2 W$ which is a conjugation on $\bigwedge^2 W$, the Plücker quadric $Q(*)$ and the holomorphic isometry $f_4 : G_2(W) \to Q(*)$ from Theorem 8.7. We now use f_4 to transfer the quaternionic Kähler structure $\mathfrak{J}$ of $G_2(W)$ onto $Q(*)$, thereby obtaining the quaternionic Kähler structure $\mathfrak{J}^Q$ on $Q(*)$ given by

$$\forall U \in G_2(W) \;:\; \mathfrak{J}^Q_{f_4(U)} = \{\, T_U f_4 \circ J \circ (T_U f_4)^{-1} \,|\, J \in \mathfrak{J}_U \,\} \,.$$

In this way also $Q(*)$ becomes a $\mathbb{C}$&$\mathbb{H}$-Kähler manifold.

8.15 Proposition. *The quaternionic Kähler structure $\mathfrak{J}^Q$ is compatible with the $\mathbb{C}$Q-structures on $TQ(*)$ in the sense that we have*

$$\forall p \in Q(*),\; A \in \mathfrak{A}(Q(*),p),\; J \in \mathfrak{J}^Q_p \;:\; J \circ A = A \circ J \,. \tag{8.37}$$

Proof. Let $p \in Q(*)$ and $A \in \mathfrak{A}(Q(*),p)$ be given, and choose $z \in \pi^{-1}(\{p\})$ so that

$$\forall v \in \mathcal{H}^\pi_z Q(*) \;:\; \overrightarrow{((\pi_*|\mathcal{H}^\pi_z Q(*))^{-1} \circ A \circ \pi_*)v} = * \,\overrightarrow{v} \tag{8.38}$$

holds (see Theorem 1.16 and Proposition 1.15).

We consider the surjective map $\widetilde{f}_4 : \mathrm{St}_2(W) \to \widetilde{Q}(*),\; u \mapsto u_1 \wedge u_2$ which we already used in the proof of Proposition 8.6, and choose $u \in \mathrm{St}_2(W)$ so that $\widetilde{f}_4(u) = z$ holds. The map $\widetilde{f}_4$ satisfies $\pi \circ \widetilde{f}_4 = f_4 \circ \theta$ and

$$\forall\, \xi \in T_u \mathrm{St}_2(W) \;:\; \overrightarrow{(\widetilde{f}_4)_* \xi} = (\overrightarrow{\xi}\,)_1 \wedge u_2 + u_1 \wedge (\overrightarrow{\xi}\,)_2 \tag{8.39}$$

(see Equation (8.19)), moreover $(T_u \widetilde{f}_4|\mathcal{H}^\theta_u) : \mathcal{H}^\theta_u \to \mathcal{H}^\pi_z Q(*)$ is a $\mathbb{C}$-linear isometry (see (8.16)).

In the sequel three isomorphic spaces are of importance: First, the horizontal space $\mathcal{H}^\pi_z Q(*)$ belonging to π. Second, the horizontal space $\mathcal{H}^\theta_u$ belonging to θ, which is related to $\mathcal{H}^\pi_z Q(*)$ by the $\mathbb{C}$-linear isometry $(T_u \widetilde{f}_4|\mathcal{H}^\theta_u) : \mathcal{H}^\theta_u \to \mathcal{H}^\pi_z Q(*)$. Third, the $\mathbb{C}$-linear space $\mathrm{End}(\mathbb{C}^2)$, which is isomorphic to $\mathcal{H}^\theta_u$ in the following way: Let us extend the unitary system (u_1, u_2) to a positively oriented unitary basis (u_1, u_2, u_3, u_4) of W and consider the element $u^\perp \in \mathrm{St}_2(W)$ characterized by $(u^\perp)_1 = u_3$ and $(u^\perp)_2 = u_4$. Then $u^\perp : \mathbb{C}^2 \to \theta(u)^\perp$ is a $\mathbb{C}$-linear isometry, and the map $L_\theta : \mathrm{End}(\mathbb{C}^2) \to \mathcal{H}^\theta_u$ determined by

$$\forall X \in \mathrm{End}(\mathbb{C}^2) \;:\; \overrightarrow{L_\theta(X)} = u^\perp \circ X$$

is an isomorphism of $\mathbb{C}$-linear spaces (see Equation (8.10) and Proposition 8.4(b)).

We now use the linear isomorphisms L_θ and $L_\pi := (T_u \widetilde{f}_4|\mathcal{H}^\theta_u) \circ L_\theta : \mathrm{End}(\mathbb{C}^2) \to \mathcal{H}^\pi_z Q(*)$ to transfer the relevant transformations onto $\mathrm{End}(\mathbb{C}^2)$. Specifically, we define for $q \in \mathrm{Im}(\mathbb{H})$ the $\mathbb{C}$-linear map $\widehat{J}_q : \mathrm{End}(\mathbb{C}^2) \to \mathrm{End}(\mathbb{C}^2)$ by

$$\forall X \in \mathrm{End}(\mathbb{C}^2) \;:\; L_\theta(\widehat{J}_q(X)) = J^L_q(L_\theta(X)) \,;$$

we also define the anti-linear map $\widehat{A} : \mathrm{End}(\mathbb{C}^2) \to \mathrm{End}(\mathbb{C}^2)$ by

$$\forall X \in \mathrm{End}(\mathbb{C}^2) \ : \ \overrightarrow{L_\pi(\widehat{A}(X))} = *\,\overrightarrow{L_\pi(X)}\,. \tag{8.40}$$

For the proof of Equation (8.37), it then suffices to show

$$\widehat{J}_q \circ \widehat{A} = \widehat{A} \circ \widehat{J}_q \tag{8.41}$$

because of Equation (8.38).

To prove Equation (8.41), we derive explicit representations of $\widehat{J}_q$ and of $\widehat{A}$. We have

$$\forall X \in \mathrm{End}(\mathbb{C}^2) \ : \ \widehat{J}_q(X) = X \circ \overline{q} \tag{8.42}$$

because of the definition of J_q^L (see Equations (8.30) and (8.29)), and we will show below that

$$\forall X \in \mathrm{End}(\mathbb{C}^2) \ : \ \widehat{A}(X) = \alpha \circ D \circ X \circ D \circ \alpha \tag{8.43}$$

holds, where $\alpha : \mathbb{C}^2 \to \mathbb{C}^2$ is the canonical conjugation of $\mathbb{C}^2$, and $D : \mathbb{C}^2 \to \mathbb{C}^2$ is the $\mathbb{C}$-linear map characterized by $De_1 = e_2$ and $De_2 = -e_1$. From Equations (8.42) and (8.43) we then obtain for arbitrary $X \in \mathrm{End}(\mathbb{C}^2)$

$$(\widehat{J}_q \circ \widehat{A})(X) = \alpha \circ D \circ X \circ D \circ \alpha \circ \overline{q} \quad \text{and} \quad (\widehat{A} \circ \widehat{J}_q)(X) = \alpha \circ D \circ X \circ \overline{q} \circ D \circ \alpha\,.$$

For the verification of Equation (8.41), it therefore suffices to show

$$D \circ \alpha \circ \overline{q} = \overline{q} \circ D \circ \alpha\,.$$

Because both sides of the latter equation are anti-linear, it only has to be checked for e_1 and e_2, and this is easily done via a direct calculation involving the definitions of D and α, and Equation (8.23).

Thus, it only remains to prove Equation (8.43). For this purpose let $X \in \mathrm{End}(\mathbb{C}^2)$ be given; suppose that X is represented by the matrix $\left(\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{smallmatrix}\right)$ with respect to the canonical basis of $\mathbb{C}^2$. Then $\xi := L_\theta(X) \in \mathcal{H}^\theta_u$ is given by

$$(\overrightarrow{\xi}\,)_1 = c_{11}\, u_3 + c_{21}\, u_4 \quad \text{and} \quad (\overrightarrow{\xi}\,)_2 = c_{12}\, u_3 + c_{22}\, u_4\,,$$

and hence, $v := L_\pi(X) \in \mathcal{H}^\pi_z Q(*)$ is given by

$$\begin{aligned} \overrightarrow{v} \ &= \ \overrightarrow{(\widetilde{f}_4)_*(\xi)} \\ &\overset{(8.39)}{=} \ (c_{11}\, u_3 + c_{21}\, u_4) \wedge u_2 + u_1 \wedge (c_{12}\, u_3 + c_{22}\, u_4) \\ &= \ -c_{11}\, u_2 \wedge u_3 - c_{21}\, u_2 \wedge u_4 + c_{12}\, u_1 \wedge u_3 + c_{22}\, u_1 \wedge u_4\,. \end{aligned}$$

By Example B.3, we now obtain

$$\begin{aligned} \overrightarrow{(\widetilde{f}_4)_*\, L_\theta(\widehat{A}(X))} &= \overrightarrow{L_\pi(\widehat{A}(X))} \overset{(8.40)}{=} *\,\overrightarrow{v} \\ &= -\overline{c_{11}}\, u_1 \wedge u_4 + \overline{c_{21}}\, u_1 \wedge u_3 - \overline{c_{12}}\, u_2 \wedge u_4 + \overline{c_{22}}\, u_2 \wedge u_3 \\ &= (-\overline{c_{22}}\, u_3 + \overline{c_{12}}\, u_4) \wedge u_2 + u_1 \wedge (\overline{c_{21}}\, u_3 - \overline{c_{11}}\, u_4) \overset{(8.39)}{=} \overrightarrow{(\widetilde{f}_4)_* \eta}\,, \end{aligned} \tag{8.44}$$

where $\eta \in T_u\mathrm{St}_2(W)$ is characterized by

$$(\overrightarrow{\eta})_1 = -\overline{c_{22}}\,u_3 + \overline{c_{12}}\,u_4 \quad \text{and} \quad (\overrightarrow{\eta})_2 = \overline{c_{21}}\,u_3 - \overline{c_{11}}\,u_4 \;,$$

and therefore by

$$\overrightarrow{\eta} = u^\perp \circ \alpha \circ D \circ X \circ D \circ \alpha \;. \tag{8.45}$$

The latter representation shows that $\eta \in \mathcal{H}_u^\theta$ holds, and from (8.44) we see

$$L_\theta(\widehat{A}(X)) = \eta \;. \tag{8.46}$$

Equation (8.43) now follows from (8.46) and (8.45) by the definition of L_θ. □

8.4 Q^3 is isomorphic to $\mathrm{Sp}(2)/\mathrm{U}(2)$

The subject of this section is the series of Hermitian symmetric spaces $\mathrm{Sp}(r)/\mathrm{U}(r)$ and the fact that this series intersects with the series of complex quadrics at $\mathrm{Sp}(2)/\mathrm{U}(2) \cong Q^3$. First, we show how the Hermitian symmetric space $\mathrm{Sp}(r)/\mathrm{U}(r)$ can be realized as a $\mathrm{Sp}(r)$-orbit M in $G_r(\mathbb{C}^{2r})$; M is a Hermitian symmetric subspace of that complex Grassmannian. Then we specialize to the case $r = 2$. In this case, M is contained in the Grassmannian $G_2(\mathbb{C}^4)$, which is mapped holomorphic isometrically onto the Plücker quadric $Q(*)$ by the Plücker embedding $\mathcal{P} : G_2(\mathbb{C}^4) \to \mathbb{P}(\bigwedge^2\mathbb{C}^4)$, as we saw in Section 8.2. We will show that $\mathcal{P}$ maps the orbit $M \cong \mathrm{Sp}(2)/\mathrm{U}(2)$ onto a 3-dimensional, totally geodesic complex subquadric of $Q(*)$.

For the first part of the section, we return to the general situation of Section 8.2, where W is a unitary space of arbitrary dimension. But now we suppose $\dim W = n = 2r$ to be even. We fix some symplectic (i.e. non-degenerate, alternating, $\mathbb{C}$-bilinear) form ω on W which is coupled to the inner product of W in the following way: The anti-linear map $\tau : W \to W$ characterized by

$$\forall w, w' \in W \;:\; \omega(w,w') = \langle w, \tau(w')\rangle_{\mathbb{C}} \tag{8.47}$$

is orthogonal with respect to the real inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}} := \mathrm{Re}(\langle\cdot,\cdot\rangle_{\mathbb{C}})$ on W and therefore satisfies

$$\forall w, w' \in W \;:\; \langle \tau(w), \tau(w')\rangle_{\mathbb{C}} = \overline{\langle w, w'\rangle_{\mathbb{C}}} \;. \tag{8.48}$$

Then τ also satisfies $\tau^2 = -\mathrm{id}_W$, because we have for every $w, w' \in W$

$$\langle w, \tau^2(w')\rangle_{\mathbb{C}} \overset{(8.47)}{=} \omega(w, \tau(w')) = -\omega(\tau(w'), w) \overset{(8.47)}{=} -\langle \tau(w'), \tau(w)\rangle_{\mathbb{C}} \overset{(8.48)}{=} -\overline{\langle w', w\rangle_{\mathbb{C}}} = -\langle w, w'\rangle_{\mathbb{C}} \;.$$

In order to establish that such a symplectic form ω exists, we choose a unitary basis $(w_1, \dots, w_{2r})$ of W, define an anti-linear bijection $\tau : W \to W$ by

$$\forall \mu \in \{1, \dots, r\} \;:\; \big(\, \tau(w_\mu) = w_{r+\mu} \quad \text{and} \quad \tau(w_{r+\mu}) = -\tau(w_\mu) \,\big) \;,$$

and a $\mathbb{C}$-bilinear form ω by

$$\omega : W \times W \to \mathbb{C},\; (w, w') \mapsto \langle w, \tau(w')\rangle_{\mathbb{C}} \;.$$

Then τ satisfies $\tau^2 = -\mathrm{id}_W$ and Equation (8.48), and therefore we have for any $w, w' \in W$

$$\omega(w,w') = \langle w, \tau(w')\rangle_{\mathbb{C}} = \overline{\langle \tau(w), \tau^2(w')\rangle_{\mathbb{C}}} = -\overline{\langle \tau(w), w'\rangle_{\mathbb{C}}} = -\langle w', \tau(w)\rangle_{\mathbb{C}} = -\omega(w',w) \; ,$$

whence it follows that ω is a symplectic form on W with the desired property.

Through the symplectic form ω, W becomes a right-linear space over the skew-field of quaternions in the following way: As in Section 8.3, we denote by $\mathbb{H}$ the model of the skew-field of quaternions as a subalgebra of $\mathrm{End}(\mathbb{C}^2)$, see Equation (8.22). We regard $\mathbb{C}$ as a subfield of $\mathbb{H}$ by identifying $z \in \mathbb{C}$ with $\left(\begin{smallmatrix} z & 0 \\ 0 & \overline{z} \end{smallmatrix}\right) \in \mathbb{H}$. Then, with

$$i = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) \; , \quad j := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \quad \text{and} \quad k := i \cdot j = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right) \; ,$$

(i,j,k) is a canonical basis of $\mathrm{Im}(\mathbb{H})$, and we have

$$\mathbb{H} = \{\, z + z' \cdot j \,|\, z, z' \in \mathbb{C} \,\}$$

and

$$\forall z \in \mathbb{C} \; : \; j \cdot z = \overline{z} \cdot j \; .$$

W becomes an $\mathbb{H}$-right-linear space by the definition

$$\forall w \in W, \; z, z' \in \mathbb{C} \; : \; w \cdot (z + z'\, j) := z\, w + \overline{z'}\, \tau(w) \; . \tag{8.49}$$

Note that this multiplication extends the multiplication of the $\mathbb{C}$-linear space W and that multiplication with j is equivalent to the application of τ. Regarded in this way, the dimension of W over $\mathbb{H}$ is r.

Furthermore, we introduce a *symplectic inner product* $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ on W by

$$\forall w, w' \in \mathbb{H} \; : \; \langle w, w'\rangle_{\mathbb{H}} := \overline{\langle w, w'\rangle_{\mathbb{C}}} - \overline{\omega(w,w')} \cdot j \; . \tag{8.50}$$

It should be noted that $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ is $\mathbb{H}$-linear in its second entry (as is customary for symplectic inner products), whereas with regard to the first entry it is additive and satisfies

$$\forall w, w' \in W, \; q \in \mathbb{H} \; : \; \langle w\, q, w'\rangle_{\mathbb{H}} = \overline{q} \cdot \langle w, w'\rangle_{\mathbb{H}} \; .$$

A basis $(w_1, \dots, w_r)$ of the $\mathbb{H}$-linear space W is called a *symplectic basis* of $(W, \langle\cdot,\cdot\rangle_{\mathbb{H}})$ if

$$\forall \mu, \nu \in \{1, \dots, r\} \; : \; \langle w_\mu, w_\nu\rangle_{\mathbb{H}} = \delta_{\mu\nu}$$

holds (where $\delta_{\mu\nu}$ is the Kronecker symbol). Such bases of W do indeed exist, see [Art57], p. 136f.

The symplectic inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ gives rise to the *symplectic group*

$$\mathrm{Sp}(W) := \{\, B : W \to W \,|\, B \text{ is } \mathbb{H}\text{-linear and } \forall w, w' \in W : \langle Bw, Bw'\rangle_{\mathbb{H}} = \langle w, w'\rangle_{\mathbb{H}} \,\} \; ;$$

as is well-known, this is a connected Lie group of dimension $r(2r+1)$ (see [Che46], Chap. I, § VIII, Proposition 2, p. 23 and Chap. II, § V, Proposition 3, p. 37).

8.16 Proposition. $\mathrm{Sp}(W) = \{\, B \in \mathrm{SU}(W) \,|\, B \circ \tau = \tau \circ B \,\} = \{\, B \in \mathrm{U}(W) \,|\, B \circ \tau = \tau \circ B \,\}$; *moreover, every* $B \in \mathrm{Sp}(W)$ *leaves* ω *invariant.*

Proof. For the first part of the proposition, it suffices to show

$$\mathrm{Sp}(W) \;\subset\; \{\, B \in \mathrm{SU}(W) \,|\, B \circ \tau = \tau \circ B \,\} \tag{8.51}$$

and

$$\{\, B \in \mathrm{U}(W) \,|\, B \circ \tau = \tau \circ B \,\} \;\subset\; \mathrm{Sp}(W)\,. \tag{8.52}$$

For the proof of (8.51), let $B \in \mathrm{Sp}(W)$ be given. Because of the $\mathbb{H}$-linearity of B, we have for every $w \in W$

$$B(\tau w) = B(w \cdot j) = (Bw) \cdot j = \tau(Bw)$$

and therefore $B \circ \tau = \tau \circ B$.

Moreover, with the projection $P : \mathbb{H} \to \mathbb{C},\ z + z'\,j \mapsto z$ we have for any $w, w' \in W$

$$\langle Bw, Bw' \rangle_{\mathbb{C}} \overset{(8.50)}{=} \overline{P(\langle Bw, Bw' \rangle_{\mathbb{H}})} = \overline{P(\langle w, w' \rangle_{\mathbb{H}})} \overset{(8.50)}{=} \langle w, w' \rangle_{\mathbb{C}}$$

and therefore $B \in \mathrm{U}(W)$.

We now see that B leaves ω invariant:

$$\omega(Bw, Bw') = \langle Bw, \tau(Bw') \rangle_{\mathbb{C}} = \langle Bw, B(\tau\, w') \rangle_{\mathbb{C}} = \langle w, \tau(w') \rangle_{\mathbb{C}} = \omega(w, w')\,.$$

Therefore B also leaves the $2r$-form $\psi := \omega \wedge \ldots \wedge \omega$ (r factors) on W invariant. Because ψ is a volume form[25] on W, $B \in \mathrm{SU}(W)$ follows. Thus it is shown that B lies in the right-hand side set of (8.51).

For the proof of (8.52), let $B \in \mathrm{U}(W)$ with $B \circ \tau = \tau \circ B$ be given. Then we have for every $w \in W$ and $q \in \mathbb{H}$, say $q = z + z'\,j$ with $z, z' \in \mathbb{C}$

$$B(w \cdot q) \overset{(8.49)}{=} B(z\,w + \overline{z'}\,\tau(w)) = z\,Bw + \overline{z'}\,B(\tau(w)) = z\,Bw + \overline{z'}\,\tau(Bw) \overset{(8.49)}{=} (Bw) \cdot q\,;$$

therefore B is $\mathbb{H}$-linear. Moreover, we have for every $w, w' \in W$:

$$\begin{aligned}
\langle Bw, Bw' \rangle_{\mathbb{H}} &\overset{(8.50)}{=} \overline{\langle Bw, Bw' \rangle_{\mathbb{C}}} - \overline{\omega(Bw, Bw')} \cdot j \\
&= \overline{\langle w, w' \rangle_{\mathbb{C}}} - \overline{\langle Bw, \tau(Bw') \rangle_{\mathbb{C}}} \cdot j \;=\; \overline{\langle w, w' \rangle_{\mathbb{C}}} - \overline{\langle Bw, B(\tau w') \rangle_{\mathbb{C}}} \cdot j \\
&= \overline{\langle w, w' \rangle_{\mathbb{C}}} - \overline{\langle w, \tau w' \rangle_{\mathbb{C}}} \cdot j \;=\; \overline{\langle w, w' \rangle_{\mathbb{C}}} - \overline{\omega(w, w')} \cdot j \overset{(8.50)}{=} \langle w, w' \rangle_{\mathbb{H}}\,.
\end{aligned}$$

Ergo, $B \in \mathrm{Sp}(W)$ holds. $\square$

[25] For the proof that ψ is a volume form, we fix a symplectic basis $(u_1, \ldots, u_r)$ of W. Then $(u_1, \ldots, u_r, \tau(u_1), \ldots, \tau(u_r))$ is a unitary basis of W, and if we denote by $(\alpha_1, \ldots, \alpha_{2r})$ the dual basis of W^*, we have $\omega = -\sum_{\mu=1}^{r} \alpha_\mu \wedge \alpha_{\mu+r}$. From this representation, it follows that $\psi = \pm\, r! \cdot (\alpha_1 \wedge \cdots \wedge \alpha_{2r})$ is a volume form on W.

8.17 Remark. The symplectic structure on W singles out a complex orientation on W : There is exactly one complex orientation on W such that for every symplectic basis $(w_1, \dots, w_r)$ of W, the unitary basis $(w_1, \dots, w_r, \tau(w_1), \dots, \tau(w_r))$ of W is positively oriented.

Indeed, if ψ is a complex volume form on W, and $(w_1, \dots, w_r)$ and $(w'_1, \dots, w'_r)$ are two symplectic bases of W, then there exists $B \in \mathrm{Sp}(W)$ with $B(w_k) = w'_k$; we have $B \circ \tau = \tau \circ B$ and $\det(B) = 1$ by Proposition 8.16, whence

$$\begin{aligned}\psi(w'_1, \dots, w'_r, \tau\, w'_1, \dots, \tau\, w'_r) &= \psi(Bw_1, \dots, Bw_r, \tau(Bw_1), \dots, \tau(Bw_r)) \\ &= \psi(Bw_1, \dots, Bw_r, B(\tau\, w_1), \dots, B(\tau\, w_r)) = \psi(w_1, \dots, w_r, \tau\, w_1, \dots, \tau\, w_r)\end{aligned}$$

follows. Thus, the unitary bases $(w_1, \dots, w_r, \tau(w_1), \dots, \tau(w_r))$ and $(w'_1, \dots, w'_r, \tau(w'_1), \dots, \tau(w'_r))$ of W are of the same orientation.

8.18 Proposition. *(a) The set*

$$M := \{\, Y \in G_r(W) \,|\, \tau(Y) \perp Y \,\}$$

is an orbit of the canonical action of $\mathrm{Sp}(W)$ *on* $G_r(W)$.

(b) M *is a complex submanifold of the complex Grassmannian* $G_r(W)$. *We have* $\dim_{\mathbb{C}}(M) = \frac{1}{2}\, r(r+1)$.

(c) Let $Y \in M$ *be given and denote by* K *the isotropy group of the action of* $\mathrm{Sp}(W)$ *on* $G_r(W)$ *at* Y. *Then*

$$\Psi : K \to \mathrm{U}(Y),\ B \mapsto B|Y$$

is an isomorphism of Lie groups.

(d) M *is a Hermitian symmetric subspace of the Hermitian symmetric space* $G_r(W)$, *and therefore a connected, complete, totally geodesic submanifold of* $G_r(W)$. *As Hermitian symmetric space,* M *is isomorphic to the symmetric space* $\mathrm{Sp}(r)/\mathrm{U}(r)$ *of type CI, see [Hel78], p. 518.*

Proof. Throughout the proof, it should be kept in mind that because of $\dim W = 2r$, we have for any $Y \in G_r(W)$

$$\tau(Y) \perp Y \iff \tau(Y) = Y^{\perp} \iff W = Y \oplus \tau(Y)\,.$$

For (a). Let us consider the Stiefel manifold $\mathrm{St}_r(W) \subset L(\mathbb{C}^r, W)$ of unitary r-frames in W, the canonical projection $\theta : \mathrm{St}_r(W) \to G_r(W),\ u \mapsto u(\mathbb{C}^r)$ and

$$\widetilde{M} := \{\, u \in \mathrm{St}_r(W) \,|\, (u_1, \dots, u_r) \text{ is a symplectic basis of } (W, \langle \cdot, \cdot \rangle_{\mathbb{H}}) \,\} \neq \varnothing\,,$$

where we put $u_\mu := u(e_\mu)$ for any $u \in L(\mathbb{C}^r, W)$ and $\mu \in \{1, \dots, r\}$; $(e_1, \dots, e_r)$ is the canonical basis of $\mathbb{C}^r$. Immediately, we will show

$$\theta^{-1}(M) = \widetilde{M}\,. \tag{8.53}$$

Because $\widetilde{M}$ is an orbit of the Lie group action

$$\mathrm{Sp}(W) \times \mathrm{St}_r(W) \to \mathrm{St}_r(W),\ (B,u) \mapsto B \circ u$$

and $\theta : \mathrm{St}_r(W) \to G_r(W)$ is $\mathrm{Sp}(W)$-equivariant, it then follows from Equation (8.53) that M is an orbit of the action of $\mathrm{Sp}(W)$ on $G_r(W)$.

For the proof of Equation (8.53): First, let $u \in \theta^{-1}(M)$ be given. With $Y := \theta(u) \in M$, we then have $\tau(Y) = Y^{\perp}$ and therefore for any $\mu,\nu \in \{1,\dots,r\}$

$$\langle u_\mu, u_\nu\rangle_{\mathbb{H}} \overset{(8.50)}{=} \overline{\langle u_\mu, u_\nu\rangle_{\mathbb{C}}} - \overline{\omega(u_\mu,u_\nu)} \cdot j = \underbrace{\overline{\langle u_\mu, u_\nu\rangle_{\mathbb{C}}}}_{=\delta_{\mu\nu}} - \overline{\langle \underbrace{u_\mu}_{\in W}, \underbrace{\tau(u_\nu)}_{\in W^\perp}\rangle_{\mathbb{C}}} \cdot j = \delta_{\mu\nu}\ ,$$

whence $u \in \widetilde{M}$ follows. Conversely, let $u \in \widetilde{M}$ be given and put $Y := \theta(u)$. Then we have for every $\mu,\nu \in \{1,\dots,r\}$: $\langle u_\mu,u_\nu\rangle_{\mathbb{H}} = \delta_{\mu\nu} \in \mathbb{R}$ and therefore by Equation (8.50): $0 = \omega(u_\mu,u_\nu) = \langle u_\mu, \tau(u_\nu)\rangle_{\mathbb{C}}$. It follows that $Y \perp \tau(Y)$ and therefore $Y \in M$, hence $u \in \theta^{-1}(M)$ holds.

For (b). We remark that M is a regular submanifold of $G_r(W)$, because it is an orbit of the action of a compact Lie group. However, a more explicit proof is required to show that M is a complex submanifold of $G_r(W)$.

For this purpose, we first note that we have $M \neq \varnothing$ because of (a). We now consider the Stiefel manifold $\widehat{\mathrm{St}}_r(W) \subset L(\mathbb{C}^r, W)$ of complex r-frames in W; this is a complex manifold, and the canonical projection $\widehat{\theta} : \widehat{\mathrm{St}}_r(W) \to G_r(W),\ u \mapsto u(\mathbb{C}^r)$ is a holomorphic submersion. Moreover, we consider the holomorphic map

$$g : \widehat{\mathrm{St}}_r(W) \to \mathbb{C}^{r(r-1)/2},\ u \mapsto \big(\omega(u_\mu,u_\nu)\big)_{1\leq\mu<\nu\leq r}\ .$$

Note that because of Equation (8.47), $g^{-1}(\{0\}) = \widehat{\theta}^{-1}(M)$ holds.

Immediately, we will show that g is a submersion; it then follows that $g^{-1}(\{0\}) = \widehat{\theta}^{-1}(M)$ is a regular, complex submanifold of $\widehat{\mathrm{St}}_r(W)$ (see [Nar68], Corollary 2.5.5, p. 81). Thus we may then conclude that M is a complex submanifold of $G_r(W)$. (Local trivializations of $\widehat{\theta}$ give rise to local parameterizations of M.) Moreover, we see that the complex dimension of $\widehat{\theta}^{-1}(M)$ is equal to $r\cdot 2r - \frac{1}{2}r(r-1) = \frac{1}{2}r(3r+1)$. Because $\mathrm{GL}(\mathbb{C}^r)$ acts simply transitively on the fibres of $\widehat{\theta}$, the fibre dimension of $\widehat{\theta}$ is r^2, and thus the dimension of M is $\frac{1}{2}r(3r+1) - r^2 = \frac{1}{2}r(r+1)$.

For the proof of the submersivity of g,[26] let $u \in \widehat{\mathrm{St}}_r(W)$ be given. Then we have $\overrightarrow{T_u\widehat{\mathrm{St}}_r(W)} = L(\mathbb{C}^r, W)$ and

$$\forall \xi \in T_u\widehat{\mathrm{St}}_r(W)\ :\ \overrightarrow{T_u g(\xi)} = \big(\omega((\overrightarrow{\xi})_\mu, u_\nu) + \omega(u_\mu, (\overrightarrow{\xi})_\nu)\big)_{1\leq\mu<\nu\leq r}\ . \tag{8.54}$$

[26] The proof of the submersivity of g is to a large extent analogous to the proof of Theorem 7.11(b). It should, however, be noted that in Theorem 7.11 the function g is defined with respect to a symmetric bilinear form, whereas here it is defined with respect to a skew-symmetric bilinear form. This difference necessitates some changes in the details of the proof.

To show that $T_u g : T_u\widehat{\mathrm{St}}_r(W) \to T_u\mathbb{C}^{r(r-1)/2}$ is surjective, it is therefore sufficient to prove that the linear forms $(\lambda_{\mu\nu})_{\mu<\nu}$ with

$$\lambda_{\mu\nu} : L(\mathbb{C}^r, W) \to \mathbb{C},\ a \mapsto \omega(a_\mu, u_\nu) + \omega(u_\mu, a_\nu) \qquad (\mu < \nu)$$

are linear independent. For this we first note that because of the non-degeneracy of ω we have

$$\forall z \in \mathbb{C}^r\ \exists\, w \in W\ :\ \big(\omega(w, u_\nu)\big)_{1\leq\nu\leq r} = z\ . \tag{8.55}$$

Now let $(\alpha_{\mu\nu})_{\mu<\nu} \in \mathbb{C}^{r(r-1)/2}$ be given so that $\sum_{\mu<\nu} \alpha_{\mu\nu}\, \lambda_{\mu\nu} = 0$ holds. Further, let $\mu_0 < \nu_0$ be given. By (8.55) there exists $a \in L(\mathbb{C}^r, W)$ so that

$$\forall \mu, \nu \in \{1, \ldots, r\}\ :\ \omega(a_\mu, u_\nu) = \delta_{\mu,\mu_0} \cdot \delta_{\nu,\nu_0}$$

holds. Then we have for any $\mu < \nu$: $\lambda_{\mu\nu}(a) = \delta_{\mu,\mu_0} \cdot \delta_{\nu,\nu_0}$ and therefore

$$0 = \sum_{\mu<\nu} \alpha_{\mu\nu}\, \lambda_{\mu\nu}(a) = \alpha_{\mu_0\,\nu_0}\ .$$

This shows the linear independence of $(\lambda_{\mu\nu})$.

For (c). We have $K \subset \mathrm{Sp}(W) \subset \mathrm{U}(W)$, and by definition for any $B \in K$: $B(Y) = Y$ and thus $B|Y \in \mathrm{U}(Y)$. This shows that Ψ indeed maps into $\mathrm{U}(Y)$; it is clear that Ψ is a homomorphism of Lie groups.

For the injectivity of Ψ: Let $B \in K$ be given with $\Psi(B) = \mathrm{id}_Y$. Because of $B \in \mathrm{Sp}(W)$, we have $B \circ \tau = \tau \circ B$ by Proposition 8.16 and therefore

$$B|\tau(Y) = (\tau|Y) \circ \underbrace{(B|Y)}_{=\mathrm{id}_Y} \circ (\tau|Y)^{-1} = \mathrm{id}_{\tau(Y)}\ .$$

Because of $W = Y \oplus \tau(Y)$, we conclude $B = \mathrm{id}_W$.

For the surjectivity of Ψ: Let $D \in \mathrm{U}(Y)$ be given and define a $\mathbb{C}$-linear map $B : W \to W$ by

$$B|Y = D \quad \text{and} \quad B|\tau(Y) = (\tau|Y) \circ D \circ (\tau|Y)^{-1} \tag{8.56}$$

Note that B leaves Y and $\tau(Y)$ invariant, and that besides $B|Y \in \mathrm{U}(Y)$ we also have $B|\tau(Y) \in \mathrm{U}(\tau(Y))$ because of Equation (8.48). Therefrom $B \in \mathrm{U}(W)$ follows. Moreover, (8.56) implies $B \circ \tau = \tau \circ B$, and thus we have $B \in \mathrm{Sp}(W)$ by Proposition 8.16. Clearly, we have $B(Y) = Y$, hence $B \in K$, and $\Psi(B) = B|Y = D$.

For (d). We regard $G_r(W)$ as a Hermitian symmetric $\mathrm{SU}(W)$-space; if we fix $Y \in M$ and denote by $S : W \to W$ the linear map characterized by $S|Y = \mathrm{id}_Y$ and $S|\tau(Y) = -\mathrm{id}_{\tau(Y)}$, then the symmetric structure of $G_r(W)$ is with respect to the "origin point" Y described by the involutive Lie group automorphism

$$\sigma : \mathrm{SU}(W) \to \mathrm{SU}(W),\ B \mapsto SBS^{-1}\ .$$

By (a) and (b), M is a complex submanifold and an orbit of the Lie group $\mathrm{Sp}(W) \subset \mathrm{SU}(W)$ acting on $G_r(W)$. To prove that M is a Hermitian symmetric subspace of $G_r(W)$, it therefore suffices to show that $\mathrm{Sp}(W)$ is invariant under σ (see [KN69], Theorem XI.4.1, p. 234).

To show this, we first note that $S \in \mathrm{U}(W)$ and $\tau \circ S = -S \circ \tau$ holds. Now, let $B \in \mathrm{Sp}(W)$ be given. Then we have $B \in \mathrm{U}(W)$ by Proposition 8.16 and therefore $\sigma(B) = SBS^{-1} \in \mathrm{U}(W)$; from the equations $\tau \circ B = B \circ \tau$ (again see Proposition 8.16) and $\tau \circ S = -S \circ \tau$ we get $\tau \circ \sigma(B) = \sigma(B) \circ \tau$ and therefore $\sigma(B) \in \mathrm{Sp}(W)$ by Proposition 8.16. Thus we have shown $\sigma(\mathrm{Sp}(W)) \subset \mathrm{Sp}(W)$. Because σ is involutive, $\sigma(\mathrm{Sp}(W)) = \mathrm{Sp}(W)$ follows.

It follows from (b) and (c) that M is isomorphic to the quotient space $\mathrm{Sp}(r)/\mathrm{U}(r)$.

Finally, we note that M is connected along with $\mathrm{Sp}(W)$ and that as a symmetric subspace, it is a complete, totally geodesic submanifold of $G_r(W)$, see [KN69], Theorem XI.4.1, p. 234. □

We now specialize to the situation of Theorem 8.7, where W is a 4-dimensional unitary space (i.e. $r = 2$ holds), and W is equipped with a complex orientation. Again, we denote the Hodge operator corresponding to this situation by $* : \bigwedge^2 W \to \bigwedge^2 W$ and regard $\bigwedge^2 W$ as a $\mathbb{C}$Q-space via the $\mathbb{C}$Q-structure $\mathfrak{A} := \mathbb{S}^1 \cdot *$.

8.19 Theorem. *(a) Let $\widehat{\omega} : \bigwedge^2 W \to \mathbb{C}$ be the linear form uniquely characterized by*

$$\forall w_1, w_2 \in W \;:\; \widehat{\omega}(w_1 \wedge w_2) = \omega(w_1, w_2) \,.$$

Then $U := \ker \widehat{\omega}$ is a 5-dimensional $\mathbb{C}$Q-subspace of $(\bigwedge^2 W, \mathfrak{A})$; we denote its induced $\mathbb{C}$Q-structure by $\mathfrak{A}'$.

The vector $\widehat{\omega}^\sharp \in \bigwedge^2 W$ characterized by

$$\forall \xi \in \textstyle\bigwedge^2 W \;:\; \widehat{\omega}(\xi) = \langle \xi, \widehat{\omega}^\sharp \rangle_{\mathbb{C}}$$

is given by

$$\widehat{\omega}^\sharp = -(\, u_1 \wedge \tau(u_1) + u_2 \wedge \tau(u_2)\,) \,, \tag{8.57}$$

where (u_1, u_2) is any symplectic basis of W. Note that $U = (\widehat{\omega}^\sharp)^\perp$ holds.

(b) $f_4(M)$ is the 3-dimensional, totally geodesic subquadric $Q' := Q(\mathfrak{A}') = Q(*) \cap [U]$ of the Plücker quadric $Q(*)$ and

$$f_3 := f_4|M : M \to Q'$$

is a holomorphic isometry.

(c) For every $B \in \mathrm{Sp}(W)$, we have

$$B^{(2)}|U \in \mathrm{Aut}_s(\mathfrak{A}')_0 \quad \textit{and} \quad B^{(2)}(\widehat{\omega}^\sharp) = \widehat{\omega}^\sharp \,, \tag{8.58}$$

and

$$F_3 : \mathrm{Sp}(W) \to \mathrm{Aut}_s(\mathfrak{A}')_0,\ B \mapsto B^{(2)}|U$$

is a two-fold covering map of Lie groups with kernel $\{\pm \mathrm{id}_W\}$. Herein we recognize the well-known isomorphy of Lie groups

$$\boxed{\mathrm{Sp}(2) \cong \mathrm{Spin}(5)}\,.$$

(d) (f_3, F_3) *is an almost-isomorphism of Hermitian symmetric spaces from the* $\mathrm{Sp}(W)$*-space* M *onto the* $\mathrm{Aut}_s(\mathfrak{A}')_0$*-space* Q'. *In particular, via* F_3 Q' *can be regarded as a Hermitian symmetric* $\mathrm{Sp}(W)$*-space, and we have shown the following almost-isomorphy of Hermitian symmetric spaces:*

$$\boxed{Q^3 \cong \mathrm{Sp}(2)/\mathrm{U}(2)}\,.$$

Proof. For (a). The existence and uniqueness of $\widehat{\omega}$ follows from the universal property characterizing the exterior product $\bigwedge^2 W$, and the existence and uniqueness of $\widehat{\omega}^\sharp$ is then clear. Because ω is non-degenerate, we have $\widehat{\omega} \neq 0$, and therefore U is a 5-dimensional, complex subspace of $\bigwedge^2 W$.

Next, we verify Equation (8.57). For this purpose, we let a symplectic basis (u_1, u_2) of W be given, then (u_1, u_2, u_3, u_4) with $u_3 := \tau(u_1)$ and $u_4 := \tau(u_2)$ is a unitary basis of W, and $(u_\mu \wedge u_\nu)_{\mu<\nu}$ is a unitary basis of $\bigwedge^2 W$. For the proof of Equation (8.57), it suffices to show the equality

$$\widehat{\omega}(\xi) = -\langle\, \xi\,,\, u_1 \wedge u_3 + u_2 \wedge u_4 \,\rangle_{\mathbb{C}}$$

for the elements ξ of the latter basis, and this means because of Equation (8.47)

$$\forall \mu < \nu \;:\; \langle u_\mu, \tau(u_\nu)\rangle_{\mathbb{C}} = -\langle\, u_\mu \wedge u_\nu\,,\, u_1 \wedge u_3 + u_2 \wedge u_4 \,\rangle_{\mathbb{C}}\;;$$

the latter equation is easily verified by direct calculation.

It only remains to show that U is a $\mathbb{C}$Q-subspace of $\bigwedge^2 W$. For this purpose, we let ψ be the complex orientation on W so that the above unitary basis $(u_1, u_2, \tau(u_1), \tau(u_2))$ is positively oriented (Remark 8.17 shows that ψ does not depend on the choice of (u_1, u_2)) and $*_\psi$ be the Hodge operator on $\bigwedge^2 W$ corresponding to the orientation ψ; then we have $*_\psi \in \mathfrak{A}$. Example B.3 shows that

$$*_\psi(\widehat{\omega}^\sharp) \overset{(8.57)}{=} -\, *_\psi \left(u_1 \wedge \tau(u_1) + u_2 \wedge \tau(u_2)\right) = u_2 \wedge \tau(u_2) + u_1 \wedge \tau(u_1) \overset{(8.57)}{=} -\widehat{\omega}^\sharp$$

holds; because $*_\psi$ transforms unitary bases of $\bigwedge^2 W$ into unitary bases of $\bigwedge^2 W$, it follows that $U = (\widehat{\omega}^\sharp)^\perp$ is $*_\psi$-invariant and hence a $\mathbb{C}$Q-subspace of $\bigwedge^2 W$.

For (b). We consider the canonical projections $\widehat{\theta} : \widehat{\mathrm{St}}_2(W) \to G_2(W),\ u \mapsto u(\mathbb{C}^2)$ and $\widehat{\pi} : \bigwedge^2 W \setminus \{0\} \to \mathbb{P}(\bigwedge^2 W),\ v \mapsto [v]$, and the holomorphic map $\widehat{f}_4 : \widehat{\mathrm{St}}_2(W) \to \widehat{Q}(*),\ u \mapsto u_1 \wedge u_2$; then $f_4 \circ \widehat{\theta} = \widehat{\pi} \circ \widehat{f}_4$ holds. As we saw in the proof of Proposition 8.18(b), we have

$$\widehat{\theta}^{-1}(M) = g^{-1}(\{0\}) \tag{8.59}$$

with the holomorphic submersion $g : \widehat{\mathrm{St}}_2(W) \to \mathbb{C},\ u \mapsto \omega(u_1, u_2)$. We have $g = \widehat{\omega} \circ \widehat{f}_4$, and therefore Equation (8.59) shows that $\widehat{f}_4$ maps $\widehat{\theta}^{-1}(M)$ into $\ker \widehat{\omega} = U$. Thus f_4 maps M into $Q(*) \cap [U] = Q'$. Because M is compact, Q' is connected, and we have $\dim_{\mathbb{C}} M = 3 = \dim_{\mathbb{C}} Q'$ (see Proposition 8.18(b)), it follows that in fact $f_4(M) = Q'$ holds.

For (c). Let $B \in \mathrm{Sp}(W)$ be given. Then B leaves ω invariant by Proposition 8.16, and therefore we have $\widehat{\omega} \circ B^{(2)} = \widehat{\omega}$. Because we also have $B^{(2)} \in \mathrm{U}(\bigwedge^2 W)$, it follows that

$B^{(2)}(\widehat{\omega}^\sharp) = \widehat{\omega}^\sharp$ holds. It also follows that $B^{(2)}$ leaves $\ker\widehat{\omega} = U$ invariant. Because we have $B^{(2)} \in \mathrm{Aut}_s(\mathfrak{A})$ by Theorem 8.7(b), we conclude that $F_3 : \mathrm{Sp}(W) \to \mathrm{Aut}_s(\mathfrak{A}')$, $B \mapsto B^{(2)}|U$ is a homomorphism of Lie groups, which because of the connectedness of $\mathrm{Sp}(W)$ in fact maps into $\mathrm{Aut}_s(\mathfrak{A}')_0$. Thus we have shown (8.58).

For $B \in \mathrm{Sp}(W)$, $B^{(2)}|U = \mathrm{id}_U$ already implies $B^{(2)} = \mathrm{id}_{\bigwedge^2 W}$ because of (8.58), and therefore we have $\ker F_3 = \ker F_4 \cap \mathrm{Sp}(W) = \{\pm\mathrm{id}_W\}$ by Theorem 8.7(b). Thus, F_3 is a two-fold covering map of Lie groups over its image, which is indeed all of $\mathrm{Aut}_s(\mathfrak{A}')_0$ because we have $\dim\mathrm{Sp}(W) = 10 = \dim\mathrm{Aut}_s(\mathfrak{A}')_0$.

For (d). It is now clear that (f_3, F_3) is an almost-isomorphism of homogeneous spaces from the $\mathrm{Sp}(W)$-space M onto the $\mathrm{Aut}_s(\mathfrak{A}')_0$-space Q'. Because the Lie groups $\mathrm{Sp}(W)$ and $\mathrm{Aut}_s(\mathfrak{A}')_0$ are of compact type, Proposition A.5 shows that (f_3, F_3) is an almost-isomorphisms of affine symmetric spaces; because f_3 is also a holomorphic isometry, (f_3, F_3) is in fact an almost-isomorphism of Hermitian symmetric spaces. □

8.5 Q^6 is isomorphic to $\mathrm{SO}(8)/\mathrm{U}(4)$

The two series of Hermitian symmetric spaces Q^m and $\mathrm{SO}(2n)/\mathrm{U}(n)$ intersect for $m = 6$, $n = 4$. In the present section, we construct the corresponding isomorphism $Q^6 \cong \mathrm{SO}(8)/\mathrm{U}(4)$ explicitly.

There are several geometric realizations for the Hermitian symmetric space $\mathrm{SO}(2n)/\mathrm{U}(n)$:

- The two connected components of the congruence family $\mathfrak{F}(\mathbb{P}^{n-1}, Q^{2n-2})$ studied in Theorem 7.11 are as Hermitian symmetric spaces isomorphic to $\mathrm{SO}(2n)/\mathrm{U}(n)$ (see Theorem 7.11(c)(i)).
- If V is a $2n$-dimensional complex linear space, the submanifold of the complex Grassmannian $G_n(V)$ constituted by those $U \in G_n(V)$ which are isotropic with respect to some non-degenerate, symmetric bilinear form β on V has two connected components, both of which are isomorphic to $\mathrm{SO}(2n)/\mathrm{U}(n)$.
- The manifold of orthogonal complex structures on the Euclidean space $\mathbb{R}^{2n}$ has two connected components, both of which are isomorphic to $\mathrm{SO}(2n)/\mathrm{U}(n)$.

We will base our construction of the isomorphism $Q^6 \cong \mathrm{SO}(8)/\mathrm{U}(4)$ on the realization $\mathfrak{F}(\mathbb{P}^3, Q^6)$ of $\mathrm{SO}(8)/\mathrm{U}(4)$. In the construction, we will extensively use the theory of $\mathbb{C}$Q-spaces, as we did throughout the present dissertation.[27] Also, we will use the theory of Clifford alge-

[27] It should be noted, however, that both $\mathrm{SO}(8)/\mathrm{U}(4)$ and Q^6 can be described without use of $\mathbb{C}$Q-structures: the former space as a connected component of the manifold of orthogonal complex structures on $\mathbb{R}^8$, and the latter as a 6-dimensional algebraic complex quadric (which can then be regarded as a symmetric complex quadric via a suitable Hermitian inner product on the underlying complex linear space, see Remark 1.12(b)). Therefore, it should be possible in principle to describe the isomorphism without reference to the theory of $\mathbb{C}$Q-spaces.

bras, spin groups and its representations as described in Appendix B. The principle of triality (Section B.6) will play an important role.

The strategy is as follows: Let $(\mathbb{V}, \mathfrak{A})$ be an 8-dimensional $\mathbb{C}$Q-space and fix $A \in \mathfrak{A}$. Then A gives rise to a symmetric bilinear form β on $\mathbb{V}$, thereby to the Clifford algebra $C(\mathbb{V}, \beta)$ and to the spaces S_+ and S_- of positive resp. negative half-spinors, which are in the present situation again 8-dimensional. It will turn out that $S_\pm$ can be equipped with a $\mathbb{C}$Q-structure $\mathfrak{A}_\pm$ in a canonical way. $\mathfrak{A}_\pm$ defines the 6-dimensional complex quadric $Q_\pm := Q(\mathfrak{A}_\pm)$. We will construct isomorphisms of Hermitian symmetric spaces from $Q_\pm$ to the connected components of the congruence family $\mathfrak{F}(\mathbb{P}^3, Q)$.

For the construction of these isomorphisms, we note that for any spinor $s \neq 0$, $Z(s) := \{ v \in \mathbb{V} \,|\, \rho(v)s = 0 \}$ (where $\rho : C(\mathbb{V}, \beta) \to \mathrm{End}(S)$ denotes the spin representation of the Clifford algebra $C(\mathbb{V}, \beta)$) is an isotropic subspace of $\mathbb{V}$. As it is described in Subsection 8.5.2, $Z(s)$ is of the maximal dimension 4 if and only if $s \in \widehat{Q}(\mathfrak{A}_+) \dot\cup \widehat{Q}(\mathfrak{A}_-)$ holds. In this way, with every element of $\widehat{Q}(\mathfrak{A}_\pm)$ there is associated a 4-dimensional isotropic subspace of $\mathbb{V}$ and therefore an element of $\mathfrak{F}(\mathbb{P}^3, Q)$. It turns out in Subsection 8.5.3 that this association descends to isomorphisms of Hermitian symmetric spaces h_+ and h_- from Q_+ resp. Q_- to one resp. the other connected component of $\mathfrak{F}(\mathbb{P}^3, Q)$.

As preparation for this construction, we investigate in Subsection 8.5.1 the relationship between the vector representation χ resp. the spin representations $\rho_\pm$ of $\mathrm{Spin}(\mathbb{V}, \beta)$ and the $\mathbb{C}$Q-structures on the spaces $\mathbb{V}$ and $S_\pm$. In particular, we construct a subgroup G of $\mathrm{Spin}(\mathbb{V}, \beta)$ (isomorphic to the real spin group $\mathrm{Spin}(8)$) so that $\chi|G : G \to \mathrm{Aut}_s(\mathfrak{A})_0$ and $\rho_\pm|G : G \to \mathrm{Aut}_s(\mathfrak{A}_\pm)_0$ are two-fold covering maps of Lie groups. Via these covering maps, the quadrics Q and $Q_\pm$ can be regarded as Hermitian symmetric G-spaces.

The fundamental facts on Clifford algebras, the spin group and spin representations which are needed here are gathered in Appendix B.

8.5.1 The vector and spin representations of Spin(8)

Let $(\mathbb{V}, \mathfrak{A})$ be an 8-dimensional $\mathbb{C}$Q-space. We fix $A \in \mathfrak{A}$ and consider the non-degenerate, symmetric bilinear form

$$\beta : \mathbb{V} \times \mathbb{V} \to \mathbb{C}\ ,\ (v_1, v_2) \mapsto \langle v_1, A v_2 \rangle_{\mathbb{C}}$$

induced by A, the quadratic form $q : \mathbb{V} \to \mathbb{C},\ v \mapsto \frac{1}{2}\,\beta(v, v)$ and the complex Clifford algebra $C(\mathbb{V}, \beta)$ (see Section B.3). We also have the vector representation $\chi : \Gamma(\mathbb{V}, \beta) \to \mathrm{O}(\mathbb{V}, \beta)$ of the Clifford group $\Gamma(\mathbb{V}, \beta)$.

We now exhibit a connection between this complex situation and the situation of a real Clifford algebra. The space $\mathbb{V}' := JV(A) = V(-A)$ is a real-8-dimensional, totally real subspace of $\mathbb{V}$, which we regard in the sequel as an euclidean space, and

$$\forall v \in \mathbb{V},\ x \in \mathbb{V}' \ :\ \beta(v, x) = -\langle v, x \rangle_{\mathbb{C}} \tag{8.60}$$

holds. It follows that $\beta' := \beta|(\mathbb{V}'\times\mathbb{V}')$ is a symmetric, negative definite, real bilinear form on $\mathbb{V}'$, and we also consider the real Clifford algebra $C(\mathbb{V}',\beta')$; we denote by $\chi' : \Gamma(\mathbb{V}',\beta') \to \mathrm{SO}(\mathbb{V}')$ the vector representation of $\Gamma(\mathbb{V}',\beta')$. It should be noted that with $q' := q|\mathbb{V}'$, $(q')^{-1}(\{-1\})$ is the sphere of radius $\sqrt{2}$ in $\mathbb{V}'$.

The $\mathbb{R}$-linear map $\iota : \mathbb{V}' \hookrightarrow C(\mathbb{V},\beta)$ is a Clifford map for the Clifford algebra $C(\mathbb{V}',\beta')$, and therefore there exists one and only one algebra homomorphism $\psi : C(\mathbb{V}',\beta') \to C(\mathbb{V},\beta)$ so that $\psi(x) = x$ holds for every $x \in \mathbb{V}'$.

8.20 Proposition. *(a) ψ is injective.*

(b) Put

$$G := \{\, x_1 \cdots x_{2k} \,|\, k \leq 4,\ x_1,\dots,x_{2k} \in (q')^{-1}(\{-1\}) \,\} \ \subset\ \mathrm{Spin}(\mathbb{V},\beta)\ ,$$

where the multiplication is carried out in $C(\mathbb{V},\beta)$. G is a compact, simply connected Lie subgroup of $\mathrm{Spin}(\mathbb{V},\beta)$, and $\Psi := (\psi|\mathrm{Spin}(\mathbb{V}',\beta')) : \mathrm{Spin}(\mathbb{V}',\beta') \to G$ is an isomorphism of Lie groups. Moreover, we have

$$\forall g' \in \mathrm{Spin}(\mathbb{V}',\beta')\ :\ \chi'(g') = \chi(\Psi(g'))|\mathbb{V}'\ . \tag{8.61}$$

(c) We have

$$G = \{\, g \in \mathrm{Spin}(\mathbb{V},\beta) \,|\, \chi(g) \in \mathrm{Aut}_s(\mathfrak{A})_0 \,\}$$

and $\chi|G : G \to \mathrm{Aut}_s(\mathfrak{A})_0$ is a two-fold covering map of Lie groups with $\ker(\chi|G) = \{\pm 1\}$.

Proof. For (a). Let $(x_1,\dots,x_8)$ be an $\mathbb{R}$-basis of $\mathbb{V}'$. Then the same list of vectors is a $\mathbb{C}$-basis of $\mathbb{V}$, and ψ maps the induced $\mathbb{R}$-basis $(x_S)_{S\subset\{1,\dots,8\}}$ of $C(\mathbb{V}',\beta')$ (see Theorem B.7(b)) onto the $\mathbb{C}$-basis $(x_S)_{S\subset\{1,\dots,8\}}$ of $C(\mathbb{V},\beta)$, which is in particular linear independent over $\mathbb{R}$. Therefore ψ is injective.

For (b). Proposition B.15(c)(ii) shows that ψ maps $\mathrm{Spin}(\mathbb{V}',\beta')$ onto G. Clearly Ψ is a homomorphism of abstract groups. As restriction of an injective linear map to a compact submanifold, Ψ (regarded as a map into $\mathrm{Spin}(\mathbb{V},\beta)$) also is an embedding, and its image $G = \Psi(\mathrm{Spin}(\mathbb{V}',\beta'))$ is a compact submanifold of $\mathrm{Spin}(\mathbb{V},\beta)$. It follows that G is a Lie subgroup of $\mathrm{Spin}(\mathbb{V},\beta)$ (see [Var74], Theorem 2.12.6, p. 99) and that Ψ is an isomorphism of Lie groups onto G. Consequently G is simply connected along with $\mathrm{Spin}(\mathbb{V}',\beta')$.

For Equation (8.61): Let us denote the canonical involutions of $C(\mathbb{V},\beta)$ and $C(\mathbb{V}',\beta')$ (see Proposition B.10(a)) by α and α', respectively. α leaves $\psi(C(\mathbb{V}',\beta'))$ invariant; this fact together with the unique characterization of α' in Proposition B.10(a) shows that $\psi\circ\alpha' = \alpha\circ\psi$ holds. We thus have for any $g' \in \mathrm{Spin}(\mathbb{V}',\beta')$ and $x \in \mathbb{V}'$

$$\begin{aligned}\chi'(g')x &= \psi(\chi'(g')x) = \psi(\alpha'(g')\cdot x\cdot (g')^{-1}) \\ &= \psi(\alpha'(g'))\cdot\psi(x)\cdot\psi(g')^{-1} = \alpha(\psi(g'))\cdot x\cdot\psi(g')^{-1} = \chi(\Psi(g'))x\ .\end{aligned}$$

For (c). We put $\widetilde{G} := \{\, g \in \mathrm{Spin}(\mathbb{V},\beta) \,|\, \chi(g) \in \mathrm{Aut}_s(\mathfrak{A})_0 \,\}$.

First, we show $G \subset \widetilde{G}$. For this, let $g \in G$ be given. Then $\chi(g)$ is a $\mathbb{C}$-linear transformation of $\mathbb{V}$, and with $g' := \Psi^{-1}(g) \in \mathrm{Spin}(\mathbb{V}',\beta')$ we have $\chi(g)|\mathbb{V}' = \chi'(g') \in \mathrm{SO}(\mathbb{V}')$ by (b) and Proposition B.15(e), whence $\chi(g) \in \mathrm{Aut}_s(\mathfrak{A})_0$ follows by Proposition 2.17(a) because of $\mathbb{V}' = V(-A)$. Thus we have $g \in \widetilde{G}$.

We now consider the Lie group isomorphism $\Xi : \mathrm{SO}(\mathbb{V}') \to \mathrm{Aut}_s(\mathfrak{A})_0,\ L \mapsto L^{\mathbb{C}}$ (see Proposition 2.17(a)) and the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Spin}(\mathbb{V}',\beta') \arrow[r, "\Psi"] \arrow[d, "\chi'"'] & G \arrow[r, hook] \arrow[d, "\chi|G"'] & \mathrm{Spin}(\mathbb{V},\beta) \arrow[d, "\chi"] \\
\mathrm{SO}(\mathbb{V}') \arrow[r, "\Xi"'] & \mathrm{Aut}_s(\mathfrak{A})_0 \arrow[r, hook] & \mathrm{SO}(\mathbb{V},\beta) .
\end{tikzcd}$$

From the commutativity of the left-hand half of the diagram, we see that $\chi|G : G \to \mathrm{Aut}_s(\mathfrak{A})_0$ is a two-fold covering of Lie groups with $\ker(\chi|G) = \{\pm 1\}$, see Proposition B.15(d),(e).

Finally, we prove $\widetilde{G} \subset G$. For this, let $\widetilde{g} \in \widetilde{G}$ be given. Then we have $B := \chi(\widetilde{g}) \in \mathrm{Aut}_s(\mathfrak{A})_0$. Because both $\chi|G : G \to \mathrm{Aut}_s(\mathfrak{A})_0$ and $\chi : \mathrm{Spin}(\mathbb{V},\beta) \to \mathrm{SO}(\mathbb{V},\beta)$ are two-fold covering maps of Lie groups, the commutativity of the right-hand side of the diagram shows that both pre-images of B under χ are contained in G. In particular, we have $\widetilde{g} \in G$. $\square$

Our next aim is to obtain an analogous result as that of Proposition 8.20(c) for the half-spin representations $\rho_\pm$ belonging to $C(\mathbb{V},\beta)$.

To obtain the explicit description of the spin representation given in Section B.5, we fix a complex-4-dimensional, $\mathfrak{A}$-isotropic subspace W of $\mathbb{V}$ (see Corollary 2.22). Then $W' := A(W)$ is another complex-4-dimensional, $\mathfrak{A}$-isotropic subspace of $\mathbb{V}$, W and W' are also β-isotropic, and we have $\mathbb{V} = W \oplus W'$. Moreover, we fix a complex orientation on W and denote by $\omega \in \bigwedge^4 W$ the positive unit 4-vector of W corresponding to this orientation (see Section B.2). Note that the Hermitian inner product of W induces a Hermitian inner product on $\bigwedge W$ in the way described in Section B.2. We denote this product also by $\langle\cdot,\cdot\rangle_{\mathbb{C}}$

We apply the construction of Section B.5 in this situation. Thereby we obtain the spinor space $S = \bigwedge W$, the spaces $S_+ = \bigwedge^{\mathrm{even}} W$ and $S_- = \bigwedge^{\mathrm{odd}} W$ of even resp. of odd half-spinors, the non-degenerate bilinear form $\beta_S : S \times S \to \mathbb{C}$ of Proposition B.30 which is here symmetric (Proposition B.30(b)(v)), their restrictions $\beta_\pm := \beta_S|(S_\pm \times S_\pm)$, the spin representation $\rho : C(\mathbb{V},\beta) \to \mathrm{End}(S)$ (Theorem B.26), and the half-spin representations $\rho_\pm : \mathrm{Spin}(\mathbb{V},\beta) \to \mathrm{SO}(S_\pm,\beta_\pm)$.

8.21 Proposition. *The anti-linear map $A_S : S \to S$ characterized by*

$$\forall s_1, s_2 \in S \;:\; \langle s_1, A_S(s_2)\rangle_{\mathbb{C}} = \beta_S(s_1,s_2) \tag{8.62}$$

is a conjugation on S. In the sequel, we regard S as a $\mathbb{C}$Q-space via the $\mathbb{C}$Q-structure $\mathfrak{A}_S := \mathbb{S}^1 \cdot A_S$.

Proof. It is obvious that A_S is anti-linear, and because β_S is symmetric, A_S is self-adjoint with respect to the real inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}} := \mathrm{Re}(\langle\cdot,\cdot\rangle_{\mathbb{C}})$ on $S = \bigwedge W$. It thus only remains to show that A_S is also orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ (see Definition 2.1), and for this purpose, we derive an explicit description of A_S involving the Hodge operator (see Appendix B.2).

Let $k \in \{0,\dots,4\}$ and $s \in \bigwedge^k W$ be given. For $\widetilde{s} \in \bigwedge^\ell W$ with $\ell \in \{0,\dots,4\}$, $\ell \neq 4-k$, we have by the definition of β_S (Proposition B.30(a))

$$\langle \widetilde{s}, A_S(s)\rangle_{\mathbb{C}} \overset{(8.62)}{=} \beta_S(\widetilde{s}, s) = \varphi(\kappa(\widetilde{s}) \wedge s) = 0\ ,$$

where κ is the main anti-automorphism of $C(\mathbb{V},\beta)$ and φ is as in Proposition B.30. Therefore, $A_S(s) \in \bigwedge^{4-k} W$ holds. Now let us denote by $*$ the Hodge operator on $S = \bigwedge W$ with respect to the chosen orientation on W. Then we have for any $\widetilde{s} \in \bigwedge^{4-k} W$ by Proposition B.30(d)

$$\begin{aligned}\langle \widetilde{s}, A_S(s)\rangle_{\mathbb{C}} &\overset{(8.62)}{=} \beta_S(\widetilde{s}, s) \\ &= (-1)^{4(4-k)} \cdot (-1)^{(4-k)(4-k+1)/2} \cdot \langle \widetilde{s}, A_S(s)\rangle_{\mathbb{C}} \\ &= (-1)^{k(k-1)/2} \cdot \langle \widetilde{s}, A_S(s)\rangle_{\mathbb{C}}\ .\end{aligned}$$

Thus, we have shown

$$\forall k \in \{0,\dots,4\}\ :\ A_S|\textstyle\bigwedge^k W = (-1)^{k(k-1)/2} \cdot (*|\textstyle\bigwedge^k W)\ .$$

From this representation of A_S and Proposition B.2(e) we see that A_S is orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$. $\square$

Because the spaces S_+ and S_- are the β_S-ortho-complement of each other (Proposition B.30(c)), they are A_S-invariant, and therefore $\mathbb{C}$Q-subspaces of S. $A_\pm := A_S|S_\pm : S_\pm \to S_\pm$ is a conjugation on $S_\pm$, and $\mathfrak{A}_\pm := \mathbb{S}^1 \cdot A_\pm$ is the $\mathbb{C}$Q-structure on $S_\pm$ induced by $\mathfrak{A}_S$.

To gain insight into the behaviour of the half-spin representations $\rho_\pm$, we take advantage of the fact that they are "intertwined" with the vector representation χ in the way described by the principle of triality, see Section B.6. We denote by $\mathfrak{T} = \mathbb{V} \oplus S = \mathbb{V} \oplus S_+ \oplus S_-$ the triality algebra which was described there. But now we also equip $\mathfrak{T}$ with the complex inner product characterized by the following properties: (1) The restriction of this inner product to $\mathbb{V}\times\mathbb{V}$ and to $S \times S$ equals the inner product on $\mathbb{V}$ resp. on S we considered before. (2) The subspaces $\mathbb{V}$ and S of $\mathfrak{T}$ are orthogonal to each other. We denote also this inner product by $\langle\cdot,\cdot\rangle_{\mathbb{C}}$.

Moreover, we consider the anti-linear map $A_{\mathfrak{T}} : \mathfrak{T} \to \mathfrak{T}$ characterized by

$$A_{\mathfrak{T}}|\mathbb{V} = A \quad\text{and}\quad A_{\mathfrak{T}}|S = A_S\ .$$

Then $A_{\mathfrak{T}}$ is a conjugation on $\mathfrak{T}$, therefore $\mathfrak{A}_{\mathfrak{T}} := \mathbb{S}^1 \cdot A_{\mathfrak{T}}$ is a $\mathbb{C}$Q-structure on $\mathfrak{T}$, and we regard $\mathfrak{T}$ as a $\mathbb{C}$Q-space in this way in the sequel. It should be noted that $(\mathbb{V},\mathfrak{A})$, $(S,\mathfrak{A}_S)$, $(S_+,\mathfrak{A}_+)$ and $(S_-,\mathfrak{A}_-)$ are $\mathbb{C}$Q-subspaces of $(\mathfrak{T},\mathfrak{A}_{\mathfrak{T}})$, and that

$$\forall X, Y \in \mathfrak{T}\ :\ \langle X, A_{\mathfrak{T}} Y\rangle_{\mathbb{C}} = \beta_{\mathfrak{T}}(X,Y)$$

holds, where $\beta_{\mathfrak{T}}$ is the non-degenerate, symmetric bilinear form on $\mathfrak{T}$ from Section B.6 induced by β and β_S.

We now fix $w_1 \in \mathbb{S}(W)$ and put $w_1' := Aw_1 \in \mathbb{S}(W')$, note that $\beta(w_1, w_1') = 1$ holds. Then we consider the triality automorphism $T : \mathfrak{T} \to \mathfrak{T}$ corresponding to this choice of (w_1, w_1') as described in Theorem B.34 and by $\vartheta : \mathrm{Spin}(\mathbb{V}, \beta) \to \mathrm{Spin}(\mathbb{V}, \beta)$ the corresponding triality automorphism of $\mathrm{Spin}(\mathbb{V}, \beta)$, see Theorem B.35. We summarize the fundamental properties of T and ϑ:

$$T^3 = \mathrm{id}_{\mathfrak{T}}\,, \quad \vartheta^3 = \mathrm{id}_{\mathrm{Spin}(\mathbb{V},\beta)}\,, \tag{8.63}$$

$$T(\mathbb{V}) = S_+\,, \quad T(S_+) = S_- \quad \text{and} \quad T(S_-) = \mathbb{V}\,, \tag{8.64}$$

$$\forall X, Y \in \mathfrak{T} \;:\; \beta_{\mathfrak{T}}(T(X), T(Y)) = \beta_{\mathfrak{T}}(X, Y)\,, \tag{8.65}$$

$$\forall g \in \mathrm{Spin}(\mathbb{V}, \beta) \;:\; \mu(\vartheta(g)) \circ T = T \circ \mu(g)\,. \tag{8.66}$$

In the last equation, $\mu : \mathrm{Spin}(\mathbb{V}, \beta) \to \mathrm{SO}(\mathfrak{T}, \beta_{\mathfrak{T}})$ is the representation of $\mathrm{Spin}(\mathbb{V}, \beta)$ on $\mathfrak{T}$ induced by χ and ρ as described in Section B.6.

As was emphasized at the end of Section B.6, it is in some regards more natural to consider restrictions of T:

$$T_{\mathbb{V}+} := T|\mathbb{V} : \mathbb{V} \to S_+\,, \quad T_{+-} := T|S_+ : S_+ \to S_- \quad \text{and} \quad T_{-\mathbb{V}} : T|S_- : S_- \to \mathbb{V}\,.$$

From Equation (8.66) and the definition of μ, we see that the following equations hold for any $g \in \mathrm{Spin}(\mathbb{V}, \beta)$:

$$\rho_+(\vartheta(g)) \circ T_{\mathbb{V}+} = T_{\mathbb{V}+} \circ \chi(g)\,, \tag{8.67}$$

$$\rho_-(\vartheta(g)) \circ T_{+-} = T_{+-} \circ \rho_+(g)\,, \tag{8.68}$$

$$\text{and} \quad \chi(\vartheta(g)) \circ T_{-\mathbb{V}} = T_{-\mathbb{V}} \circ \rho_-(g)\,. \tag{8.69}$$

The following proposition clarifies the relationship between T and the $\mathbb{C}$Q-space structure of $\mathfrak{T}$:

8.22 Proposition. *(a)* $T \in \mathrm{Aut}_s(\mathfrak{A}_{\mathfrak{T}})_0$.

(b) $T_{\mathbb{V}+}$, T_{+-} *and* $T_{-\mathbb{V}}$ *are* $\mathbb{C}$Q*-isomorphisms between the respective* $\mathbb{C}$Q*-subspaces of* $\mathfrak{T}$.

Proof. For (a). We extend w_1 into a positively oriented unitary basis $(w_1, \dots, w_4)$ of W, then $(w_1', \dots, w_4')$ with $w_k' := Aw_k$ is a unitary basis of W' so that $\beta(w_k, w_\ell') = \delta_{k\ell}$ holds. The 24 elements of $\mathfrak{T}$

$$\begin{gathered} w_1,\, w_2,\, w_3,\, w_4, \quad w_1',\, w_2',\, w_3',\, w_4' \;\in\; \mathbb{V}\,, \\ 1,\, w_1 \wedge w_2,\, w_1 \wedge w_3,\, w_1 \wedge w_4, \quad w_1 \wedge w_2 \wedge w_3 \wedge w_4,\, -w_3 \wedge w_4,\, w_2 \wedge w_4,\, -w_2 \wedge w_3 \;\in\; S_+\,, \\ w_2 \wedge w_3 \wedge w_4,\, -w_2,\, -w_3,\, -w_4, \quad w_1,\, w_1 \wedge w_3 \wedge w_4,\, -w_1 \wedge w_2 \wedge w_4,\, w_1 \wedge w_2 \wedge w_3 \;\in\; S_- \end{gathered}$$

constitute a unitary basis of $\mathfrak{T}$, which is mapped by T onto the same basis in a different ordering, as Theorem B.34 shows. Therefore T is a unitary map.

Also we have $T \in \mathrm{O}(\mathfrak{T}, \beta_{\mathfrak{T}})$ by Equation (8.65), and therefore $\det_{\mathbb{C}}(T) \in \{\pm 1\}$. Moreover, we have

$$(\det{}_{\mathbb{C}}(T))^3 = \det{}_{\mathbb{C}}(T^3) \overset{(8.63)}{=} \det{}_{\mathbb{C}}(\mathrm{id}_{\mathfrak{T}}) = 1$$

and therefore $\det_{\mathbb{C}}(T) = 1$, hence $T \in \mathrm{SO}(\mathfrak{T}, \beta_{\mathfrak{T}})$.

Thus we have shown $T \in \mathrm{U}(\mathfrak{T}) \cap \mathrm{SO}(\mathfrak{T}, \beta_{\mathfrak{T}}) = \mathrm{Aut}_s(\mathfrak{A}_{\mathfrak{T}})_0$, see Proposition 2.18(b).

For (b). This follows from (a) and the fact that V, S_+ and S_- are $\mathbb{C}$Q-subspaces of $\mathfrak{T}$ which are permuted cyclically by T. □

8.23 Proposition. *(a) For any $g \in G$, we have $\rho_\pm(g) \in \mathrm{Aut}_s(\mathfrak{A}_\pm)_0$, and the maps*

$$\rho_+|G : G \to \mathrm{Aut}_s(\mathfrak{A}_+)_0 \quad \textit{and} \quad \rho_-|G : G \to \mathrm{Aut}_s(\mathfrak{A}_-)_0$$

are two-fold covering maps of Lie groups.

(b) $\vartheta(G) = G$. Thus $\vartheta|G : G \to G$ is a Lie group automorphism of order 3.

Proof. Let $g \in G$ be given. Below, we show $\rho(g) \in \mathrm{Aut}_s(\mathfrak{A}_S)$; because $S_\pm$ is a $\mathbb{C}$Q-subspace of S, $\rho_\pm(g) \in \mathrm{Aut}_s(\mathfrak{A}_\pm)$ follows; because G is connected and $\rho_\pm|G : G \to \mathrm{Aut}_s(\mathfrak{A}_\pm)$ is continuous, we then see that in fact, $\rho_\pm(g) \in \mathrm{Aut}_s(\mathfrak{A}_\pm)_0$ holds.

For the proof of $\rho(g) \in \mathrm{Aut}_s(\mathfrak{A}_S)$: We have $\mathrm{Aut}_s(\mathfrak{A}_S) = \mathrm{U}(S) \cap \mathrm{O}(S, \beta_S)$ by Proposition 2.18(a) and $\rho(g) \in \mathrm{O}(S, \beta_S)$ by Proposition B.30(b)(iii). Thus, it remains to show $\rho(g) \in \mathrm{U}(S)$. For this we note that by the definition of G, there exist $k \leq 4$ and $x_1, \dots, x_{2k} \in (q')^{-1}(\{-1\})$ such that $g = x_1 \cdots x_{2k}$ and hence $\rho(g) = \rho(x_1) \circ \dots \circ \rho(x_{2k})$ holds. It is therefore sufficient to show $\rho(x) \in \mathrm{U}(S)$ for any $x \in (q')^{-1}(\{-1\})$.

For this purpose, let $x \in (q')^{-1}(\{-1\}) \subset \mathbb{V}'$ be given. By Proposition 2.20(g) there exists $a_1 \in W$ so that $x = a_1 - Aa_1$ holds, and $q'(x) = -1$ implies $\|a_1\| = 1$. We extend a_1 to a positively oriented, unitary basis $(a_1, \dots, a_4)$ of W.

By Theorem B.26, we have for any $s \in S$

$$\rho(x)s = \rho(a_1)s - \rho(Aa_1)s = a_1 \wedge s - \nu_{\beta(\cdot, Aa_1)}s = a_1 \wedge s - \nu_{\langle \cdot, a_1 \rangle_{\mathbb{C}}} s\,. \tag{8.70}$$

Using this equation, we calculate

$s \in S$	1	$a_1 \wedge a_2$	$a_1 \wedge a_3$	$a_1 \wedge a_4$	$a_2 \wedge a_3$	$a_2 \wedge a_4$	$a_3 \wedge a_4$	$a_1 \wedge a_2 \wedge a_3 \wedge a_4$
$\rho(x)s$	a_1	$-a_2$	$-a_3$	$-a_4$	$a_1 \wedge a_2 \wedge a_3$	$a_1 \wedge a_2 \wedge a_4$	$a_1 \wedge a_3 \wedge a_4$	$-a_2 \wedge a_3 \wedge a_4$

In the Clifford algebra $C(\mathbb{V}, \beta)$, we have $x = -x^{-1}$ and therefore $\rho(x) = -\rho(x)^{-1}$. Thus, we obtain from the previous table also the following values:

$s \in S$	a_1	a_2	a_3	a_4	$a_1 \wedge a_2 \wedge a_3$	$a_1 \wedge a_2 \wedge a_4$	$a_1 \wedge a_3 \wedge a_4$	$a_2 \wedge a_3 \wedge a_4$
$\rho(x)s$	-1	$a_1 \wedge a_2$	$a_1 \wedge a_3$	$a_1 \wedge a_4$	$-a_2 \wedge a_3$	$-a_2 \wedge a_4$	$-a_3 \wedge a_4$	$a_1 \wedge a_2 \wedge a_3 \wedge a_4$

These two tables show that $\rho(x)$ transforms the unitary basis $(w_N)_{N \subset \{1, \dots, 4\}}$ of S (see (B.21)) into another unitary basis of S. Therefore, we have $\rho(x) \in \mathrm{U}(S)$.

Next, we show $\vartheta(G) = G$. Let $g \in G$ be given. Then Equation (8.69) shows that we have

$$\chi(\vartheta(g)) = T_{-\mathbb{V}} \circ \rho_-(g) \circ T_{-\mathbb{V}}^{-1} .$$

As we saw above, we have $\rho_-(g) \in \mathrm{Aut}_s(\mathfrak{A}_-)_0$; because $T_{-\mathbb{V}} : S_- \to \mathbb{V}$ is a $\mathbb{C}$Q-isomorphism by Proposition 8.22(b), it follows that $\chi(\vartheta(g)) \in \mathrm{Aut}_s(\mathfrak{A})_0$ holds. This fact implies $\vartheta(g) \in G$ by Proposition 8.20(c). Thus we have shown $\vartheta(G) \subset G$. We then also have $G = \vartheta^3(G) \subset \vartheta^2(G) \subset \vartheta(G)$, see Equation (8.63).

It remains to show that $\rho_\pm|G : G \to \mathrm{Aut}_s(\mathfrak{A}_\pm)_0$ is a two-fold covering map of Lie groups. For this, we note that $\vartheta|G : G \to G$ and $\widetilde{T}_{\mathbb{V}+} : \mathrm{Aut}_s(\mathfrak{A})_0 \to \mathrm{Aut}_s(\mathfrak{A}_+)_0,\ B \mapsto T_{\mathbb{V}+} \circ B \circ T_{\mathbb{V}+}^{-1}$ are isomorphisms of Lie groups, that the diagram

$$\begin{tikzcd} G \arrow[r, "\vartheta|G"] \arrow[d, "\chi|G"'] & G \arrow[d, "\rho_+|G"] \\ \mathrm{Aut}_s(\mathfrak{A})_0 \arrow[r, "\widetilde{T}_{\mathbb{V}+}"'] & \mathrm{Aut}_s(\mathfrak{A}_+)_0 \end{tikzcd} \tag{8.71}$$

commutes by Equation (8.67), and that $\chi|G : G \to \mathrm{Aut}_s(\mathfrak{A})_0$ is a two-fold covering of Lie groups by Proposition 8.20(c).

A repetition of this argument with the isomorphism of Lie groups $\widetilde{T}_{+-} : \mathrm{Aut}_s(\mathfrak{A}_+)_0 \to \mathrm{Aut}_s(\mathfrak{A}_-)_0,\ B \mapsto T_{+-} \circ B \circ T_{+-}^{-1}$ then shows that $\rho_-|G$ is a two-fold covering of Lie groups, too. □

8.24 Remark. It also follows from the commutativity of Diagram (8.71) that $\ker(\rho_+|G) = \{1, \vartheta(-1)\}$ and likewise $\ker(\rho_-|G) = \{1, \vartheta^2(-1)\}$ holds. The elements $\vartheta(-1), \vartheta^2(-1) \in G$ are described explicitly in Proposition B.36(e).

8.5.2 Pure spinors

To establish the isomorphy between the complex quadrics $Q(\mathfrak{A}_\pm)$ and the connected components of $\mathfrak{F}(\mathbb{P}^3, Q)$ which we advertised above, we need a correspondence between the set of 4-dimensional isotropic subspaces of $\mathbb{V}$ and a certain subset of S, the set of *pure spinors*. We describe the results concerning pure spinors following the exposition of Lawson/Michelsohn, see [LM89], §IV.9, p. 335ff. Theorem 8.31, which states that the set of pure spinors is the union of two quadratic cones in our situation, can be found in Chevalley, [Che54], IV.1.1, p. 113.

At first, we consider the general situation of Section B.5, where V is a complex linear space of even dimension $n = 2r$ equipped with some symmetric, non-degenerate bilinear form β. Then we have the Clifford algebra $C := C(V, \beta)$ and the vector representation $\chi : \Gamma(V, \beta) \to \mathrm{O}(V, \beta)$ of the Clifford group. As in Section B.5, we fix a decomposition $V = W \oplus W'$ of V into isotropic, r-dimensional subspaces and a "unit volume" $\omega \in \bigwedge^r W \setminus \{0\}$. Then we consider the space $S = \bigwedge W$ of spinors, the spin representation $\rho : C(V, \beta) \to \mathrm{End}(S)$, the spaces

$S_+ = \bigwedge^{\text{even}} W$ and $S_- = \bigwedge^{\text{odd}} W$ of even resp. odd half-spinors, and their representations $\rho_\pm : \mathrm{Spin}(V,\beta) \to \mathrm{GL}(S_\pm)$.

Moreover, we denote by $\mathfrak{I} \subset G_r(V)$ the set of complex r-dimensional β-isotropic subspaces of V. $\Gamma(V,\beta)$ acts on $\mathfrak{I}$ via $\check{\chi} : \Gamma(V,\beta) \times \mathfrak{I} \to \mathfrak{I},\ (g,U) \mapsto \chi(g)U$. We regard $\mathfrak{I}$ as a $\Gamma(V,\beta)$-space in this way.

8.25 Proposition. *$\check{\chi}$ is transitive.*

Proof. By Lemma B.21 there exists a complex inner product $\langle\cdot,\cdot\rangle$ on V and a conjugation A on $(V,\langle\cdot,\cdot\rangle)$ so that the elements of $\mathfrak{I}$ are precisely the complex-r-dimensional, $\mathfrak{A}$-isotropic subspaces of V, and

$$\beta|(V(A) \times V(A)) = \langle\cdot,\cdot\rangle|(V(A) \times V(A)) \tag{8.72}$$

holds. Therefore, Proposition 2.20(e),(f) shows that for any given $U_1, U_2 \in \mathfrak{I}$, there exist orthogonal complex structures $\tau_1, \tau_2 : V(A) \to V(A)$ such that $U_k = \{\, x + J\tau_k x \,|\, x \in V(A) \,\}$ for $k \in \{1,2\}$. Then there exists $L \in \mathrm{O}(V(A))$ with $\tau_2 = L \circ \tau_1 \circ L^{-1}$ and therefore $U_2 = L^{\mathbb{C}}(U_1)$; we have $L^{\mathbb{C}} \in \mathrm{O}(V,\beta)$ because of Equation (8.72). Proposition B.12(e) shows that there exists $g \in \Gamma(V,\beta)$ with $\chi(g) = L^{\mathbb{C}}$ and therefore $U_2 = \check{\chi}(g, U_1)$. □

We now associate with every $s \in S$ a linear map

$$j_s : V \to S,\ v \mapsto \rho(v)s\ .$$

For a "generic" choice of $s \in S$, j_s is injective. However, there exist particular $s \in S$ for which j_s has a non-trivial kernel, and these spinors will play an important role in the following.

8.26 Proposition. *For $s \in S \setminus \{0\}$, $\ker j_s$ is an isotropic subspace of V.*

Proof. Let $v, w \in \ker j_s$ be given. Then we have $\rho(v)s = \rho(w)s = 0$ and therefore

$$0 = \rho(v)\rho(w)s + \rho(w)\rho(v)s = \rho(v\cdot w + w \cdot v)s = \rho(\beta(v,w)\cdot 1_C)s = \beta(v,w)\cdot s\ ,$$

hence $\beta(v,w) = 0$. □

8.27 Definition. *$s \in S \setminus \{0\}$ is called* pure *if $\ker j_s \in \mathfrak{I}$ holds. We denote the set of pure spinors by $\mathfrak{P}(S)$. For $U \in \mathfrak{I}$, we call any $s \in \mathfrak{P}(S)$ with $\ker j_s = U$ a* representative spinor *of U.*

8.28 Example. We have $\ker j_{1_S} = W'$ and $\ker j_\omega = W$, consequently $1_S, \omega \in \mathfrak{P}(S)$ holds.

8.29 Proposition. *The ρ-action of $\Gamma(V,\beta)$ on S leaves $\mathfrak{P}(S)$ invariant and the action*

$$\check{\rho} : \Gamma(V,\beta) \times \mathfrak{P}(S) \to \mathfrak{P}(S),\ (g,s) \mapsto \rho(g)s$$

is transitive; we regard $\mathfrak{P}(S)$ as a homogeneous $\Gamma(V,\beta)$-space in this way. The map $Z : \mathfrak{P}(S) \to \mathfrak{I},\ s \mapsto \ker j_s$ is equivariant with respect to the actions of $\Gamma(V,\beta)$ on $\mathfrak{P}(S)$ and on $\mathfrak{I}$, and we have for $s, s' \in \mathfrak{P}(S)$

$$\mathbb{C}^\times \cdot s \subset \mathfrak{P}(S) \quad \textit{and} \quad \big(\, Z(s) = Z(s') \iff \exists\, \lambda \in \mathbb{C}^\times : s = \lambda\, s' \,\big)\ . \tag{8.73}$$

Proof. We first show
$$\forall g \in \Gamma(V,\beta),\ s \in \mathfrak{P}(S)\ :\ \ker j_{\rho(g)s} = \chi(g)(\ker j_s)\ . \tag{8.74}$$

Let $g \in \Gamma(V,\beta)$ and $s \in \mathfrak{P}(S)$ be given, then we have for any $v \in \ker j_s$
$$j_{\rho(g)s}(\chi(g)v) = \rho(\chi(g)v)\,\rho(g)\,s = \rho(\alpha(g)vg^{-1}g)\,s = \rho(\alpha(g))\,\rho(v)\,s = \rho(\alpha(g))\,j_s(v) = 0\ ,$$
and therefore
$$\chi(g)(\ker j_s) \subset \ker j_{\rho(g)s} \tag{8.75}$$
holds. We have
$$r = \dim(\ker j_s) = \dim(\,\chi(g)(\ker j_s)\,) \overset{(8.75)}{\leq} \dim(\ker j_{\rho(g)s}) \overset{(*)}{\leq} r\ ,$$
where the inequality marked $(*)$ follows from the fact that $\ker j_{\rho(g)s}$ is isotropic, see Proposition 8.26. This chain of inequalities implies $\dim(\,\chi(g)(\ker j_s)\,) = \dim(\ker j_{\rho(g)s})$, and therefore in fact equality holds in (8.75). This completes the proof of Equation (8.74).

Equation (8.74) shows that $\mathfrak{P}(S)$ is invariant under the ρ-action of $\Gamma(V,\beta)$ on S and that the map Z is equivariant with respect to $\check{\rho}$ and $\check{\chi}$.

We next show (8.73). Let $s \in \mathfrak{P}(S)$ and $\lambda \in \mathbb{C}^\times$ be given. Then we have $j_{\lambda s} = \lambda \cdot j_s$ and therefore $\lambda\, s \in \mathfrak{P}(S)$ and $Z(\lambda\, s) = \ker j_{\lambda s} = \ker j_s = Z(s)$. Conversely, we suppose that $s, s' \in \mathfrak{P}(S)$ are given with $Z(s) = Z(s')$. Because $\mathfrak{I}$ is a homogeneous $\Gamma(V,\beta)$-space (see Proposition 8.25) and Z is equivariant (by Equation (8.74)), we may suppose without loss of generality that $Z(s) = Z(s') = W'$ and $s' = 1_S$ holds (see Example 8.28). Because of $\ker j_s = W'$ we have
$$\forall w' \in W'\ :\ \rho(w')s = 0\ . \tag{8.76}$$
Let us fix a basis $(w_1,\ldots,w_r)$ of W and consider the basis $(w_1',\ldots,w_r')$ of W' so that $\beta(w_k,w_\ell') = \delta_{k\ell}$ holds (see Proposition B.23). We use the notation w_N from (B.21) with respect to the basis $(w_1,\ldots,w_r)$ and denote by I the power set of $\{1,\ldots,r\}$. Then $(w_N)_{N\in I}$ is a basis of S, therefore there exist numbers $c_N \in \mathbb{C}$ with $s = \sum_{N\in I} c_N \cdot w_N$. We will now show by induction on k that
$$\forall k \in \{0,\ldots,r-1\},\ N \in I\ :\big(\,\#N = r-k \implies c_N = 0\,\big) \tag{8.77}$$
holds; it follows that $s = c_\varnothing \cdot w_\varnothing = c_\varnothing \cdot 1_S$ holds. Because we have $s \neq 0$, we then conclude $s \in \mathbb{C}^\times \cdot 1$.

For the proof of (8.77): First, suppose $k = 0$. The only $N \in I$ with $\#N = r-0$ is $N = \{1,\ldots,r\}$ and then we have $c_N = \rho(w_n')\cdots\rho(w_1')s = 0$ by Equation (8.76).

Now, let $k \leq r-1$ be given and suppose that (8.77) holds for all $k' < k$. Then let $N \in I$ be given with $\#N = r-k$, say $N = \{\ell_1,\ldots,\ell_{r-k}\}$ with $\ell_1 < \ldots < \ell_{r-k}$. With $\xi := w_{\ell_{r-k}}'\cdots w_{\ell_1}' \in C$ the following equations hold:

$$c_{N'} = 0 \quad \text{for every } N' \in I \text{ with } \#N' > r-k\ , \tag{8.78}$$
$$\rho(\xi)w_{N'} = 0 \quad \text{for every } N' \in I \text{ with } \#N' \leq r-k \text{ and } N' \neq N\ , \tag{8.79}$$
$$\rho(\xi)w_N = 1_S\ . \tag{8.80}$$

In fact, Equation (8.78) is simply the induction hypothesis and Equation (8.80) follows from the definition of ρ. For the proof of Equation (8.79) we handle the cases $\#N' < r-k$ and $\#N' = r-k$ separately. First, let $N' \in I$ be given with $\#N' < r-k$. Then $\rho(\xi)w_{N'} = 0$ follows from the definition of ξ and the fact that for any $w' \in W'$, $\rho(w')$ is an anti-derivation of degree (-1). Now let $N' \in I$ be given with $\#N' = r-k$ and $N' \neq N$. Then there exists $\ell \in N \setminus N'$. We have $\rho(w_\ell')w_{N'} = 0$ by the definition of ρ and therefore also $\rho(\xi)w_N = 0$.

From Equations (8.78), (8.79) and (8.80) we see that $\rho(\xi)s = c_N\,1_S$ holds. But on the other hand Equation (8.76) shows that we have $\rho(\xi)s = 0$. From these two equalities, $c_N = 0$ follows, and this completes the proof of (8.77).

Finally, we show that $\check{\rho}$ is transitive. Let $s, s' \in \mathfrak{P}(S)$ be given. Then we have $Z(s), Z(s') \in \mathfrak{I}$. By Proposition 8.25 there exists $g_0 \in \Gamma(V,\beta)$ with $Z(s') = \chi(g_0)Z(s) = Z(\rho(g_0)s)$, and therefore by (8.73) there exists $\lambda \in \mathbb{C}^\times$ so that $s' = \lambda\,\rho(g_0)s$ holds. Thus we have $s' = \check{\rho}(g,s)$ with $g := \lambda\, g_0 \in \Gamma(V,\beta)$. □

8.30 Corollary. $\mathfrak{P}(S) \subset S_+ \cup S_-$.

Proof. Let $s \in \mathfrak{P}(S)$ be given. Because $\Gamma(V,\beta)$ acts transitively on $\mathfrak{P}(S)$ (Proposition 8.29) and we have $1_S \in \mathfrak{P}_S$ (Example 8.28), there exists $g \in \Gamma(V,\beta)$ so that $s = \rho(g)1_S$ holds. By Proposition B.12(f), we have either $g \in C^+(V,\beta)$ and then $s = \rho(g)1_S \in S_+$, or else $g \in C^-(V,\beta)$ and then $s = \rho(g)1_S \in S_-$. □

We put $\mathfrak{P}(S_+) := \mathfrak{P}(S) \cap S_+$ and $\mathfrak{P}(S_-) := \mathfrak{P}(S) \cap S_-$; on these spaces $\Gamma^+(V,\beta)$ acts transitively via ρ, and we have $\mathfrak{P}(S) = \mathfrak{P}(S_+)\dot{\cup}\mathfrak{P}(S_-)$ by Corollary 8.30.

8.31 Theorem. (Chevalley, [Che54], IV.1.1, p. 113.) *In the situation of Subsection 8.5.1, where $\mathbb{V}$ is an 8-dimensional $\mathbb{C}$Q-space and the corresponding half-spinor spaces $S_\pm$ are regarded as $\mathbb{C}$Q-spaces $(S_\pm, \mathfrak{A}_\pm)$ in the way described there, we have*

$$\mathfrak{P}(S_\pm) = \widehat{Q}(\mathfrak{A}_\pm) .$$

Proof. It suffices to show that

$$\mathfrak{P}(S) = \{\, s \in (S_+ \cup S_-) \setminus \{0\} \,|\, q_S(s) = 0 \,\} =: \widetilde{\mathfrak{P}}$$

holds with the quadratic form $q_S : S \to \mathbb{C},\ s \mapsto \frac{1}{2}\,\beta_S(s,s)$.

We first note that both $\mathfrak{P}(S)$ and $\widetilde{\mathfrak{P}}$ are invariant under $\rho(\Gamma(\mathbb{V},\beta))$. For $\mathfrak{P}(S)$ this was shown in Proposition 8.29. For $s \in \widetilde{\mathfrak{P}}$ and $g \in \Gamma(\mathbb{V},\beta)$, we have either $\rho(g)s \in S_+ \setminus \{0\}$ or $\rho(g)s \in S_- \setminus \{0\}$, and we have by Proposition B.30(b)(ii): $q_S(\rho(g)s) = \varepsilon(g)\lambda(g)\cdot q_S(s) = 0$. Thus we have shown $\rho(g)s \in \widetilde{\mathfrak{P}}$.

Because $\mathfrak{P}(S)$ is the orbit through 1_S of the action of $\rho(\Gamma(\mathbb{V},\beta))$ on S (see Proposition 8.29 and Example 8.28) and we have $1_S \in \widetilde{\mathfrak{P}}$, the $\rho(\Gamma(\mathbb{V},\beta))$-invariance of $\widetilde{\mathfrak{P}}$ already implies $\mathfrak{P}(S) \subset \widetilde{\mathfrak{P}}$.

For the converse inclusion, let $s \in \widetilde{\mathfrak{P}}$ be given. We will show that there exists $g \in \Gamma(\mathbb{V},\beta)$ so that

$$\exists\, a,b \in \mathbb{C}\ :\ s' := \rho(g)s = a\cdot 1_S + b\cdot\omega \tag{8.81}$$

holds. Then $s \in \widetilde{\mathfrak{P}}$ implies $s' \in \widetilde{\mathfrak{P}}$, whence we see

$$0 = q_S(s') = \tfrac{1}{2}\cdot\beta_S(a\cdot 1_S + b\cdot\omega,\ a\cdot 1_S + b\cdot\omega) = a\,b\ .$$

Therefore, either $a = 0$ and then $s' \in \mathbb{C}^\times\cdot\omega$, or else $b = 0$ and then $s' \in \mathbb{C}^\times\cdot 1_S$ holds. In either case, we have $s' \in \mathfrak{P}(S)$ (see Example 8.28) and therefore also $s = \rho(g^{-1})s' \in \mathfrak{P}(S)$.

It remains to prove the existence of $g \in \Gamma(\mathbb{V},\beta)$ so that (8.81) holds. We have $s \in S_+ \cup S_-$. Let us first consider the case $s \in S_+$. We choose a basis $(w_1,\dots,w_4)$ of W and denote by $(w_1',\dots,w_4')$ the basis of W' so that $\beta(w_j,w_\ell') = \delta_{j\ell}$ holds, see Proposition B.23. Then $\{1_S,\omega\} \cup \{\, w_k \wedge w_\ell \,|\, k < \ell \,\}$ is a basis of S_+. Therefore there exist $a,d \in \mathbb{C}$ and $c_{k\ell} \in \mathbb{C}$ for $k < \ell$ so that $s = a\,1_S + \sum_{k<\ell} c_{k\ell}\,w_k \wedge w_\ell + d\,\omega$ holds, and because of $s \neq 0$ at least one of these coefficients is non-zero. In fact, we may suppose without loss of generality that $a = 1$ holds, as the following argument shows: If $a \neq 0$ holds, then the homogeneous component of degree 0 of $\rho(g')s$ is 1_S, where $g' := \frac{1}{a}\cdot 1_C \in \Gamma(\mathbb{V},\beta)$. If $c_{k\ell} \neq 0$ holds for some $k < \ell$, then the homogeneous component of degree 0 of $\rho(g')s$ is 1_S, where $g' := \frac{1}{c_{k\ell}}(w_\ell + w_\ell')\cdot(w_k + w_k') \in \Gamma(\mathbb{V},\beta)$ (see Proposition B.25(a)). If $d \neq 0$, then the homogeneous component of degree 0 of $\rho(g')s$ is 1_S, where $g' := \frac{1}{d}(w_4 + w_4')\cdots(w_1 + w_1') \in \Gamma(\mathbb{V},\beta)$ (again, see Proposition B.25(a)). Because of the $\rho(\Gamma(\mathbb{V},\beta))$-invariance of $\widetilde{\mathfrak{P}}$, we may replace s by $\rho(g')s$ in any of these cases.

For every $k < \ell$ we now put $g_{k\ell} := 1 - c_{k\ell}\,w_k\cdot w_\ell \in \Gamma(\mathbb{V},\beta)$ (see Proposition B.25(b)) and $g := \prod_{k<\ell} g_{k\ell} \in \Gamma(\mathbb{V},\beta)$ (where the factors are ordered lexicographically). Let $k < \ell$ be given and denote by $\tilde{k} < \tilde{\ell}$ the two elements of

$\{1,\ldots,4\}\setminus\{k,\ell\}$. We then have

$$\begin{aligned}\rho(g_{k\ell})s &= \rho(1-c_{k\ell}\,w_k\cdot w_\ell)s = s - c_{k\ell}\,w_k\wedge w_\ell\wedge(1+\sum_{k'<\ell'} c_{k'\ell'}w_{k'}\wedge w_{\ell'} + d\cdot\omega)\\ &= s - c_{k\ell}\,w_k\wedge w_\ell - c_{k\ell}\,c_{\tilde k\tilde\ell}\,\underbrace{w_k\wedge w_\ell\wedge w_{\tilde k}\wedge w_{\tilde\ell}}_{=\pm\omega}\\ &= s - c_{k\ell}\,w_k\wedge w_\ell + d'\cdot\omega\end{aligned}$$

with a suitable $d'\in\mathbb{C}$. It follows that

$$\rho(g)s = s - \sum_{k<\ell} c_{k\ell}\,w_k\wedge w_\ell + d''\cdot\omega$$

holds with some $d''\in\mathbb{C}$. Therefore the homogeneous component of degree 2 of $\rho(g)s$ is zero, and hence $\rho(g)s$ is of the form of (8.81). This completes the proof of (8.81) for the case $s\in S_+$.

If on the other hand $s\in S_-$ holds, we fix $v\in\mathbb{V}$ with $q(v)=1$, then we have $v\in\Gamma(\mathbb{V},\beta)$ and hence $\rho(v)s\in\widetilde{\mathfrak{P}}\cap S_+$. By the preceding arguments there exists $g'\in\Gamma(\mathbb{V},\beta)$ so that $\rho(g')(\rho(v)s)=\rho(g'\cdot v)s$ is of the form of (8.81). Hence (8.81) is satisfied with $g:=g'\cdot v\in\Gamma(\mathbb{V},\beta)$. □

8.5.3 The construction of the isomorphism

In the situation of Subsection 8.5.1, we now consider the 6-dimensional complex quadrics $Q := Q(\mathfrak{A})\subset\mathbb{P}(\mathbb{V})$ and $Q_\pm := Q(\mathfrak{A}_\pm)\subset\mathbb{P}(S_\pm)$. We will construct isomorphisms of Hermitian symmetric spaces from Q_+ and Q_- to one and the other of the two connected components of the congruence family $\mathfrak{F}(\mathbb{P}^3,Q)$ which was studied in Theorem 7.11.

At first, we consider the spaces involved only as homogeneous spaces, not as symmetric spaces. Originally, the quadric Q is a Hermitian homogeneous $\mathrm{Aut}_s(\mathfrak{A})_0$-space (see Corollary 3.4), likewise $Q_\pm$ is a Hermitian homogeneous $\mathrm{Aut}_s(\mathfrak{A}_\pm)_0$-space, and the connected components of $\mathfrak{F}(\mathbb{P}^3,Q)$ are Hermitian homogeneous $\mathrm{Aut}_s(\mathfrak{A})_0$-spaces (see Theorem 7.11(c)). But now, we regard all these spaces as Hermitian homogeneous G-spaces via the actions

$$\begin{aligned}\widetilde{\chi}: G\times Q\to Q,\ (g,p)\mapsto \underline{\chi(g)}(p)\ ,\\ \widetilde{\rho}_\pm: G\times Q_\pm\to Q_\pm,\ (g,[s])\mapsto \underline{\rho_\pm(g)}([s])\\ \text{and}\quad \widetilde{\chi}_{\mathfrak{F}}: G\times\mathfrak{F}(\mathbb{P}^3,Q)\to\mathfrak{F}(\mathbb{P}^3,Q),\ (g,\Lambda)\mapsto\underline{\chi(g)}(\Lambda)\ .\end{aligned}$$

(Here, we again used the notation $\underline{B}$ for the holomorphic isometry on $\mathbb{P}(V)$ induced by some unitary transformation B of a unitary space V.) The actions $\widetilde{\chi}$ and $\widetilde{\rho}_\pm$ are indeed transitive, and the orbits of $\widetilde{\chi}_{\mathfrak{F}}$ are indeed the connected components of $\mathfrak{F}(\mathbb{P}^3,Q)$ because $\chi|G : G\to\mathrm{Aut}_s(\mathfrak{A})_0$ and $\rho_\pm|G : G\to\mathrm{Aut}_s(\mathfrak{A}_\pm)_0$ are surjective, see Propositions 8.20(c) and 8.23(b).

In the following constructions, we will also use the quadratic cones $\widehat{Q}_\pm := \widehat{Q}(\mathfrak{A}_\pm) = \mathfrak{P}(S_\pm)$ (see Theorem 8.31) and the manifolds $\widetilde{Q}_\pm := \widetilde{Q}(\mathfrak{A}_\pm) = \widehat{Q}_\pm\cap\mathbb{S}(S_\pm)$. Moreover, we consider the Hopf fibrations $\pi:\mathbb{S}(\mathbb{V})\to\mathbb{P}(\mathbb{V}),\ v\mapsto[v]$ and $\pi_\pm:\mathbb{S}(S_\pm)\to\mathbb{P}(S_\pm),\ s\mapsto[s]$.

8.32 Proposition. *There exists a map* $h_\pm : Q_\pm \to \mathfrak{F}(\mathbb{P}^3, Q)$ *characterized by*

$$\forall s \in \widetilde{Q}_\pm \;:\; h_\pm([s]) = [\{\, v \in \mathbb{V} \,|\, \rho(v)s = 0 \,\}] \,. \tag{8.82}$$

$h_\pm$ is a holomorphic embedding. The images $\mathfrak{F}(\mathbb{P}^3, Q)_+ := h_+(Q_+)$ *and* $\mathfrak{F}(\mathbb{P}^3, Q)_- := h_-(Q_-)$ *are the two connected components of* $\mathfrak{F}(\mathbb{P}^3, Q)$. *With the transitive* G*-action* $\widetilde{\chi}_{\mathfrak{F}\pm} := \widetilde{\chi}_{\mathfrak{F}}|(G \times \mathfrak{F}(\mathbb{P}^3, Q)_\pm)$ *on* $\mathfrak{F}(\mathbb{P}^3, Q)_\pm$, $h_\pm$ *is* $(\widetilde{\rho}_\pm, \widetilde{\chi}_{\mathfrak{F}\pm})$*-equivariant, meaning that* $(h_\pm, \mathrm{id}_G)$ *is an isomorphism of homogeneous spaces from* $Q_\pm$ *onto* $\mathfrak{F}(\mathbb{P}^3, Q)_\pm$.

Proof. We use the concept of pure spinors and the corresponding notations introduced in Subsection 8.5.2. By Theorem 8.31, we have $\widehat{Q}_\pm = \mathfrak{P}(S_\pm)$, and hence, $Z(s)$ is a 4-dimensional isotropic subspace of $\mathbb{V}$ for every $s \in \widetilde{Q}_\pm$ by Definition 8.27. Therefore, the maps

$$\widetilde{h}_\pm : \widetilde{Q}_\pm \to \mathfrak{F}(\mathbb{P}^3, Q),\; s \mapsto [Z(s)]$$

indeed have values in $\mathfrak{F}(\mathbb{P}^3, Q)$. Because $\widetilde{Q}_\pm$ is connected, $\widetilde{h}_\pm$ in fact maps into a connected component of $\mathfrak{F}(\mathbb{P}^3, Q)$, which we denote by $\mathfrak{F}(\mathbb{P}^3, Q)_\pm$. No 4-dimensional isotropic subspace of $\mathbb{V}$ has representative spinors in both S_+ and S_- (by (8.73) in Proposition 8.29), whence $\mathfrak{F}(\mathbb{P}^3, Q)_+ \neq \mathfrak{F}(\mathbb{P}^3, Q)_-$ follows. Because $\mathfrak{F}(\mathbb{P}^3, Q)$ has exactly two connected components, we see that these are $\mathfrak{F}(\mathbb{P}^3, Q)_+$ and $\mathfrak{F}(\mathbb{P}^3, Q)_-$.

From (8.73) we see that there exists a map $h_\pm : Q_\pm \to \mathfrak{F}(\mathbb{P}^3, Q)_\pm$ with $h_\pm \circ \pi_\pm = \widetilde{h}_\pm$ and that this map is injective. $h_\pm$ obviously satisfies Equation (8.82), and Proposition 8.29 shows that it is $(\widetilde{\rho}_\pm, \widetilde{\chi}_{\mathfrak{F}\pm})$-equivariant. Proposition A.1 thus shows that $h_\pm$ is a diffeomorphism.

It remains to show the holomorphy of $h_\pm$. We consider the holomorphic bundle map $\widehat{\pi}_\pm : \widehat{Q}_\pm \to Q_\pm,\; s \mapsto [s]$, the trivial complex vector bundles

$$\tau : \widehat{Q}_\pm \times \mathbb{V} \to \widehat{Q}_\pm,\; (s,v) \mapsto s \quad \text{and} \quad \tau' : \widehat{Q}_\pm \times S \to \widehat{Q}_\pm,\; (s,s') \mapsto s \,,$$

and the holomorphic vector bundle morphism

$$j : \widehat{Q}_\pm \times \mathbb{V} \to \widehat{Q}_\pm \times S,\; (s,v) \mapsto (s, j_s(v)) = (s, \rho(v)s)$$

between τ and τ' over $\widehat{Q}_\pm$. For every $s \in \widehat{Q}_\pm$, the kernel of $j(s,\cdot) : v \mapsto j(s,v)$ is the complex 4-dimensional linear subspace $Z(s)$, and therefore $\ker j$ is a complex subbundle of τ of complex fibre dimension 4.

Now let $p \in Q_\pm$ be given and choose an open neighbourhood $U \in \mathfrak{U}^o(p, Q_\pm)$ such that there exists a holomorphic local section $f : U \to \widehat{Q}_\pm$ of $\widehat{\pi}_\pm$. By reducing the size of U if necessary, we can further arrange that there exists an open neighborhood $\widehat{U}$ of $f(U)$ in $\widehat{Q}_\pm$ on which there exists a holomorphic frame field $(b_1, \dots, b_4)$ of the complex vector bundle $\ker j$. If we then denote by $\widehat{\mathrm{St}}_4(\mathbb{V})$ the Stiefel manifold of complex 4-frames in $\mathbb{V}$, by $G_4(\mathbb{V})$ the complex 4-Grassmannian over $\mathbb{V}$, and consider the holomorphic projection

$$\mathrm{pr} : \ker j \to \mathbb{V},\; (s,v) \mapsto v \,,$$

the holomorphic map

$$g : \widehat{\mathrm{St}}_4(\mathbb{V}) \to G_4(\mathbb{V}),\; (v_1, \dots, v_4) \mapsto \mathrm{span}_{\mathbb{C}}\{v_1, \dots, v_4\}$$

and the holomorphic map $\theta : G_4(\mathbb{V}) \to \mathfrak{F}(\mathbb{P}^3, \mathbb{P}(\mathbb{V}))$ from Theorem 7.4, then $h_\pm|U$ is equal to the following composition of holomorphic maps

$$U \xrightarrow{f} \widehat{U} \xrightarrow{(b_1,\dots,b_4)} (\ker j) \times_{\widehat{Q}_\pm} \dots \times_{\widehat{Q}_\pm} (\ker j) \xrightarrow{(\mathrm{pr})^4} \widehat{\mathrm{St}}_4(\mathbb{V}) \xrightarrow{g} G_4(\mathbb{V}) \xrightarrow{\theta} \mathfrak{F}(\mathbb{P}^3, \mathbb{P}(\mathbb{V}))$$

and therefore holomorphic. □

Now we wish to regard Q, $Q_\pm$ and $\mathfrak{F}(\mathbb{P}^3, Q)_\pm$ also as Hermitian symmetric G-spaces. Originally, Q is a Hermitian symmetric $\mathrm{Aut}_s(\mathfrak{A})_0$-space (Proposition 3.9(c)), likewise $Q_\pm$ is a Hermitian symmetric $\mathrm{Aut}_s(\mathfrak{A}_\pm)_0$-space, and $\mathfrak{F}(\mathbb{P}^3, Q)_\pm$ is a Hermitian symmetric $\mathrm{Aut}_s(\mathfrak{A})_0$-space (Theorem 7.11(c)(i)). To regard these spaces as symmetric G-spaces, we apply Proposition A.2 to the situations given by the following table:

Proposition A.2	M	φ	G	$\widetilde{G}$	$\widetilde{\varphi}$	τ
here	Q	$(B,p) \mapsto \underline{B}(p)$	$\mathrm{Aut}_s(\mathfrak{A})_0$	G	$\widetilde{\chi}$	$\chi\|G$
here	$Q_\pm$	$(B,p) \mapsto \underline{B}(p)$	$\mathrm{Aut}_s(\mathfrak{A}_\pm)_0$	G	$\widetilde{\rho}_\pm$	$\rho_\pm\|G$
here	$\mathfrak{F}(\mathbb{P}^3, Q)_\pm$	$(B,\Lambda) \mapsto \underline{B}(\Lambda)$	$\mathrm{Aut}_s(\mathfrak{A})_0$	G	$\widetilde{\chi}_{\mathfrak{F}\pm}$	$\chi\|G$

.

Regarding the hypotheses of Proposition A.2, we note that G is simply connected (Proposition 8.20(b)) and that $\chi|G : G \to \mathrm{Aut}_s(\mathfrak{A})_0$ and $\rho_\pm|G : G \to \mathrm{Aut}_s(\mathfrak{A}_\pm)_0$ are covering maps of Lie groups (Propositions 8.20(c) and 8.23(b)). It remains to show:

8.33 Proposition. *The isotropy groups of the G-actions on Q, $Q_\pm$ and $\mathfrak{F}(\mathbb{P}^3, Q)_\pm$ are connected.*

Proof. For $\widetilde{\chi}$. Let $p \in Q$ be given, and let $K_p \subset \mathrm{Aut}_s(\mathfrak{A})_0$ be the isotropy group of the action of $\mathrm{Aut}_s(\mathfrak{A})_0$ on Q at p. Then $\widetilde{K}_p := \chi^{-1}(K_p) \subset G$ is the isotropy group of the action $\widetilde{\chi}$ at p. Because K_p is connected (Proposition 3.9(a)) and $\chi|G : G \to \mathrm{Aut}_s(\mathfrak{A})_0$ is a two-fold covering map (Proposition 8.20(c)), it suffices for the proof of the connectedness of $\widetilde{K}_p$ to show that the two pre-images 1 and -1 of $\mathrm{id}_{\mathbb{V}} \in K_p$ under χ can be connected in $\widetilde{K}_p$.

For this purpose, remember that $\mathbb{V}' = V(-A)$ and the subgroup G were constructed with respect to a fixed $A \in \mathfrak{A}$. We now consider the $\mathbb{C}$Q-subspace $U := \mathrm{span}_{\mathfrak{A}}\{z\} = \mathrm{span}_{\mathbb{C}}\{z, Az\}$ (with $z \in \pi^{-1}(\{p\})$) of $\mathbb{V}$; we denote its induced $\mathbb{C}$Q-structure by $\mathfrak{A}_U$. Then we have by Proposition 3.9(a)

$$K_p = \{\, B \in \mathrm{Aut}_s(\mathfrak{A})_0 \,\big|\, B|U \in \mathrm{Aut}_s(\mathfrak{A}_U)_0 \,\} \,. \tag{8.83}$$

$U^{\perp,\mathbb{V},\beta}$ is a 6-dimensional $\mathbb{C}$Q-subspace of $\mathbb{V}$, and $\mathbb{V}'$ is a maximal, totally real subspace of $\mathbb{V}$. Consequently, $\mathbb{V}' \cap U^{\perp,\mathbb{V},\beta}$ is a real-6-dimensional, totally real subspace of $\mathbb{V}$; moreover the restriction of β to this space is a negative definite, symmetric, real bilinear form. Therefore, there exist $x_1, x_2 \in \mathbb{V}' \cap U^{\perp,\mathbb{V},\beta}$ with $q(x_1) = q(x_2) = -1$ and $\beta(x_1, x_2) = 0$. Then we have

$$x_1 \cdot x_1 = x_2 \cdot x_2 = -1 \quad \text{and} \quad x_1 \cdot x_2 = -x_2 \cdot x_1 \,. \tag{8.84}$$

We consider the curve

$$c : \mathbb{R} \to C(\mathbb{V}, \beta),\ t \mapsto \cos(t) \cdot 1 - \sin(t)\, x_1 \cdot x_2 \,.$$

We have $c(0) = 1$ and $c(\pi) = -1$, and by a calculation involving Equations (8.84) one sees that for any $t \in \mathbb{R}$,

$$c(t) = v_-(t) \cdot v_+(t) \quad \text{holds with} \quad v_\pm(t) := \sin(\tfrac{t}{2})\, x_2 \pm \cos(\tfrac{t}{2})\, x_1 \ ;$$

because we have $q'(v_\pm(t)) = -1$, the definition of G shows that the curve c runs entirely in G.[28] Moreover, any $u \in U$ is β-orthogonal to $v_+(t)$ and $v_-(t)$; because $\chi(v_\pm(t))$ is the β-orthogonal reflection in the hyperplane $(\mathbb{C}\, v_\pm(t))^{\perp,\mathbb{V},\beta}$ by Proposition B.12(c), we see that $\chi(v_\pm(t))u = u$ and therefore also $\chi(c(t))\, u = \chi(v_-(t))\, \chi(v_+(t))\, u = u$ holds. Thus we have shown $\chi(c(t))|U = \mathrm{id}_U \in \mathrm{Aut}_s(\mathfrak{A}_U)_0$. It follows by Equation (8.83) that c runs entirely in $\widetilde{K}_p$.

For $\widetilde{\rho}_+$. Let $[s] \in Q_+$ be given and denote by $\widetilde{K}_{[s]}$ the isotropy group of $\widetilde{\rho}_+$ at $[s]$. Because $T_{\mathbb{V}+} : \mathbb{V} \to S_+$ is a $\mathbb{C}$Q-isomorphism (Proposition 8.22(b)), $\underline{T_{\mathbb{V}+}}|Q : Q \to Q_+$ is a holomorphic isometry. If we put $p := (\underline{T_{\mathbb{V}+}})^{-1}([s]) \in Q$, we have by Equation (8.67)

$$\forall g \in G \ : \ \widetilde{\rho}_+(\vartheta(g), [s]) = \underline{T_{\mathbb{V}+}}(\widetilde{\chi}(g,p)) \ .$$

It follows that $\widetilde{K}_{[s]} = \vartheta(\widetilde{K}_p)$ holds, and is therefore connected by the previous result.

For $\widetilde{\rho}_-$. Let $[s] \in Q_-$ be given. A similar argument as that for $\widetilde{\rho}_+$ shows that the isotropy group $\widetilde{K}_{[s]}$ of $\widetilde{\rho}_-$ at $[s]$ satisfies $\widetilde{K}_{[s]} = \vartheta^2(\widetilde{K}_p)$ with $p := (\underline{T_{+-} \circ T_{\mathbb{V}+}})^{-1}([s]) = \underline{T}^{-2}([s]) \in Q$ and is therefore also connected.

For $\widetilde{\chi}_{\mathfrak{F}\pm}$. Let $\Lambda \in \mathfrak{F}(\mathbb{P}^3, Q)_\pm$ be given and denote by $\widetilde{K}_\Lambda$ the isotropy group of $\widetilde{\chi}_{\mathfrak{F}\pm}$ at Λ. We have $[s] := h_\pm^{-1}(\Lambda) \in Q_\pm$, and because of the $(\widetilde{\rho}_\pm, \widetilde{\chi}_{\mathfrak{F}\pm})$-equivariance of the diffeomorphism $h_\pm$ (Proposition 8.32), we have $\widetilde{K}_\Lambda = \widetilde{K}_{[s]}$. Therefore $\widetilde{K}_\Lambda$ is connected by the preceding results. □

We now regard Q, $Q_\pm$ and $\mathfrak{F}(\mathbb{P}^3, Q)_\pm$ as Hermitian symmetric G-spaces in the way described above.

8.34 Theorem. $(h_\pm, \mathrm{id}_G)$ *is an isomorphism of Hermitian symmetric spaces from the 6-dimensional complex quadric* $Q_\pm$ *onto* $\mathfrak{F}(\mathbb{P}^3, Q)_\pm$. *Because the latter space is isomorphic to* $\mathrm{SO}(8)/\mathrm{U}(4)$, *we therefore have the following isomorphy of Hermitian symmetric spaces:*

$$\boxed{Q^6 \cong \mathrm{SO}(8)/\mathrm{U}(4)}\ .$$

Proof. $(h_\pm, \mathrm{id}_G)$ is an isomorphism of homogeneous G-spaces from $Q_\pm$ to $\mathfrak{F}(\mathbb{P}^3, Q)_\pm$ by Proposition 8.32. Because the Lie group G is of compact type, Proposition A.5 shows that $(h_\pm, \mathrm{id}_G)$ is an isomorphism of affine symmetric spaces. Because the symmetric spaces involved are irreducible Hermitian symmetric spaces, $(h_\pm, \mathrm{id}_G)$ is in fact an isomorphism of Hermitian symmetric spaces.

The isomorphy $\mathfrak{F}(\mathbb{P}^3, Q)_\pm \cong \mathrm{SO}(8)/\mathrm{U}(4)$ has been shown in Theorem 7.11(c)(i). □

[28] In fact, c is a 1-parameter subgroup of G. — We also mention that the curve $-c$ is used in [LM89] to prove the connectedness of the spin group, see [LM89], the proof of Theorem 2.10, p. 20.

8.35 Remark. The preceding construction of the isomorphism $Q^6 \cong \mathfrak{F}(\mathbb{P}^3, Q^6)_\pm$ is not totally pleasing because different models of the 6-dimensional complex quadric are involved on the left-hand side and on the right-hand side of the isomorphy (namely $Q_\pm$ and Q, respectively). However, it is easy to remedy this shortcoming by use of the triality automorphisms.

Indeed, consider the holomorphic isometries

$$\underline{T_{\mathbb{V}+}}|Q : Q \to Q_+ \;, \quad \underline{T_{+-}}|Q_+ : Q_+ \to Q_- \quad \text{and} \quad \underline{T_{-\mathbb{V}}}|Q_- : Q_- \to Q \;.$$

Then Equations (8.67)–(8.69) show that $(\underline{T_{\mathbb{V}+}}|Q, \vartheta)$, $(\underline{T_{+-}}|Q_+, \vartheta)$ and $(\underline{T_{-\mathbb{V}}}, \vartheta)$ are isomorphisms of Hermitian homogeneous G-spaces between the respective quadrics; they are in fact isomorphisms of Hermitian symmetric spaces because of Proposition A.5.

We now put

$$f_6^+ := h_+ \circ (\underline{T_{\mathbb{V}+}}|Q) : Q \to \mathfrak{F}(\mathbb{P}^3, Q)_+ \quad \text{and} \quad f_6^- := h_- \circ (\underline{T_{-\mathbb{V}}}|Q_-)^{-1} : Q \to \mathfrak{F}(\mathbb{P}^3, Q)_- \;,$$

also $F_6^+ := \vartheta$ and $F_6^- := \vartheta^{-1} = \vartheta^2$. Then Theorem 8.34 shows that $(f_6^\pm, F_6^\pm)$ is an isomorphism of Hermitian symmetric G-spaces from Q to $\mathfrak{F}(\mathbb{P}^3, Q)$, as we desired.

Moreover, with

$$g := h_- \circ (\underline{T_{+-}}|Q) \circ h_+^{-1} : \mathfrak{F}(\mathbb{P}^3, Q)_+ \to \mathfrak{F}(\mathbb{P}^3, Q)_- \;,$$

(g, ϑ) is an isomorphism of Hermitian symmetric G-spaces from $\mathfrak{F}(\mathbb{P}^3, Q)_+$ to $\mathfrak{F}(\mathbb{P}^3, Q)_-$, and because of Equations (8.63), the following diagram commutes (in which we abbreviate $\mathfrak{F}_\pm := \mathfrak{F}(\mathbb{P}^3, Q)_\pm$):

$$\begin{tikzcd}
 & & G \times Q \arrow[d, "\widetilde{\chi}"] \arrow[ddll, "F_6^+ \times f_6^+"'] \arrow[ddrr, "F_6^- \times f_6^-"] & & \\
 & & Q \arrow[dl, "f_6^+"'] \arrow[dr, "f_6^-"] & & \\
 & \mathfrak{F}_+ \arrow[rr, "g"'] & & \mathfrak{F}_- & \\
G \times \mathfrak{F}_+ \arrow[ur, "\widetilde{\chi}_{\mathfrak{F}+}"] \arrow[rrrr, "\vartheta \times g"'] & & & & G \times \mathfrak{F}_- \arrow[ul, "\widetilde{\chi}_{\mathfrak{F}-}"']
\end{tikzcd} \;.$$

8.36 Remark. It is possible to reconstruct the Plücker embedding and the corresponding two-fold covering map of Lie groups $\mathrm{SU}(4) \to \mathrm{Aut}_s(\mathfrak{A}^6)_0$, $B \mapsto B^{(2)}$, which we used in Section 8.2 to establish the isomorphism $Q^4 \cong G_2(\mathbb{C}^4)$, from the objects of the present section by a suitable reduction of dimension.

For this purpose, we note that the subspace $\bigwedge^2 W$ of $S_+ = \bigwedge^{\mathrm{even}} W$ is invariant under $A_+|\bigwedge^2 W = (-*)|\bigwedge^2 W$ and therefore a $\mathbb{C}$Q-subspace of the $\mathbb{C}$Q-space S_+; we denote its induced $\mathbb{C}$Q-structure by $\mathfrak{A}'$. It gives rise to the 4-dimensional complex quadric $Q' := Q(\mathfrak{A}') \subset$

$\mathbb{P}(\bigwedge^2 W)$. Q' is the Plücker quadric $\{\,[u_1 \wedge u_2]\,|\, u \in \widehat{\mathrm{St}}_2(W)\,\}$, which was denoted by $Q(*)$ in Section 8.2. Moreover, we consider the Lie subgroup

$$G' := \vartheta(\{\, g \in G \,|\, \chi(g)w_1 = w_1 \,\})$$

of G, where w_1 is the vector which was used to define the triality automorphisms in Subsection 8.5.1. In this setting, I can prove the following facts:

(a) For any $s \in \widehat{Q}' := \widehat{Q}(\mathfrak{A}')$, say $s = u_1 \wedge u_2$, we have

$$Z(s) \cap W = \mathbb{C}u_1 \oplus \mathbb{C}u_2 \;\in\; G_2(W)\;.$$

Consequently, the map $f : Q' \to G_2(W)$ uniquely characterized by

$$\forall s \in \widehat{Q}' \;:\; f([s]) = Z(s) \cap W$$

is the inverse of the Plücker embedding.

(b) For any $g' \in G'$, we have $\chi(g')|W \in \mathrm{SU}(W)$ and the map

$$\chi' : G' \to \mathrm{SU}(W),\; g' \mapsto \chi(g')|W$$

is an isomorphism of Lie groups.

Also, for any $g' \in G'$, we have $\rho_+(g')|\bigwedge^2 W \in \mathrm{Aut}_s(\mathfrak{A}')_0$ and the map

$$\rho' : G' \to \mathrm{Aut}_s(\mathfrak{A}')_0,\; g \mapsto \rho_+(g)|\textstyle\bigwedge^2 W$$

is a two-fold covering map of Lie groups.

(c) $\Phi := \rho' \circ (\chi')^{-1}$ is the two-fold covering map of Lie groups

$$\Phi : \mathrm{SU}(W) \to \mathrm{Aut}_s(\mathfrak{A}')_0,\; B \mapsto B^{(2)}\;.$$

Therefore (f^{-1}, Φ) is the almost-isomorphism of Hermitian symmetric spaces from $G_2(W)$ to Q' described in Theorem 8.7(c).

Chapter 9

Perspectives

The results on the complex quadric presented in this dissertation give rise to various questions which are still open. Among them are the following:

- *Is it possible to apply the methods used to classify the totally geodesic submanifolds of the complex quadric to other symmetric spaces?*

  The classification of totally geodesic submanifolds of the complex quadric is essentially equivalent to the classification of the curvature-invariant subspaces of its tangent spaces, and the solution for the latter problem was based on the combination of two important results: First, the general relations between the roots of a symmetric spaces and the roots of a Lie triple system in it (Section 4.2), and second, the "geometric" description of the roots and root spaces of the complex quadric via the theory of $\mathbb{C}$Q-structures (Theorem 3.15).

  Let us first discuss the extension of these methods to other irreducible Riemannian symmetric spaces of rank 2. Then the results of Section 4.2 will still give full insight into the relationship between the root spaces of the symmetric space studied and the root spaces of its symmetric subspaces. Therefore, the main problem in the case of rank 2 is to find a replacement for the theory of $\mathbb{C}$Q-structures which is suited to give a description of the roots and root spaces for the symmetric space.

  The geometry of the complex 2-Grassmannians $G_2(\mathbb{C}^n)$, which are irreducible Hermitian symmetric spaces of rank 2, has been studied by J. Berndt, see [Ber97]. The cited paper gives in particular the eigenvalues and eigenspaces of the Jacobi operator of $G_2(\mathbb{C}^n)$ in terms of the complex structure and the quaternionic structure of the Kähler-quaternionic-Kähler manifold $G_2(\mathbb{C}^n)$ (see also Section 8.3). It seems reasonable to hope that this description might make it possible to expand the method here applied to Q^n also to $G_2(\mathbb{C}^n)$.

  But for the quaternionic 2-Grassmannians $G_2(\mathbb{H}^n)$ I know of no similar approach; the same is true of the irreducible symmetric spaces of rank 2 and compact type not yet mentioned, namely those locally isometric to $\mathrm{SU}(3)/\mathrm{SO}(3)$, $\mathrm{SU}(6)/\mathrm{Sp}(3)$, $\mathrm{SO}(10)/\mathrm{SU}(5)$, EIII, EIV or G (see [Hel78], p. 518).

For symmetric spaces of rank ≥ 3, the problem of the classification of totally geodesic submanifolds becomes much more difficult. It seems likely to me that for this case, a more powerful description of the relations betweens the roots and root spaces of the ambient space and of the subspace than that of Section 4.2 is needed. Some similar investigations for the case of Lie algebras have been carried out by ESCHENBURG, see [Esc84].

- *What can be said about the equipment of congruence families with symmetric structures and complex structures?*

  In Chapter 7 we were concerned with congruence families $\mathfrak{F}(N_0, M)$ induced by a homogeneous subspace N_0 of a Riemannian symmetric space M of compact type. We saw that such families are always naturally reductive Riemannian homogeneous spaces. On the other hand, among the specific examples of congruence families we studied, there are some which are in fact Riemannian symmetric spaces, while others can not be equipped with such a structure. Also some of the families can be equipped with a complex structure, while others can not be (even though the ambient space M is Hermitian symmetric).

  These observations raise the following questions: Are there (necessary or sufficient) criteria – formulated in terms of properties of M and N_0 – for the existence of a Riemannian symmetric space structure on $\mathfrak{F}(N_0, M)$ which induces the original naturally reductive structure? Are there criteria for the possibility to equip $\mathfrak{F}(N_0, M)$ with a complex structure?

  It would of course also be of interest to study further examples of congruence families, for example those induced in the complex quadric by the other types of totally geodesic submanifolds.

- *Which hypersurfaces of other Hermitian manifolds than $\mathbb{P}^n$ are analogous to complex quadrics?*

  One possible way to generalize the concept of a complex quadric is as follows:

  Let M be a Hermitian manifold and N be a complex hypersurface of M with parallel second fundamental form. We call N a $\mathbb{C}$Q-*hypersurface of* M if the shape operator $A_\eta^{N \hookrightarrow M}$ is a conjugation on the unitary space T_pN for every $p \in N$ and $\eta \in \perp_p^1(N \hookrightarrow M)$.

  If $\mathbb{V}$ is a unitary space, then any complex quadric in $\mathbb{P}(\mathbb{V})$ is a $\mathbb{C}$Q-hypersurface of $\mathbb{P}(\mathbb{V})$, as Theorem 1.16 shows. Conversely, because of the rigidity of submanifolds with parallel second fundamental form (see [Rec99], Abschnitt 17.15, Theorem 1) one can show that any $\mathbb{C}$Q-hypersurface of $\mathbb{P}(\mathbb{V})$ is a complex quadric in $\mathbb{P}(\mathbb{V})$.

  One can now ask the following questions: What are the $\mathbb{C}$Q-hypersurfaces in other Hermitian manifolds? In what ways is their behavior analogous to that of the complex quadrics in $\mathbb{P}(\mathbb{V})$, in what ways is it different?

In my opinion, these questions are of some interest and invite further investigation.

Appendix A

Reductive homogeneous spaces and symmetric spaces

In this appendix, fundamental aspects of the theory of homogeneous and symmetric spaces are described.

The viewpoint taken here on the theory of symmetric spaces has been strongly influenced by an unpublished exposition by H. Reckziegel. For the description of the root theory for symmetric spaces given in Section A.4, an unpublished lecture script by G. Thorbergsson was of help, as were the books [Hel78] and [Loo69].

The following notation should be kept in mind: If M_1, M_2, M are sets, $f : M_1 \times M_2 \to M$ is a map and $p_0 \in M_1$, $q_0 \in M_2$ holds, we consider the maps $f_{p_0} : M_2 \to M,\ q \mapsto f(p_0, q)$ and $f^{q_0} : M_1 \to M,\ p \mapsto f(p, q_0)$.

A.1 Reductive homogeneous spaces

Homogeneous spaces. Let M be a manifold and G be a Lie group acting on M via the action $\varphi : G \times M \to M$. In this setting (M, φ) is called a *G-space*. (M, φ) is called *homogeneous* if the action of φ on M is transitive. Let (M, φ) be a homogeneous G-space. Then we denote for every $p \in M$ by G_p the isotropy group of M at p. G_p is closed in G, on the quotient space G/G_p there exists one and only one differentiable structure so that the canonical action

$$\psi : G \times (G/G_p) \to (G/G_p),\ (g_1, g_2 \cdot G_p) \mapsto (g_1 g_2) \cdot G_p$$

is differentiable, and then the map $G/G_p \to M,\ g \cdot G_p \mapsto \varphi(g, p)$ becomes a G-equivariant diffeomorphism (see [Var74], Theorem 2.9.4, p. 77).

Homomorphisms of homogeneous spaces. Suppose that (M, φ) is a homogeneous G-space and (M', φ') is a homogeneous G'-space. Let $p \in M$ and $p' \in M'$ be given, and suppose that $F : G \to G'$ is a homomorphism of Lie groups such that $F(G_p) \subset G'_{p'}$ holds. Then there exists one and only one map $f : M \to M'$ with $f(p) = p'$ so that the diagram

$$\begin{tikzcd} G \times M \arrow[r, "\varphi"] \arrow[d, "F\times f"'] & M \arrow[d, "f"] \\ G' \times M' \arrow[r, "\varphi'"'] & M' \end{tikzcd} \tag{A.1}$$

commutes. Conversely, if $F : G \to G'$ is a homomorphism of Lie groups and $f : M \to M'$ is a map (which we do not suppose a priori to be differentiable) so that diagram (A.1) commutes, then we have $F(G_p) \subset G'_{f(p)}$ for any $p \in M$; also f is necessarily differentiable as Proposition A.1(a) below shows.

In this spirit, we call a pair (f, F) consisting of a map $f : M \to M'$ and a homomorphism of Lie groups $F : G \to G'$ a *homomorphism of homogeneous spaces* if Diagram (A.1) commutes. We call (f, F) an *almost-isomorphism of homogeneous spaces* if f is injective and F is a covering map of Lie groups, then f necessarily is a diffeomorphism (see Proposition A.1(b) below). We call (f, F) an *isomorphism of homogeneous spaces* if moreover F is in fact an isomorphism of Lie groups. In the case $G' = G$, $F = \mathrm{id}_G$, we also call simply f (instead of (f, id_G)) a homomorphism (isomorphism) of homogeneous spaces.

A homogeneous G'-space (M', φ') is called a *homogeneous subspace* of the homogeneous G-space (M, φ), if M' is a submanifold of M, G' is a Lie subgroup of G and $(M' \hookrightarrow M, G' \hookrightarrow G)$ is a homomorphism of homogeneous spaces.

It is occasionally useful also to speak of homomorphisms of non-homogeneous spaces: If (M, φ) is a G-space and (M', φ') is a G'-space, we call a pair (f, F) consisting of a map $f : M \to M'$ and a homomorphism of Lie groups $F : G \to G'$ a *homomorphism of spaces*, if diagram (A.1) commutes. Again, we do not require f to be differentiable; it should be noted that in this setting the differentiability of f does not necessarily follow from the differentiability of F.

A.1 Proposition. *Let (M, φ) be a homogeneous G-space, (M', φ') a G'-space and (f, F) a homomorphism from (M, φ) to (M', φ').*

(a) f is differentiable and of constant rank.

(b) If also (M', φ') is homogeneous, f is injective and $F : G \to G'$ is a surjective submersion[29], then f is a diffeomorphism onto M'.

Proof. For (a). Fix $p \in M$. Then we have $f \circ \varphi^p = (\varphi')^{f(p)} \circ F$ and therefore $f \circ \varphi^p$ is differentiable. Because (M, φ) is a homogeneous space, $\varphi^p : G \to M$ is a surjective submersion ([Var74], Lemma 2.9.2, p. 76), whence it follows that f is differentiable. For any $g \in G$, the maps $\varphi_g : M \to M$ and $\varphi'_{F(g)} : M' \to M'$ are diffeomorphisms and $\varphi'_{F(g)} \circ f = f \circ \varphi_g$ holds; the constancy of the rank of f follows by differentiation of the latter equation.

[29] If G has countable topology, then the submersivity of the Lie group homomorphism F is already implied by its surjectivity because of the theorem of Sard.

For (b). f is differentiable by (a) and surjective because of the equation $f \circ \varphi^p = (\varphi')^{f(p)} \circ F$. It also follows from that equation that $f^{-1} \circ (\varphi')^{f(p)} \circ F = \varphi^p$ is differentiable; because $(\varphi')^{f(p)} \circ F : G \to M'$ is a surjective submersion, we conclude that f^{-1} is differentiable. □

Riemannian homogeneous spaces. A (homogeneous) G-space (M, φ) is called a *Riemannian (homogeneous) G-space* if M is a Riemannian manifold and $\varphi_g : M \to M$ is an isometry for every $g \in G$.

If (M, φ) and (M', φ') are Riemannian homogeneous spaces, we call a homomorphism (almost-isomorphism, isomorphism) (f, F) of homogeneous spaces an *homomorphism (almost-isomorphism, isomorphism) of Riemannian homogeneous spaces*, if f is a homothetic immersion, i.e. if

$$\exists\, c \in \mathbb{R}_+ \; \forall\, p \in M,\; v, w \in T_pM \; : \; \langle f_* v, f_* w\rangle_{M'} = c \cdot \langle v, w\rangle_M$$

holds.[30] Because of the homogeneity of M it suffices to verify this equation for a fixed $p \in M$.

A Riemannian homogeneous G'-space (M', φ') is called a *Riemannian homogeneous subspace* of the Riemannian homogeneous G-space (M, φ), if M' is a submanifold of M, G' is a Lie subgroup of G and $(M' \hookrightarrow M, G' \hookrightarrow G)$ is a homomorphism of Riemannian homogeneous spaces.

Reductive homogeneous spaces. Suppose that (M, φ) is a homogeneous G-space and denote by $\mathfrak{g}$ the Lie algebra of G, by $\mathfrak{k}_p$ the Lie algebra of the isotropy group G_p at $p \in M$. A *reductive structure* on (M, φ) is a family $(\mathfrak{m}_p)_{p \in M}$ of linear subspaces of $\mathfrak{g}$ so that

$$\forall p \in M,\; g \in G \; : \big(\; \mathfrak{g} = \mathfrak{k}_p \oplus \mathfrak{m}_p \quad \text{and} \quad \mathrm{Ad}(g)\mathfrak{m}_p = \mathfrak{m}_{\varphi(g,p)} \; \big) \tag{A.2}$$

holds. In this situation, we call $(M, \varphi, (\mathfrak{m}_p))$ or simply M a *reductive homogeneous G-space.*

If $(\mathfrak{m}_p)$ is a reductive structure and $p_0 \in M$, then $(\mathfrak{m}_p)$ is already determined by the datum $(p_0, \mathfrak{m}_{p_0})$ via (A.2). Conversely, if $p_0 \in M$ and an $\mathrm{Ad}(G_{p_0})$-invariant linear subspace $\mathfrak{m} \subset \mathfrak{g}$ with $\mathfrak{g} = \mathfrak{k}_{p_0} \oplus \mathfrak{m}$ is given, there exists one and only one reductive structure $(\mathfrak{m}_p)$ on (M, φ) with $\mathfrak{m}_{p_0} = \mathfrak{m}$. Therefore, we call such a pair $(p_0, \mathfrak{m})$ a *reductive datum* for the homogeneous space (M, φ).

If $(M, \varphi, (\mathfrak{m}_p))$ and $(M', \varphi', (\mathfrak{m}'_p))$ are reductive homogeneous spaces, we call a homomorphism (almost-isomorphism, isomorphism) (f, F) of homogeneous spaces a *homomorphism (almost-isomorphism, isomorphism) of reductive homogeneous spaces*, if

$$\forall p \in M \; : \; F_L(\mathfrak{m}_p) \subset \mathfrak{m}'_{f(p)}$$

holds. $(M', \varphi', (\mathfrak{m}'_p))$ is called a *reductive homogeneous subspace* of $(M, \varphi, (\mathfrak{m}_p))$ if (M', φ') is a homogeneous subspace of (M, φ) and $(M' \hookrightarrow M, G' \hookrightarrow G)$ is a homomorphism of reductive homogeneous spaces.

[30] We do not require f to be an isometry, because the case $c \neq 1$ occurs necessarily for example in Theorem 7.10.

Suppose $(M, \varphi, (\mathfrak{m}_p))$ is a reductive homogeneous G-space. Then $\tau_p : \mathfrak{m}_p \to T_pM,\ X \mapsto (\varphi^p)_* X_e$ is an isomorphism of linear spaces; here we interpret the elements of $\mathfrak{g} \supset \mathfrak{m}_p$ as left-invariant vector fields on G, and e denotes the neutral element of G.

As was seen by NOMIZU, $(\mathfrak{m}_p)$ induces two covariant derivatives on M of particular importance: the torsion-free *canonical covariant derivative of the first kind* ∇ (see [Nom54], Theorem 10.1), characterized by

$$\forall p \in M,\ X, Y \in \mathfrak{m}_p\ :\ \nabla_X\, \varphi^p_* Y = \tfrac{1}{2} \cdot \varphi^p_*[X, Y] \tag{A.3}$$

and the *canonical covariant derivative of the second kind* ∇^0 (see [Nom54], Theorem 10.2), characterized by

$$\forall p \in M,\ X, Y \in \mathfrak{m}_p\ :\ \nabla^0_X\, \varphi^p_* Y \equiv 0\ . \tag{A.4}$$

For every $X \in \mathfrak{m}_p$ we denote by $\gamma_X : \mathbb{R} \to G$ the 1-parameter-subgroup of G induced by X. Then $\varphi^p \circ \gamma_X : \mathbb{R} \to M$ is a geodesic with respect to ∇^0, and every ∇^0-geodesic γ of M with $\gamma(0) = p$ can be obtained in this way (see [KN69], Corollary X.2.5, p. 192). Because the difference tensor $\nabla - \nabla^0$ is skew-symmetric, the ∇^0-geodesics and the ∇-geodesics in M coincide.

If $(M', \varphi', (\mathfrak{m}'_p))$ is a reductive homogeneous subspace of $(M, \varphi, (\mathfrak{m}_p))$, then M' is an affine submanifold of M (where M and M' are either both equipped with the canonical derivative of the first kind or are both equipped with the canonical derivative of the second kind). Indeed, for any $p \in M$ it follows from Equation (A.3) resp. (A.4) that the second fundamental form of the inclusion $M' \hookrightarrow M$ vanishes at p.

Naturally reductive homogeneous spaces. $(M, \varphi, (\mathfrak{m}_p))$ is called a *naturally reductive homogeneous space* if (M, φ) is a Riemannian homogeneous space, $(M, \varphi, (\mathfrak{m}_p))$ is a reductive homogeneous space, and the canonical covariant derivative of the first kind of $(M, \varphi, (\mathfrak{m}_p))$ coincides with the Levi-Civita derivative induced by the Riemannian metric on M.

If M and M' are naturally reductive homogeneous spaces, we call a pair (f, F) a *homomorphism (almost-isomorphism, isomorphism) of naturally reductive homogeneous spaces* if it is both a homomorphism (almost-isomorphism, isomorphism) of Riemannian homogeneous spaces and a homomorphism (almost-isomorphism, isomorphism) of reductive homogeneous spaces from M to M'. The G'-space M' is called a *naturally reductive homogeneous subspace*, if $(M' \hookrightarrow M, G' \hookrightarrow G)$ is a homomorphism of naturally reductive homogeneous spaces.

A.2 Affine symmetric spaces

There are two different ways to look at symmetric spaces, a geometric one and a more Lie theoretical one. Let us call these in mind:

The geometric approach. (see [KN69], Section XI.1, p. 223.) A connected affine manifold (M, ∇) is called an *affine-symmetric space*, if for every $p \in M$ there exists an affine diffeomorphism $s_p : M \to M$ with $s_p^2 = \mathrm{id}_M$ so that p is an isolated fixed point of s_p. It follows that $T_p s_p = -\mathrm{id}_{T_pM}$ holds, therefore s_p is – if it exists – uniquely determined by these conditions. For every geodesic $\gamma :]-\varepsilon, \varepsilon[\to M$ one has

$$\forall t \in]-\varepsilon, \varepsilon[: \; s_{\gamma(0)} \circ \gamma(t) = \gamma(-t) \; ; \tag{A.5}$$

because of this fact, s_p is called the *geodesic symmetry* at p. As a consequence of Equation (A.5), M is geodesically complete.

Every tensor field of type $(1, 2r)$ or $(0, 2r+1)$ which is invariant under all geodesic symmetries s_p vanishes identically. In particular, (M, ∇) is torsion-free and its curvature tensor is parallel.

The group $\mathfrak{A}(M)$ of affine transformations $f : M \to M$ is a Lie group (see [Kob72], Theorem II.1.3, p. 41). Because M is complete and connected, already $\mathfrak{A}(M)_0$ acts transitively on M, thus (M, φ) is a homogeneous $\mathfrak{A}(M)_0$-space with $\varphi : \mathfrak{A}(M)_0 \times M \to M, \; (f, p) \mapsto f(p)$. If we fix $p_0 \in M$ and denote the isotropy group of the action of $\mathfrak{A}(M)_0$ on M at p_0 by K, then

$$\sigma : \mathfrak{A}(M)_0 \to \mathfrak{A}(M)_0, \; f \mapsto s_{p_0} \circ f \circ s_{p_0} \tag{A.6}$$

is an involutive automorphism of the Lie group $\mathfrak{A}(M)_0$ and

$$K \subset \mathrm{Fix}(\sigma) := \{\, g \in \mathfrak{A}(M)_0 \,|\, \sigma(g) = g \,\} \quad \text{and} \quad \dim K = \dim \mathrm{Fix}(\sigma)$$

holds (see [KN69], Theorem XI.1.5, p. 224). As we will see in the Lie theoretical approach below, the geometry of (M, ∇) can be recovered completely from the "datum" $(M, \varphi, p_0, \sigma)$.

The Lie theoretical approach. (see [KN69], Section XI.2, p. 225.) We now start with the situation we obtained in the geometric approach above. We suppose that a "datum" $(M, \varphi, p_0, \sigma)$ is given, where $\varphi : G \times M \to M, \; (g, p) \mapsto gp$ is an action of a connected Lie group G on a manifold M, (M, φ) is a homogeneous G-space, $p_0 \in M$ is called the *origin point* and $\sigma : G \to G$ is an involutive Lie group automorphism with

$$K \subset \mathrm{Fix}(\sigma) := \{\, g \in G \,|\, \sigma(g) = g \,\} \quad \text{and} \quad \dim K = \dim \mathrm{Fix}(\sigma) \; ,$$

where K denote the isotropy group of φ at p_0.

Then there exists one and only one differentiable map $s : M \times M \to M$ characterized by

$$\forall g_1, g_2 \in M \; : \; s(g_1 p_0, g_2 p_0) = g_1 \sigma(g_1^{-1} g_2) p_0 \; . \tag{A.7}$$

For every $p \in M$, the map

$$s_p := s(p, \cdot) : M \to M$$

is a diffeomorphism of M satisfying

$$(s_p)^2 = \mathrm{id}_M \; , \quad s_p(p) = p \quad \text{and} \quad T_p s_p = -\mathrm{id}_{T_pM} \; .$$

We call s_p the *symmetry of M at p*. The following diagram commutes for any $g \in G$:

$$\begin{array}{ccc} M \times M & \xrightarrow{s} & M \\ {\scriptstyle \varphi_g \times \varphi_g} \downarrow & & \downarrow {\scriptstyle \varphi_g} \\ M \times M & \xrightarrow[s]{} & M\ . \end{array}$$

As Nomizu has shown (see [Nom54], Theorem 15.3, p. 54), there exists one and only one covariant derivative ∇ on M with regard to which the maps s_p are affine for every $p \in M$. Then (M, ∇) is an affine-symmetric space in the sense of the geometric approach. ∇ is called the *canonical covariant derivative of the symmetric space* M. Furthermore, the diffeomorphisms $\varphi_g : M \to M$ are all affine.

If M was already equipped with a covariant derivative so that it is an affine-symmetric space in the sense of the geometric approach, and we perform the construction of the Lie theoretic approach with $G = \mathfrak{A}(M)_0$, an arbitrary origin point $p_0 \in M$ and σ given by (A.6), then the canonical covariant derivative of M obtained thereby is identical to the original covariant derivative on M. For this reason we call in the "Lie theoretic" situation $(M, \varphi, p_0, \sigma)$ (or simply M, when the other components can be inferred) a *symmetric* G*-space*; remember that in this situation we did not require $G = \mathfrak{A}(M)_0$.

Symmetric spaces as reductive homogeneous spaces; canonical decomposition. Let $(M, \varphi, p_0, \sigma)$ be a symmetric G-space, K the isotropy group of G at p_0, and $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Then the linearization of the involutive Lie group automorphism $\sigma : G \to G$ is an involutive Lie algebra automorphism $\sigma_L : \mathfrak{g} \to \mathfrak{g}$ with $\mathfrak{k} = \mathrm{Eig}(\sigma_L, 1)$. If we put $\mathfrak{m} := \mathrm{Eig}(\sigma_L, -1)$, we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$; this decomposition is called the *canonical decomposition* of $\mathfrak{g}$ with respect to σ. It should be noted that generally, $\mathfrak{m}$ is not a Lie subalgebra of $\mathfrak{g}$; however we have

$$\mathrm{Ad}_G(K)\mathfrak{m} \subset \mathfrak{m} \quad \text{and} \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}\ , \tag{A.8}$$

and therefore in particular

$$[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subset \mathfrak{m}\ . \tag{A.9}$$

Equation (A.8) shows that $(p_0, \mathfrak{m})$ is a reductive datum for M, and therefore the affine symmetric space M can be regarded as a reductive homogeneous space in a canonical way. In this situation, the two canonical covariant derivatives of the reductive homogeneous space M are equal and they coincide with the canonical covariant derivative of the symmetric space M described above via the symmetries s_p.

The map

$$\tau : \mathfrak{m} \to T_{p_0}M,\ X \mapsto (\varphi^{p_0})_* X_e \tag{A.10}$$

is an isomorphism of linear spaces (here, we interpret the elements of $\mathfrak{m} \subset \mathfrak{g}$ as left-invariant vector fields on G; e denotes the neutral element of G). We have

$$\forall g \in K,\ X \in \mathfrak{m}\ :\ \tau(\mathrm{Ad}_G(g)X) = (\varphi_g)_* \tau(X)\ ; \tag{A.11}$$

also if we denote the curvature tensor of M by R and for $w \in T_{p_0}M$ by $R_w := R(\,\cdot\,, w)w : T_{p_0}M \to T_{p_0}M$ the Jacobi operator corresponding to w, we have

$$\forall X, Y, Z \in \mathfrak{m} \; : \; R(\tau X, \tau Y)\tau Z = -\tau([[X,Y],Z]) \; , \tag{A.12}$$

in particular

$$\forall X, Z \in \mathfrak{m} \; : \; R_{\tau Z}(\tau X) = -\operatorname{ad}(Z)^2\, X \tag{A.13}$$

(see [KN69], Theorem XI.3.2(1), p. 231). τ is called the *canonical isomorphism* between $\mathfrak{m}$ and $T_{p_0}M$.

We call the symmetric space M *irreducible*, if the isotropy representation $K \to \mathrm{GL}(T_{p_0}M)$, $g \mapsto T_{p_0}\varphi_g$ is irreducible (or equivalently, if the adjoint representation $K \to \mathrm{GL}(\mathfrak{m})$, $g \mapsto \mathrm{Ad}_G(g)|\mathfrak{m}$ is irreducible).

Homomorphisms of symmetric spaces. Let $(M, \varphi, p_0, \sigma)$ be a symmetric G-space and $(M', \varphi', p_0', \sigma')$ be a symmetric G'-space. If (f, F) is a homomorphism from the homogeneous G-space (M, φ) to the homogeneous G'-space (M', φ') so that

$$f(p_0) = p_0' \quad \text{and} \quad \sigma' \circ F = F \circ \sigma$$

holds, then (f, F) is in fact a homomorphism of reductive homogeneous spaces and f is affine with respect to the canonical covariant derivatives of the symmetric spaces. Therefore, it is reasonable to call the pair (f, F) a *homomorphism of symmetric spaces.* If in this setting (f, F) is in fact an (almost-)isomorphism of homogeneous spaces, we call (f, F) an *(almost-)isomorphism of symmetric spaces.* In the case $G = G'$, $F = \mathrm{id}_G$ we also call simply f (in the place of (f, id_G)) a homomorphism resp. isomorphism of symmetric spaces.

Replacing the group G by its universal cover. Under a condition a symmetric G-space can also be interpreted as a symmetric $\widetilde{G}$-space, where $\widetilde{G}$ is the universal covering Lie group of G.

A.2 Proposition. *Let $(M, \varphi, p_0, \sigma)$ be a symmetric G-space, $\tau : \widetilde{G} \to G$ be a universal Lie group covering of the Lie group G (thus $\widetilde{G}$ is simply connected; remember that G was supposed to be connected) and $K \subset G$ the isotropy group of the G-action φ at p_0. We require that $\tau^{-1}(K) \subset \widetilde{G}$ is connected.*

Then $\widetilde{\varphi} := \varphi \circ (\tau \times \mathrm{id}_M)$ is a transitive Lie group action of $\widetilde{G}$ on M, there exists one and only one involutive Lie group automorphism $\widetilde{\sigma} : \widetilde{G} \to \widetilde{G}$ with $\tau \circ \widetilde{\sigma} = \sigma \circ \tau$, and $(M, \widetilde{\varphi}, p_0, \widetilde{\sigma})$ is a symmetric $\widetilde{G}$-space.

$(M, \varphi, p_0, \sigma)$ and $(M, \widetilde{\varphi}, p_0, \widetilde{\sigma})$ induce the same canonical covariant derivative on M.

Proof. $(M, \widetilde{\varphi})$ is a homogeneous $\widetilde{G}$-space because φ is transitive and $\tau : \widetilde{G} \to G$ is surjective.

Because $\widetilde{G}$ is simply connected, there exists one and only one Lie group homomorphism $\widetilde{\sigma} : \widetilde{G} \to \widetilde{G}$ with $\tau \circ \widetilde{\sigma} = \sigma \circ \tau$. We have $\tau \circ \widetilde{\sigma} \circ \widetilde{\sigma} = \sigma \circ \sigma \circ \tau = \tau$, and therefore the Lie group homomorphisms $\widetilde{\sigma} \circ \widetilde{\sigma}, \mathrm{id}_{\widetilde{G}} : \widetilde{G} \to \widetilde{G}$

are both lifts of the Lie group homomorphism $\tau : \widetilde{G} \to G$ with respect to the Lie group covering $\tau : \widetilde{G} \to G$. It follows that $\widetilde{\sigma} \circ \widetilde{\sigma} = \mathrm{id}_{\widetilde{G}}$ holds, hence $\widetilde{\sigma}$ is an involutive automorphism.

It is clear that $\widetilde{K} := \tau^{-1}(K)$ is the isotropy group of $\widetilde{\varphi}$ at p_0. We now show

$$\widetilde{K} \subset \mathrm{Fix}(\widetilde{\sigma}) \quad \text{and} \quad \dim \widetilde{K} = \dim \mathrm{Fix}(\widetilde{\sigma}) \,. \tag{A.14}$$

For $\widetilde{K} \subset \mathrm{Fix}(\widetilde{\sigma})$. For every $g \in \widetilde{K}$ we have $\tau(g) \in K \subset \mathrm{Fix}(\sigma)$, hence $\tau(\widetilde{\sigma}(g)) = \sigma(\tau(g)) = \tau(g)$ and therefore $\widetilde{\sigma}(g) \cdot g^{-1} \in \ker(\tau)$. Because $\widetilde{K}$ is connected and $\ker(\tau)$ is discrete, we therefore have $\widetilde{\sigma}(g) \cdot g^{-1} = e_{\widetilde{G}}$ for every $g \in \widetilde{K}$. Thus, $\widetilde{K} \subset \mathrm{Fix}(\widetilde{\sigma})$ holds.

For $\dim \widetilde{K} = \dim \mathrm{Fix}(\widetilde{\sigma})$. We have $\widetilde{K} \subset \mathrm{Fix}(\widetilde{\sigma})$ and therefore $\dim \widetilde{K} \leq \dim \mathrm{Fix}(\widetilde{\sigma})$. On the other hand, we have $\tau(\mathrm{Fix}(\widetilde{\sigma})_0) \subset \mathrm{Fix}(\sigma)_0 \subset K$ and therefore $\mathrm{Fix}(\widetilde{\sigma})_0 \subset \widetilde{K}$, hence $\dim \mathrm{Fix}(\widetilde{\sigma}) = \dim \mathrm{Fix}(\widetilde{\sigma})_0 \leq \dim \widetilde{K}$.

From (A.14) it follows that $(M, \widetilde{\varphi}, p_0, \widetilde{\sigma})$ is a symmetric $\widetilde{G}$-space. The symmetries of the symmetric spaces $(M, \varphi, p_0, \sigma)$ and $(M, \widetilde{\varphi}, p_0, \widetilde{\sigma})$ are equal at every $p \in M$ as Equation (A.7) shows, and therefore these spaces induce the same canonical covariant derivative on M. □

A.3 Riemannian and Hermitian symmetric spaces

We continue to use the notations of the preceding sections with respect to a symmetric G-space $(M, \varphi, p_0, \sigma)$. We now further suppose that (M, φ) is *almost effective*, meaning that the Lie subgroup $\{\, g \in G \,|\, \varphi_g = \mathrm{id}_M \,\}$ of G is discrete.

Examples for almost effective actions. If M is a Riemannian homogeneous space only, $\underline{G} \subset I(M)$ a Lie subgroup which still acts transitively on M and $\tau : G \to \underline{G}$ a covering map of Lie groups, then the action $\varphi : G \times M \to M,\ (g, p) \mapsto \tau(g)p$ is transitive and almost effective. The actions considered in Chapter 7 for the construction of congruence families are all constructed in this way.

Riemannian symmetric spaces. We call a Riemannian metric on the symmetric G-space M *G-invariant* if $\varphi_g : M \to M$ is an isometry with respect to this metric for every $g \in G$; in this case (M, φ) is in particular a Riemannian homogeneous space.

As Equation (A.11) shows, any G-invariant Riemannian metric on M gives rise to a $\mathrm{Ad}_G(K)$-invariant inner product on $\mathfrak{m}$, which is characterized by the fact that $\tau : \mathfrak{m} \to T_{p_0}M$ is a linear isometry. Conversely, any $\mathrm{Ad}_G(K)$-invariant inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $\mathfrak{m}$ defines via

$$\forall\, v, w \in T_{p_0}M,\ g \in G \ :\ \langle (\varphi_g)_* v, (\varphi_g)_* w \rangle := \langle\!\langle \tau^{-1} v, \tau^{-1} w \rangle\!\rangle$$

a G-invariant Riemannian metric on M. The G-invariant Riemannian metrics on M and the $\mathrm{Ad}_G(K)$-invariant inner products on $\mathfrak{m}$ are in an one-to-one correspondence in this way.

There exist $\mathrm{Ad}_G(K)$-invariant inner products on $\mathfrak{m}$ and therefore G-invariant Riemannian metrics on M if and only if the Lie group $\mathrm{Ad}_G(K) \subset \mathrm{GL}(\mathfrak{g})$ is compact (see [Hel78], Proposition IV.3.4, p. 209).

If the symmetric space M is irreducible in this situation, two G-invariant Riemannian metrics on M differ only by a constant factor $c \in \mathbb{R}_+$.

We call $(M, \varphi, p_0, \sigma)$ a *Riemannian symmetric G-space* if (M, φ) is almost effective and M is equipped with a fixed G-invariant Riemannian metric. We call a homomorphism (almost-isomorphism, isomorphism) of symmetric spaces between two Riemannian symmetric spaces a *homomorphism (almost-isomorphism, isomorphism) of Riemannian symmetric spaces*, if it also is a homomorphism of the underlying Riemannian homogeneous spaces.

If M and M' are Riemannian symmetric spaces, either of M and M' is irreducible and (f, F) is an almost-isomorphism of affine symmetric spaces from M to M', then both M and M' are irreducible and (f, F) already is an almost-isomorphism of Riemannian symmetric spaces.

Let $(M, \varphi, p_0, \sigma)$ be a Riemannian symmetric G-space. Regarded as a reductive homogeneous space in the canonical way, M is then naturally reductive (see [KN69], Theorem XI.3.3, p. 232). Recall that this means that the canonical covariant derivative of the symmetric space M is identical to the Levi-Civita derivative corresponding to the Riemannian metric of M.

Hermitian symmetric spaces. Suppose that M is a Riemannian symmetric space. We call a complex structure J on M (i.e. J is an endomorphism field on TM so that $J^2 = -\mathrm{id}_{TM}$ holds) *adapted to* $\langle\cdot,\cdot\rangle$, if $J_p : T_pM \to T_pM$ is orthogonal for every $p \in M$; we call J *G-invariant*, if the isometry $\varphi_g : M \to M$ is J-holomorphic for every $g \in G$. If M is irreducible, two G-invariant complex structures on M which are adapted to $\langle\cdot,\cdot\rangle$ differ only in sign. A *Hermitian symmetric space* is a Riemannian symmetric space M which is simultaneously a complex manifold in such a way that its complex structure J is adapted to $\langle\cdot,\cdot\rangle$ and G-invariant.

If M and M' are Hermitian symmetric spaces, we call a homomorphism (almost-isomorphism, isomorphism) of Riemannian symmetric spaces (f, F) between them a *homomorphism (almost-isomorphism, isomorphism) of Hermitian symmetric spaces*, if f is either holomorphic or anti-holomorphic. If M or M' is irreducible, any almost-isomorphism of affine symmetric spaces already is an almost-isomorphism of Hermitian symmetric spaces.

A.3 Proposition. *If $(M, \varphi, p_0, \sigma)$ is a Riemannian (Hermitian) symmetric G-space in the situation of Proposition A.2, then $(M, \widetilde{\varphi}, p_0, \widetilde{\sigma})$ also is a Riemannian (Hermitian) symmetric $\widetilde{G}$-space.*

Proof. A G-invariant Riemannian metric (Hermitian structure) on M clearly also is $\widetilde{G}$-invariant. □

A.4 The root space decomposition of a symmetric space of compact type

The principal sources for this section are an unpublished lecture script by G. Thorbergsson, and [Loo69], Section VI.1.

Let $(M, \varphi, p_0, \sigma)$ be a Riemannian symmetric G-space; remember that this means in particular that (M, φ) is almost effective. We consider the isotropy group K of φ at p_0, the canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G with respect to σ and the *Killing form*

$$\varkappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},\ (X, Y) \mapsto \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))\ .$$

It can be shown that $\varkappa$ is a symmetric bilinear form, that we have for any Lie algebra automorphism $L : \mathfrak{g} \to \mathfrak{g}$

$$\forall X, Y \in \mathfrak{g} \; : \; \varkappa(L(X),\, L(Y)) = \varkappa(X, Y)\;, \tag{A.15}$$

and that the almost-effectivity of (M, φ) causes $\varkappa|(\mathfrak{k} \times \mathfrak{k})$ to be negative definite.

Riemannian symmetric spaces of compact type.

A.4 Definition. *The Riemannian symmetric G-space $(M, \varphi, p_0, \sigma)$ is said to be of* compact type *if the Killing form $\varkappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is negative definite.*

The importance of the concept of a symmetric space of compact type is exemplified by the following proposition:

A.5 Proposition. *Let $(M, \varphi, p_0, \sigma)$ and $(M', \varphi', p_0', \sigma')$ be Riemannian symmetric spaces of compact type over the group G resp. G'. (Remember that this means in particular that G is connected.) Moreover, suppose that (f, F) is an (almost-)isomorphism of the underlying homogeneous spaces with $f(p_0) = p_0'$. Then (f, F) already is an (almost-)isomorphism of symmetric spaces. (However, note that f does not need to be a homothety.)*

Proof. We have to show that $\sigma' \circ F = F \circ \sigma$ holds, and because G is connected this equality is already implied by its linearization

$$\forall X \in \mathfrak{g} \; : \; \sigma_L'(F_L(X)) = F_L(\sigma_L(X))\;, \tag{A.16}$$

where $\mathfrak{g}$ denotes the Lie algebra of G.

Let us consider the canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of $\mathfrak{g}$ with respect to σ and the canonical decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{m}'$ of the Lie algebra $\mathfrak{g}'$ of G' with respect to σ'. It clearly suffices to show Equation (A.16) for $X \in \mathfrak{k}$ and for $X \in \mathfrak{m}$.

Because (f, F) is an almost-isomorphism of homogeneous spaces, F maps the isotropy group K of φ at p_0 onto the isotropy group K' of φ' at p_0'. $\mathfrak{k}$ and $\mathfrak{k}'$ are the Lie algebras of K and K' respectively, and therefore it follows that $F_L(\mathfrak{k}) = \mathfrak{k}'$ holds; because of $\mathfrak{k} = \mathrm{Eig}(\sigma_L, 1)$ and $\mathfrak{k}' = \mathrm{Eig}(\sigma_L', 1)$ we conclude that Equation (A.16) holds for $X \in \mathfrak{k}$.

We now denote the Killing forms of $\mathfrak{g}$ and $\mathfrak{g}'$ by $\varkappa$ resp. $\varkappa'$; these bilinear forms are negative definite. σ_L is an orthogonal involution of $(\mathfrak{g}, \varkappa)$ by Equation (A.15), and therefore $\mathfrak{m} = \mathrm{Eig}(\sigma_L, -1)$ is the $\varkappa$-orthogonal complement of $\mathfrak{k} = \mathrm{Eig}(\sigma_L, 1)$. Similarly, $\mathfrak{m}'$ is the $\varkappa'$-orthogonal complement of $\mathfrak{k}'$. Because $F_L : \mathfrak{g} \to \mathfrak{g}'$ is an isomorphism of Lie algebras, it is a linear isometry with respect to $\varkappa$ and $\varkappa'$, and thus we have

$$F_L(\mathfrak{m}) = F_L(\mathfrak{k}^{\perp,\varkappa}) = (\mathfrak{k}')^{\perp,\varkappa'} = \mathfrak{m}'\,.$$

This equality shows that Equation (A.16) holds for $X \in \mathfrak{m}$. □

Cartan subalgebras. From here on, we suppose $(M, \varphi, p_0, \sigma)$ to be a Riemannian symmetric G-space of compact type.

A.6 Proposition. *Let $\mathfrak{a} \subset \mathfrak{m}$ be a linear subspace. Then the following statements are equivalent:*

(a) $\forall X, Y, Z \in \mathfrak{a} \; : \; [[X, Y], Z] = 0$.

(b) $\forall X, Y \in \mathfrak{a} \; : \; [X, Y] = 0$.

(c) $\forall X, Y \in \mathfrak{a} \; : \; \mathrm{ad}(X) \circ \mathrm{ad}(Y) = \mathrm{ad}(Y) \circ \mathrm{ad}(X)$.

If these statements hold, we call $\mathfrak{a}$ a flat subspace *of $\mathfrak{m}$.*

Characterization (a) of this proposition reveals the geometric significance of flat subspaces: If $\mathfrak{a} \subset \mathfrak{m}$ is a flat subspace and $\tau : \mathfrak{m} \to T_{p_0}M$ is the canonical isomorphism (see (A.10)), then the curvature tensor of M vanishes on $\tau(\mathfrak{a}) \subset T_{p_0}M$ by Equation (A.12). Consequently there exists a (connected, complete) totally geodesic submanifold N of M with $p_0 \in N$ and $T_{p_0}N = \tau(\mathfrak{a})$, and N is *flat*, meaning that the curvature tensor of N vanishes identically. Such flat submanifolds of M are called *tori* in M .

As will become apparent below, property (c) in the proposition is the reason why flat subspaces play a fundamental role in the theory of root systems.

Proof of Proposition A.6. For (a) ⇒ (b). Let $X, Y \in \mathfrak{a}$ be given. Then we have

$$0 = \varkappa(\,[[X,Y],X]\,,\,Y\,) = -\varkappa(\,\mathrm{ad}(X)([X,Y])\,,\,Y\,) = \varkappa(\,[X,Y]\,,\,\mathrm{ad}(X)Y\,) = \varkappa(\,[X,Y]\,,\,[X,Y]\,)\,,$$

whence $[X, Y] = 0$ follows by the negative definity of $\varkappa$. *For (b) ⇒ (c).* This is an immediate consequence of the fact that $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is an homomorphism of Lie algebras. *For (c) ⇒ (a).* Let $X, Y, Z \in \mathfrak{a}$ be given. Then we have

$$[[X,Y],Z] = \mathrm{ad}([X,Y])Z = (\mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X))Z \overset{(c)}{=} 0\,.$$

□

A.7 Definition. *(a) The maximal dimension of flat subspaces of $\mathfrak{m}$ is called the* rank *of M and is denoted by* $\mathrm{rk}(M)$ *.*

(b) A flat subspace $\mathfrak{a} \subset \mathfrak{m}$ with $\dim \mathfrak{a} = \mathrm{rk}(M)$ is called a Cartan subalgebra *of $\mathfrak{m}$.*

A.8 Theorem. *(a) Let $\mathfrak{a}, \mathfrak{a}'$ be two Cartan subalgebras of $\mathfrak{m}$. Then there exists $g \in K$ (where K is the isotropy group of the G-action φ on M at p_0) with $\mathfrak{a}' = \mathrm{Ad}(g)\mathfrak{a}$.*

(b) Let $X \in \mathfrak{m}$ be given. Then there exists a Cartan subalgebra $\mathfrak{a} \subset \mathfrak{m}$ with $X \in \mathfrak{a}$.

Proof. See [Hel78], Theorem V.6.2, p. 246. □

Roots and the root space decomposition. Because of Theorem A.8(a), any two Cartan subalgebras in $\mathfrak{m}$ are geometrically equivalent. Keeping this in mind, we now fix a Cartan subalgebra $\mathfrak{a} \subset \mathfrak{m}$.

The central point of the root theory is to derive direct sum decompositions

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_k \quad \text{resp.} \quad \mathfrak{k} = \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_k$$

such that there exist linear forms $\lambda_1, \ldots, \lambda_k : \mathfrak{a} \to \mathbb{R}$ so that we have

$$\mathfrak{m}_j, \mathfrak{k}_j \; \subset \; \mathrm{Eig}(-\operatorname{ad}(Z)^2, \lambda_j(Z)^2)$$

for all $Z \in \mathfrak{a}$ and every $j \in \{1, \ldots, k\}$. In this context the linear forms λ_j are called *roots*, and the $\mathfrak{m}_j$ resp. $\mathfrak{k}_j$ are called *roots spaces* of $\mathfrak{m}$ resp. $\mathfrak{k}$. The endomorphisms $-\operatorname{ad}(Z)^2|\mathfrak{m}$ are of importance because under the canonical isomorphism $\tau : \mathfrak{m} \to T_{p_0}M$ they are conjugate to the Jacobi operators of M, see Equation (A.13). In the sequel we will show how to obtain decompositions with the desired property.

Because M is of compact type, $\langle \cdot, \cdot \rangle := -c\,\varkappa$ is a positive definite inner product on $\mathfrak{g}$ for any fixed $c \in \mathbb{R}_+$, and we regard $\mathfrak{g}$ as a euclidean space in this way.[31] Then the endomorphism $\operatorname{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is skew-adjoint for every $X \in \mathfrak{g}$.

We now apply the eigenspace theory to the family $(\operatorname{ad}(Z))_{Z \in \mathfrak{a}}$ of pairwise commuting (see Proposition A.6(c)), skew-adjoint endomorphisms of $\mathfrak{g}$. For this purpose we have to take an excursion into the complex. We consider the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\,\mathfrak{g}$ of the linear space $\mathfrak{g}$ and denote for any $X \in \mathfrak{g}^{\mathbb{C}}$ by $\mathrm{Re}(X), \mathrm{Im}(X) \in \mathfrak{g}$ the elements uniquely characterized by $X = \mathrm{Re}(X) + i\,\mathrm{Im}(X)$. By extending the Lie bracket of $\mathfrak{g}$ to a $\mathbb{C}$-bilinear map (which we again denote by $[\cdot, \cdot]$), $\mathfrak{g}^{\mathbb{C}}$ becomes a complex Lie algebra. We also use the notation $\operatorname{ad}(X) := [X, \cdot\,] : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ for $X \in \mathfrak{g}^{\mathbb{C}}$. It should be noted that the complexification of the involutive Lie algebra automorphism σ_L of $\mathfrak{g}$ is an involutive Lie algebra automorphism $\sigma_L^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ with $\mathrm{Eig}(\sigma_L^{\mathbb{C}}, 1) = \mathfrak{k} \oplus i\,\mathfrak{k} =: \mathfrak{k}^{\mathbb{C}}$ and $\mathrm{Eig}(\sigma_L^{\mathbb{C}}, -1) = \mathfrak{m} \oplus i\,\mathfrak{m} =: \mathfrak{m}^{\mathbb{C}}$. Herewith $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$ holds.

Moreover, we define a Hermitian inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ on $\mathfrak{g}^{\mathbb{C}}$ by putting for every $X, Y \in \mathfrak{g}^{\mathbb{C}}$, say $X = X_1 + i\,X_2$ and $Y = Y_1 + i\,Y_2$ with $X_k, Y_k \in \mathfrak{g}$,

$$\langle X, Y \rangle_{\mathbb{C}} := \langle X_1, Y_1 \rangle + \langle X_2, Y_2 \rangle + i\left(\langle X_2, Y_1 \rangle - \langle X_1, Y_2 \rangle\right) .$$

Then $\operatorname{ad}(X)$ is skew-Hermitian for every $X \in \mathfrak{g}^{\mathbb{C}}$.

The family $(\operatorname{ad}(Z))_{Z \in \mathfrak{a}}$ consists of pairwise commuting, skew-Hermitian endomorphisms of $\mathfrak{g}^{\mathbb{C}}$. Consequently, these endomorphisms are jointly diagonalizable in $\mathfrak{g}^{\mathbb{C}}$, and their eigenvalues are purely imaginary. It follows that if we put for every $\mathbb{R}$-linear form $\lambda \in \mathfrak{a}^* := L(\mathfrak{a}, \mathbb{R})$

$$\mathfrak{g}_\lambda^{\mathbb{C}} := \{\, X \in \mathfrak{g}^{\mathbb{C}} \,|\, \forall Z \in \mathfrak{a} : \operatorname{ad}(Z)X = i\lambda(Z)\,X \,\} ,$$

[31] The choice of the factor c is of no geometric relevance for any of the following constructions. We permit such a factor only because it occurs naturally in the construction of the $\mathbb{C}$Q-space structure on the space $\mathfrak{m}$ in the case where M is a complex quadric, compare Equation (3.17) in Proposition 3.12(b).

in particular

$$\mathfrak{g}_0^{\mathbb{C}} = \{\, X \in \mathfrak{g}^{\mathbb{C}} \,|\, [X, \mathfrak{a}] = \{0\} \,\} \;,$$

and

$$\Delta := \{\, \lambda \in \mathfrak{a}^* \setminus \{0\} \,|\, \mathfrak{g}_\lambda^{\mathbb{C}} \neq \{0\} \,\} \;, \tag{A.17}$$

we uniquely obtain the decomposition of $\mathfrak{g}^{\mathbb{C}}$ into a direct sum of linear spaces

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0^{\mathbb{C}} \;\oplus\; \bigoplus_{\lambda\in\Delta} \mathfrak{g}_\lambda^{\mathbb{C}} \;. \tag{A.18}$$

Δ is called the *root system* of $\mathfrak{g}^{\mathbb{C}}$ (or of $\mathfrak{g}$) with respect to $\mathfrak{a}$; its elements are called the *roots* of $\mathfrak{g}^{\mathbb{C}}$ (or of $\mathfrak{g}$) with respect to $\mathfrak{a}$.

In the sequel, two different involutions on $\mathfrak{g}$ are of importance: (1) the involutive, $\mathbb{C}$-linear Lie algebra automorphism $\sigma_L^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$, and (2) the conjugation $\mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$, $X \mapsto \overline{X} := \mathrm{Re}(X) - i\ \mathrm{Im}(X)$; this is an anti-linear Lie algebra automorphism of $\mathfrak{g}^{\mathbb{C}}$. These two involutions commute with each other.

A.9 Proposition. *(a) For every $\lambda \in \mathfrak{a}^*$ we have $\overline{\mathfrak{g}_\lambda^{\mathbb{C}}} = \mathfrak{g}_{-\lambda}^{\mathbb{C}} = \sigma_L^{\mathbb{C}}(\mathfrak{g}_\lambda^{\mathbb{C}})$, in particular*

$$-\lambda \in \Delta \iff \lambda \in \Delta \;.$$

(b) We have $\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{z}_{\mathfrak{k}^{\mathbb{C}}}(\mathfrak{a}) \oplus \mathfrak{a}^{\mathbb{C}}$, where $\mathfrak{z}_{\mathfrak{k}^{\mathbb{C}}}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}^{\mathbb{C}}$, i.e.

$$\mathfrak{z}_{\mathfrak{k}^{\mathbb{C}}}(\mathfrak{a}) = \{\, X \in \mathfrak{k}^{\mathbb{C}} \,|\, [X, \mathfrak{a}] = \{0\} \,\} \;,$$

and $\mathfrak{a}^{\mathbb{C}} := \mathfrak{a} \oplus i\,\mathfrak{a}$.

(c) For every $\lambda, \mu \in \mathfrak{a}^*$ we have $[\mathfrak{g}_\lambda^{\mathbb{C}}, \mathfrak{g}_\mu^{\mathbb{C}}] \subset \mathfrak{g}_{\lambda+\mu}^{\mathbb{C}}$.

Proof. For (a). Let $X \in \mathfrak{g}_\lambda^{\mathbb{C}}$ be given, say $X = X_1 + i\,X_2$ with $X_1, X_2 \in \mathfrak{g}$. Then we have for every $Z \in \mathfrak{a}$

$$\mathrm{ad}(Z)X_1 + i\ \mathrm{ad}(Z)X_2 = \mathrm{ad}(Z)X = i\lambda(Z)X = -\lambda(Z)X_2 + i\,\lambda(Z)X_1$$

and therefore

$$\mathrm{ad}(Z)X_1 = -\lambda(Z)X_2 \quad\text{and}\quad \mathrm{ad}(Z)X_2 = \lambda(Z)X_1 \;.$$

Via these equations we see that

$$\mathrm{ad}(Z)\overline{X} = \mathrm{ad}(Z)X_1 - i\ \mathrm{ad}(Z)X_2 = -\lambda(Z)X_2 - i\,\lambda(Z)X_1 = -i\,\lambda(Z)\overline{X}$$

and hence $\overline{X} \in \mathfrak{g}_{-\lambda}^{\mathbb{C}}$ holds. For $Z \in \mathfrak{a} \subset \mathfrak{m}^{\mathbb{C}}$ we also have $\sigma_L^{\mathbb{C}}(Z) = -Z$ and therefore

$$\mathrm{ad}(Z)(\sigma_L^{\mathbb{C}}(X)) = -\,\mathrm{ad}(\sigma_L^{\mathbb{C}}(Z))(\sigma_L^{\mathbb{C}}(X)) = -\sigma_L^{\mathbb{C}}(\,\mathrm{ad}(Z)X\,) = -\sigma_L^{\mathbb{C}}(\,i\lambda(Z)\,X\,) = -i\lambda(Z)\,\sigma_L^{\mathbb{C}}(X) \;,$$

hence $\sigma_L^{\mathbb{C}}(X) \in \mathfrak{g}_{-\lambda}^{\mathbb{C}}$. Thus we have shown $\overline{\mathfrak{g}_\lambda^{\mathbb{C}}}, \sigma_L^{\mathbb{C}}(\mathfrak{g}_\lambda^{\mathbb{C}}) \subset \mathfrak{g}_{-\lambda}^{\mathbb{C}}$. We also have $\overline{\mathfrak{g}_{-\lambda}^{\mathbb{C}}}, \sigma_L^{\mathbb{C}}(\mathfrak{g}_{-\lambda}^{\mathbb{C}}) \subset \mathfrak{g}_\lambda^{\mathbb{C}}$ and therefore by the involutivity of $\overline{\Box}$ and $\sigma_L^{\mathbb{C}}$: $\mathfrak{g}_{-\lambda}^{\mathbb{C}} \subset \overline{\mathfrak{g}_\lambda^{\mathbb{C}}}, \sigma_L^{\mathbb{C}}(\mathfrak{g}_\lambda^{\mathbb{C}})$.

For (b). It is clear from the definition of $\mathfrak{z}_{\mathfrak{k}^{\mathbb{C}}}(\mathfrak{a})$ and the flatness of $\mathfrak{a}$ that $\mathfrak{z}_{\mathfrak{k}^{\mathbb{C}}}(\mathfrak{a}) \oplus \mathfrak{a} \subset \mathfrak{g}_0^{\mathbb{C}}$ holds. For the converse direction, let $X \in \mathfrak{g}_0^{\mathbb{C}}$ be given. Then there exist $X_{\mathfrak{k}} \in \mathfrak{k}^{\mathbb{C}}$ and $X_{\mathfrak{m}} \in \mathfrak{m}^{\mathbb{C}}$ with $X = X_{\mathfrak{k}} + X_{\mathfrak{m}}$, and we have for every $Z \in \mathfrak{a}$

$$0 = \mathrm{ad}(Z)X = \underbrace{[Z, X_{\mathfrak{k}}]}_{\in\mathfrak{m}^{\mathbb{C}}} + \underbrace{[Z, X_{\mathfrak{m}}]}_{\in\mathfrak{k}^{\mathbb{C}}} \;,$$

hence $[Z, X_{\mathfrak{k}}] = 0$ and $[Z, X_{\mathfrak{m}}] = 0$. The first equality shows that $X_{\mathfrak{k}} \in \mathfrak{z}_{\mathfrak{k}^{\mathbb{C}}}(\mathfrak{a})$ holds. From the second equality we derive

$$0 = [Z, X_{\mathfrak{m}}] = [Z, \mathrm{Re}(X_{\mathfrak{m}})] + i\,[Z, \mathrm{Im}(X_{\mathfrak{m}})]$$

and therefore

$$[Z, \mathrm{Re}(X_{\mathfrak{m}})] = [Z, \mathrm{Im}(X_{\mathfrak{m}})] = 0\ .$$

Because of $\mathrm{Re}(X_{\mathfrak{m}}), \mathrm{Im}(X_{\mathfrak{m}}) \in \mathfrak{m}$ and the fact that $\mathfrak{a}$ is a *maximal* flat subspace of $\mathfrak{m}$, $\mathrm{Re}(X_{\mathfrak{m}}), \mathrm{Im}(X_{\mathfrak{m}}) \in \mathfrak{a}$ and hence $X_{\mathfrak{m}} \in \mathfrak{a}^{\mathbb{C}}$ holds. Hence we have shown $X = X_{\mathfrak{k}} + X_{\mathfrak{m}} \in \mathfrak{z}_{\mathfrak{k}^{\mathbb{C}}}(\mathfrak{a}) \oplus \mathfrak{a}^{\mathbb{C}}$.

For (c). Let $X \in \mathfrak{g}^{\mathbb{C}}_{\lambda}$ and $Y \in \mathfrak{g}^{\mathbb{C}}_{\mu}$ be given. Then we have for every $Z \in \mathfrak{a}$:

$$\begin{aligned}\mathrm{ad}(Z)([X,Y]) &= [Z,[X,Y]] = -[X,[Y,Z]] - [Y,[Z,X]] = [X, \mathrm{ad}(Z)Y] - [Y, \mathrm{ad}(Z)X] \\ &= [X, i\mu(Z)Y] - [Y, i\lambda(Z)X] = i(\lambda(Z)+\mu(Z))[X,Y]\ .\end{aligned}$$

Thus we see that $[X,Y] \in \mathfrak{g}^{\mathbb{C}}_{\lambda+\mu}$ holds. □

From the preceding results we now derive the promised root space decompositions for the real linear spaces $\mathfrak{k}$ and $\mathfrak{m}$. For this purpose, we put for every $\lambda \in \mathfrak{a}^*$

$$\mathfrak{k}_{\lambda} := (\mathfrak{g}^{\mathbb{C}}_{\lambda} + \mathfrak{g}^{\mathbb{C}}_{-\lambda}) \cap \mathfrak{k} \quad \text{and} \quad \mathfrak{m}_{\lambda} := (\mathfrak{g}^{\mathbb{C}}_{\lambda} + \mathfrak{g}^{\mathbb{C}}_{-\lambda}) \cap \mathfrak{m}\ ; \tag{A.19}$$

obviously the sum of linear spaces $\mathfrak{g}^{\mathbb{C}}_{\lambda} + \mathfrak{g}^{\mathbb{C}}_{-\lambda}$ is direct for $\lambda \neq 0$.

A.10 Proposition. *(a) We have* $\mathfrak{k}_0 = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ *and* $\mathfrak{m}_0 = \mathfrak{a}$. *Here*

$$\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) = \{\, X \in \mathfrak{k} \,|\, [X, \mathfrak{a}] = \{0\} \,\}$$

is the centralizer of $\mathfrak{a}$ *in* $\mathfrak{k}$.

(b) There exist subsets $\Delta_+ \subset \Delta$ *so that*

$$\Delta_+ \cup (-\Delta_+) = \Delta \quad \textit{and} \quad \Delta_+ \cap (-\Delta_+) = \varnothing$$

holds. We call any such set Δ_+ *a* set of positive roots *of* $\mathfrak{m}$.

(c) Let $\Delta_+ \subset \Delta$ *be a set of positive roots. Then we have*

$$\mathfrak{k} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \;\oplus\; \bigoplus_{\lambda \in \Delta_+} \mathfrak{k}_{\lambda} \quad \textit{and} \quad \mathfrak{m} = \mathfrak{a} \;\oplus\; \bigoplus_{\lambda \in \Delta_+} \mathfrak{m}_{\lambda}\ . \tag{A.20}$$

The decomposition in (A.20) *is called the* root space decomposition *of* $\mathfrak{k}$ *resp.* $\mathfrak{m}$ *with respect to* $\mathfrak{a}$*; the elements of* Δ *are also called* roots of $\mathfrak{k}$ resp. $\mathfrak{m}$ *with respect to* $\mathfrak{a}$.

(d) For $\lambda \in \mathfrak{a}^*$ *we have*

$$\mathfrak{k}_{\lambda} = \{\, X \in \mathfrak{k} \,|\, \forall Z \in \mathfrak{a} : \mathrm{ad}(Z)^2 X = -\lambda(Z)^2\, X \,\} \tag{A.21}$$

$$\textit{and} \quad \mathfrak{m}_{\lambda} = \{\, X \in \mathfrak{m} \,|\, \forall Z \in \mathfrak{a} : \mathrm{ad}(Z)^2 X = -\lambda(Z)^2\, X \,\}\ . \tag{A.22}$$

(e) $\Delta = \{\, \lambda \in \mathfrak{a}^* \setminus \{0\} \,|\, \mathfrak{m}_{\lambda} \neq \{0\} \,\}$.

Proof. For (a). By Equation (A.19) and Proposition A.9(b) we have $\mathfrak{k}_0 = \mathfrak{g}_0^{\mathbb{C}} \cap \mathfrak{k} = (\mathfrak{z}_{\mathfrak{k}^{\mathbb{C}}}(\mathfrak{a}) \oplus \mathfrak{a}^{\mathbb{C}}) \cap \mathfrak{k} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ and $\mathfrak{m}_0 = \mathfrak{g}_0^{\mathbb{C}} \cap \mathfrak{m} = (\mathfrak{z}_{\mathfrak{k}^{\mathbb{C}}}(\mathfrak{a}) \oplus \mathfrak{a}^{\mathbb{C}}) \cap \mathfrak{m} = \mathfrak{a}$.

For (b). This is simply a consequence of the fact that we have $(\lambda \in \Delta \Leftrightarrow -\lambda \in \Delta)$ for every $\lambda \in \mathfrak{a}^*$, see Proposition A.9(a).

For (c). We prove the decomposition for $\mathfrak{k}$; the decomposition for $\mathfrak{m}$ is verified analogously. Because of (a), the decomposition equation to be proved for $\mathfrak{k}$ is equivalent to

$$\mathfrak{k} = \bigoplus_{\lambda \in \Delta_+ \cup \{0\}} \mathfrak{k}_\lambda \,. \tag{A.23}$$

It is clear from the definition of $\mathfrak{k}_\lambda$ that the sum in (A.23) is indeed direct and that the inclusion "$\supset$" holds. For the converse inclusion, we let $X \in \mathfrak{k}$ be given. By the decomposition equation (A.18) there exist $X_\lambda \in \mathfrak{g}_\lambda^{\mathbb{C}}$ for $\lambda \in \Delta \cup \{0\}$ so that

$$X = X_0 + \sum_{\lambda \in \Delta} X_\lambda$$

holds. Because of $X \in \mathfrak{k}$, we have $\overline{X} = X = \sigma_L^{\mathbb{C}}(X)$ and therefore

$$\begin{aligned}
4\,X &= X + \overline{X} + \sigma_L^{\mathbb{C}}(X) + \sigma_L^{\mathbb{C}}(\overline{X}) \\
&= \underbrace{X_0 + \overline{X_0} + \sigma_L^{\mathbb{C}}(X_0) + \sigma_L^{\mathbb{C}}(\overline{X_0})}_{=:Z_0} + \sum_{\lambda \in \Delta} \big(X_\lambda + \overline{X_\lambda} + \sigma_L^{\mathbb{C}}(X_\lambda) + \sigma_L^{\mathbb{C}}(\overline{X_\lambda}) \big) \\
&= Z_0 + \sum_{\lambda \in \Delta_+} \big(\underbrace{X_\lambda + \sigma_L^{\mathbb{C}}(\overline{X_\lambda}) + \overline{X_{-\lambda}} + \sigma_L^{\mathbb{C}}(X_{-\lambda})}_{=:Y_\lambda} \big) + \sum_{\lambda \in \Delta_+} \big(\underbrace{\overline{X_\lambda} + \sigma_L^{\mathbb{C}}(X_\lambda) + X_{-\lambda} + \sigma_L^{\mathbb{C}}(\overline{X_{-\lambda}})}_{=:Y_{-\lambda}} \big) \\
&= Z_0 + \sum_{\lambda \in \Delta_+} \big(Y_\lambda + Y_{-\lambda} \big)\,.
\end{aligned}$$

By Proposition A.9(a) we have $Z_0 \in \mathfrak{g}_0^{\mathbb{C}}$ and $Y_\lambda \in \mathfrak{g}_\lambda^{\mathbb{C}}$ for $\lambda \in \Delta$. Moreover we see that $\overline{Z_0} = Z_0 = \sigma_L^{\mathbb{C}}(Z_0)$, hence $Z_0 \in \mathfrak{k}$, and $\overline{Y_\lambda} = Y_{-\lambda} = \sigma_L^{\mathbb{C}}(Y_\lambda)$ holds. It follows by Equation (A.19) that we have $Z_0 \in \mathfrak{k}_0$ and $Z_\lambda := Y_\lambda + Y_{-\lambda} \in \mathfrak{k}_\lambda$ for $\lambda \in \Delta_+$. Because we have

$$X = \tfrac{1}{4} \sum_{\lambda \in \Delta_+ \cup \{0\}} Z_\lambda\,,$$

this completes the proof of Equation (A.23).

For (d). We prove Equation (A.21); Equation (A.22) is again shown analogously. In Equation (A.21), the inclusion "$\subset$" is obvious. For the converse inclusion, we let $\lambda \in \mathfrak{a}^*$ and $X \in \mathfrak{k}$ be given so that

$$\forall Z \in \mathfrak{a} \;:\; \mathrm{ad}(Z)^2 X = -\lambda(Z)^2\, X \tag{A.24}$$

holds. We fix a system of positive roots $\Delta_+ \subset \Delta$, then by (c) there exist $X_\mu \in \mathfrak{k}_\mu$ for $\mu \in \Delta_+ \cup \{0\}$ so that

$$X = \sum_{\mu \in \Delta_+ \cup \{0\}} X_\mu \tag{A.25}$$

holds. We now calculate $\mathrm{ad}(Z)^2 X$ for $Z \in \mathfrak{a}$ in two different ways:

$$\mathrm{ad}(Z)^2 X \begin{cases} \overset{(A.25)}{=} \mathrm{ad}(Z)^2 \big(\sum\limits_{\mu \in \Delta_+ \cup \{0\}} X_\mu \big) = \sum\limits_{\mu \in \Delta_+ \cup \{0\}} \mathrm{ad}(Z)^2 X_\mu = - \sum\limits_{\mu \in \Delta_+ \cup \{0\}} \mu(Z)^2\, X_\mu \\ \overset{(A.24)}{=} -\lambda(Z)^2\, X \overset{(A.25)}{=} - \sum\limits_{\mu \in \Delta_+ \cup \{0\}} \lambda(Z)^2\, X_\mu\,. \end{cases}$$

Because of the directness of the sum in (A.20), this calculation shows that whenever $X_\mu \neq 0$ holds for some $\mu \in \Delta_+ \cup \{0\}$, we have $\mu(Z)^2 = \lambda(Z)^2$ for every $Z \in \mathfrak{a}$ and therefore it is easy to prove that $\mu = \pm\lambda$. Thus Equation (A.25) shows that $X \in \mathfrak{k}_\lambda$ holds.

For (e). Let $\lambda \in \mathfrak{a}^* \setminus \{0\}$ be given. If $\{0\} \neq \mathfrak{m}_\lambda = (\mathfrak{g}_\lambda^{\mathbb{C}} \oplus \mathfrak{g}_{-\lambda}^{\mathbb{C}}) \cap \mathfrak{m}$ holds, then we have $\mathfrak{g}_\lambda^{\mathbb{C}} \oplus \mathfrak{g}_{-\lambda}^{\mathbb{C}} \neq \{0\}$ and therefore (because of $\mathfrak{g}_{-\lambda}^{\mathbb{C}} = \overline{\mathfrak{g}_\lambda^{\mathbb{C}}}$, see Proposition A.9(a)) $\mathfrak{g}_\lambda^{\mathbb{C}} \neq \{0\}$. Therefrom $\lambda \in \Delta$ follows by Equation (A.17).

Conversely, we suppose that $\lambda \in \Delta$ holds, then we have $\mathfrak{g}^{\mathbb{C}}_\lambda \neq \{0\}$ by Equation (A.17). We fix $X \in \mathfrak{g}^{\mathbb{C}}_\lambda \setminus \{0\}$ and consider the elements

$$Y_1 := X + \overline{X} - \sigma^{\mathbb{C}}_L(X) - \sigma^{\mathbb{C}}_L(\overline{X}) \quad \text{and} \quad Y_2 := i \cdot (X - \overline{X} - \sigma^{\mathbb{C}}_L(X) + \sigma^{\mathbb{C}}_L(\overline{X})) \, .$$

Y_1 and Y_2 are $\overline{\square}$-invariant and therefore members of $\mathfrak{g}$; they also satisfy $\sigma_L(Y_k) = -Y_k$ and therefore we have $Y_1, Y_2 \in \mathfrak{m}$. By Proposition A.9(a) we moreover have $Y_1, Y_2 \in \mathfrak{g}^{\mathbb{C}}_\lambda \oplus \mathfrak{g}^{\mathbb{C}}_{-\lambda}$ and therefore $Y_1, Y_2 \in \mathfrak{m}_\lambda$ by Equation (A.19). We will now show that Y_1 and Y_2 cannot simultaneously be zero; therefrom it follows that $\mathfrak{m}_\lambda \neq \{0\}$ holds.

Assume to the contrary that $Y_1 = Y_2 = 0$ holds. We thus have

$$X + \overline{X} - \sigma^{\mathbb{C}}_L(X) - \sigma^{\mathbb{C}}_L(\overline{X}) = 0 = -\,(X - \overline{X} - \sigma^{\mathbb{C}}_L(X) + \sigma^{\mathbb{C}}_L(\overline{X}))$$

and therefore $X = \sigma^{\mathbb{C}}_L(X)$, whence $X \in \mathfrak{g}^{\mathbb{C}}_\lambda \cap \mathfrak{g}^{\mathbb{C}}_{-\lambda} = \{0\}$ follows because of Proposition A.9(a). This is a contradiction to $X \neq 0$. □

At the beginning of the present subsection, we motivated our investigation of the roots and root spaces by the relationship of these objects to the eigenvalues and eigenspaces of the Jacobi operators. We now make this relationship explicit:

A.11 Proposition. *For any $Z \in \mathfrak{a}$ we put $R_Z := -\operatorname{ad}(Z)^2|\mathfrak{m} : \mathfrak{m} \to \mathfrak{m}$ (under the canonical isomorphism $\tau : \mathfrak{m} \to T_{p_0}M$ this endomorphism is conjugate to a Jacobi operator of M, see Equation* (A.13)*), and for any function $\mu : \mathfrak{a} \to \mathbb{R}$ we define*[32]

$$E_\mu := \bigcap_{Z \in \mathfrak{a}} \operatorname{Eig}(R_Z, \mu(Z)) = \{\, X \in \mathfrak{m} \,|\, \forall Z \in \mathfrak{a} : \operatorname{ad}(Z)^2 X = -\mu(Z)\, X \,\} \tag{A.26}$$

and

$$\Sigma := \{\, \mu : \mathfrak{a} \to \mathbb{R} \,|\, E_\mu \neq \{0\} \,\} \, . \tag{A.27}$$

(a) Σ is related to the root system Δ by

$$\Sigma = \{\, \lambda^2 \,|\, \lambda \in \Delta \,\} \dot\cup \{0\} \tag{A.28}$$

and conversely

$$\Delta = \{\, \lambda \in \mathfrak{a}^* \setminus \{0\} \,|\, \lambda^2 \in \Sigma \,\} \, . \tag{A.29}$$

The spaces E_μ are related to the root spaces of $\mathfrak{m}$ by

$$\forall \lambda \in \mathfrak{a}^* \; : \; \mathfrak{m}_\lambda = E_{\lambda^2} \, , \tag{A.30}$$

in particular $E_0 = \mathfrak{a} \neq \{0\}$.

(b) Σ is a finite set and we have

$$\mathfrak{m} = \bigoplus_{\mu \in \Sigma} E_\mu \, . \tag{A.31}$$

[32]For Equation (A.26) remember that we use the notation $\operatorname{Eig}(B, \lambda) := \ker(B - \lambda \operatorname{id})$ even when λ is not an eigenvalue of B, compare Section 0.2.

Proof. The family $(R_Z)_{Z\in\mathfrak{a}}$ consists of pairwise commuting, self-adjoint endomorphisms of $\mathfrak{m}$, and therefore the spaces E_μ are pairwise orthogonal. Therefrom it follows that Σ is finite and that the sum in Equation (A.31) is indeed orthogonally direct.

By Proposition A.10(d) we have for every $\lambda \in \mathfrak{a}^*$

$$\mathfrak{m}_\lambda = \bigcap_{Z\in\mathfrak{a}} \mathrm{Eig}(R_Z, \lambda(Z)^2)\ ;$$

therefore Equation (A.26) implies Equation (A.30). From Equation (A.30) we derive that the inclusion "$\supset$" in Equation (A.28) holds, and therefore we have

$$\mathfrak{m} \overset{(*)}{=} \mathfrak{a} \oplus \bigoplus_{\lambda\in\Delta_+} \mathfrak{m}_\lambda \overset{(A.30)}{=} E_0 \oplus \bigoplus_{\lambda\in\Delta_+} E_{\lambda^2} \subset \bigoplus_{\mu\in\Sigma} E_\mu \subset \mathfrak{m}$$

(where $\Delta_+ \subset \Delta$ is a positive root system; for the equals sign marked $(*)$ see Proposition A.10(c)). Therefrom it follows that Equation (A.31) holds.

From Equation (A.31) we see in the following way that also the inclusion "$\subset$" in Equation (A.28) holds: Assume that there existed some $\mu_0 \in \Sigma \setminus (\{\,\lambda^2 \,|\, \lambda \in \Delta\,\} \cup \{0\})$. Then $E_{\mu_0} \subset \mathfrak{m}$ is perpendicular to E_μ for every $\mu \in \{\,\lambda^2 \,|\, \lambda \in \Delta\,\} \cup \{0\}$ and therefore to

$$E_0 \oplus \bigoplus_{\lambda\in\Delta_+} E_{\lambda^2} \overset{(A.30)}{=} \mathfrak{a} \oplus \bigoplus_{\lambda\in\Delta_+} \mathfrak{m}_\lambda = \mathfrak{m}\ .$$

Therefore $E_{\mu_0} = \{0\}$ holds, in contradiction to $\mu_0 \in \Sigma$.

Finally, we show Equation (A.29). Its inclusion "$\subset$" is an immediate consequence of Equation (A.28). Conversely, we let $\lambda \in \mathfrak{a}^* \setminus \{0\}$ be given such that $\lambda^2 \in \Sigma$ holds. Then we have $\mathfrak{m}_\lambda = E_{\lambda^2} \neq \{0\}$ and therefore $\lambda \in \Delta$ by Proposition A.10(e). □

Related vectors.

A.12 Definition. *For $\lambda \in \Delta$, elements $X \in \mathfrak{m}_\lambda$ and $\widehat{X} \in \mathfrak{k}_\lambda$ are called* related *to each other if*

$$\forall Z \in \mathfrak{a} \ :\ \big(\ \mathrm{ad}(Z)X = \lambda(Z)\cdot\widehat{X} \quad \textit{and} \quad \mathrm{ad}(Z)\widehat{X} = -\lambda(Z)\cdot X\ \big)$$

holds.

A.13 Proposition. *We fix $\lambda \in \Delta$.*

(a) For every $X \in \mathfrak{m}_\lambda$ there exists one and only one $\widehat{X} \in \mathfrak{k}_\lambda$ which is related to X, and the map $\Phi_\lambda : \mathfrak{m}_\lambda \to \mathfrak{k}_\lambda,\ X \mapsto \widehat{X}$ is an isomorphism of linear spaces. In particular we have $\dim(\mathfrak{k}_\lambda) = \dim(\mathfrak{m}_\lambda)$*; this number is called the* multiplicity *of the root λ and is denoted by n_λ.*

(b) We have

$$\forall X \in \mathfrak{m}_\lambda \ :\ [X, \Phi_\lambda(X)] = \langle X, X\rangle \cdot \lambda^\sharp\ ,$$

where $\lambda^\sharp \in \mathfrak{a}$ is the Riesz vector of λ, i.e. the vector uniquely characterized by $\langle\cdot,\lambda^\sharp\rangle|\mathfrak{a} = \lambda$.

Proof. For (a). Let us consider the anti-linear involution $\tau : \mathfrak{g}_\lambda^{\mathbb{C}} \to \mathfrak{g}_\lambda^{\mathbb{C}},\ H \mapsto \sigma_L^{\mathbb{C}}(\overline{H})$ (see Proposition A.9(a)). Putting $\mathfrak{g}_\lambda^{\pm} := \mathrm{Eig}(\tau, \pm 1)$, we then have

$$\mathfrak{g}_\lambda^{\mathbb{C}} = \mathfrak{g}_\lambda^{+} \oplus \mathfrak{g}_\lambda^{-} \quad \text{and} \quad \mathfrak{g}_\lambda^{-} = i\,\mathfrak{g}_\lambda^{+} \; .$$

We now show

$$\forall\, X, \widehat{X} \in \mathfrak{g} \;:\; \left(\left\{ \begin{array}{c} X \in \mathfrak{m}_\lambda \text{ and } \widehat{X} \in \mathfrak{k}_\lambda \text{ holds,} \\ \text{and } X \text{ and } \widehat{X} \text{ are related to each other} \end{array} \right\} \iff \widehat{X} + iX \in \mathfrak{g}_\lambda^{+} \right) \; . \tag{A.32}$$

For the proof of (A.32), we first suppose that $X \in \mathfrak{m}_\lambda$ and $\widehat{X} \in \mathfrak{k}_\lambda$ are related to each other. Then we have for every $Z \in \mathfrak{a}$

$$\mathrm{ad}(Z)(\widehat{X} + iX) = \mathrm{ad}(Z)\widehat{X} + i\;\mathrm{ad}(Z)X = -\lambda(Z)\,X + i\,\lambda(Z)\,\widehat{X} = i\,\lambda(Z)\,(\widehat{X} + iX)$$

and therefore $\widehat{X} + iX \in \mathfrak{g}_\lambda^{\mathbb{C}}$. Moreover, we have (because of $X, \widehat{X} \in \mathfrak{g}$) $\overline{X} = X$, $\overline{\widehat{X}} = \widehat{X}$ and (because of $X \in \mathfrak{m}$, $\widehat{X} \in \mathfrak{k}$) $\sigma_L(X) = -X$, $\sigma_L(\widehat{X}) = \widehat{X}$, whence we obtain

$$\tau(\widehat{X} + iX) = \sigma_L^{\mathbb{C}}(\overline{\widehat{X}}) - i\,\sigma_L^{\mathbb{C}}(\overline{X}) = \widehat{X} + iX$$

and therefore $\widehat{X} + iX \in \mathfrak{g}_\lambda^{+}$.

For the converse direction, we let $\widehat{X}, X \in \mathfrak{g}$ be given so that $H := \widehat{X} + iX \in \mathfrak{g}_\lambda^{+}$ holds. Then we have

$$\widehat{X} + iX = H = \tau(H) = \sigma_L^{\mathbb{C}}(\overline{H}) = \sigma_L(\widehat{X}) - i\,\sigma_L(X)$$

and therefore $\sigma_L(\widehat{X}) = \widehat{X}$, $\sigma_L(X) = -X$. This shows that $\widehat{X} \in \mathfrak{k}$ and $X \in \mathfrak{m}$ holds. Moreover, because of $H \in \mathfrak{g}_\lambda^{+} \subset \mathfrak{g}_\lambda^{\mathbb{C}}$ we have for every $Z \in \mathfrak{a}$

$$\mathrm{ad}(Z)\widehat{X} + i\;\mathrm{ad}(Z)X = \mathrm{ad}(Z)H = i\,\lambda(Z)\,H = i\,\lambda(Z)\,(\widehat{X} + iX) = -\lambda(Z)\,X + i\,\lambda(Z)\,\widehat{X}$$

and therefore

$$\mathrm{ad}(Z)\widehat{X} = -\lambda(Z)\,X \quad \text{and} \quad \mathrm{ad}(Z)X = \lambda(Z)\,\widehat{X} \; . \tag{A.33}$$

Equations (A.33) show that

$$\mathrm{ad}(Z)^2\,\widehat{X} = -\lambda(Z)^2\,\widehat{X} \quad \text{and} \quad \mathrm{ad}(Z)^2\,X = -\lambda(Z)^2\,X$$

holds, whence $\widehat{X} \in \mathfrak{k}_\lambda$ and $X \in \mathfrak{m}_\lambda$ follows by Proposition A.10(d). Equations (A.33) now show that X and $\widehat{X}$ are related to each other. This completes the proof of (A.32).

(A.32) shows in particular that for every $H \in \mathfrak{g}_\lambda^{+}$ we have $\mathrm{Re}(H) \in \mathfrak{k}_\lambda$ and $\mathrm{Im}(H) \in \mathfrak{m}_\lambda$, so that we may consider the $\mathbb{R}$-linear maps

$$\mathcal{R} := (\mathrm{Re}\,|\mathfrak{g}_\lambda^{+} : \mathfrak{g}_\lambda^{+} \to \mathfrak{k}_\lambda) \quad \text{and} \quad \mathcal{I} := (\mathrm{Im}\,|\mathfrak{g}_\lambda^{+} : \mathfrak{g}_\lambda^{+} \to \mathfrak{m}_\lambda) \; .$$

Immediately, we will show that $\mathcal{R}$ and $\mathcal{I}$ are linear isomorphisms. Therefrom follows the existence and uniqueness statement of (a) because of (A.32) ; moreover it follows that $\Phi_\lambda = \mathcal{R} \circ \mathcal{I}^{-1} : \mathfrak{m}_\lambda \to \mathfrak{k}_\lambda$ is a linear isomorphism and hence $\dim(\mathfrak{k}_\lambda) = \dim(\mathfrak{m}_\lambda)$ holds.

We now show that $\mathcal{R}$ is a linear isomorphism; analogously one shows that also $\mathcal{I}$ is a linear isomorphism. For the proof of the injectivity of $\mathcal{R}$, suppose that $H \in \mathfrak{g}_\lambda^{+}$ is given such that $\mathcal{R}(H) = 0$ holds. Thus we have $H = iX$ with some $X \in \mathfrak{g}$. By (A.32) we have $X \in \mathfrak{m}_\lambda$ and X is related to $0 \in \mathfrak{k}_\lambda$. Therefore we have for every $Z \in \mathfrak{a}$

$$0 = \mathrm{ad}(Z)\,0 = -\lambda(Z)\,X \; .$$

Because of $\lambda \neq 0$, therefrom $X = 0$ and thus $H = 0$ follows.

For the proof of the surjectivity of $\mathcal{R}$, let $\widehat{X} \in \mathfrak{k}_\lambda$ be given. By definition of $\mathfrak{k}_\lambda$, there exist $\widehat{X}_+ \in \mathfrak{g}_\lambda^{\mathbb{C}}$ and $\widehat{X}_- \in \mathfrak{g}_{-\lambda}^{\mathbb{C}}$ so that $\widehat{X} = \widehat{X}_+ + \widehat{X}_-$ holds. We have $\overline{\widehat{X}} = \widehat{X}$ (because of $\widehat{X} \in \mathfrak{g}$); because the involution $\overline{\square}$ interchanges the spaces $\mathfrak{g}_\lambda^{\mathbb{C}}$ and $\mathfrak{g}_{-\lambda}^{\mathbb{C}}$ (Proposition A.9(a)) and we have $\mathfrak{g}_\lambda^{\mathbb{C}} \cap \mathfrak{g}_{-\lambda}^{\mathbb{C}} = \{0\}$, it follows that

$$\overline{\widehat{X}_\pm} = \widehat{X}_\mp \tag{A.34}$$

holds. Also, because of $\widehat{X} \in \mathfrak{k}$ we have $\sigma_L^{\mathbb{C}}(\widehat{X}) = \widehat{X}$, whence we deduce similarly

$$\sigma_L^{\mathbb{C}}(\widehat{X}_\pm) = \widehat{X}_\mp \ . \tag{A.35}$$

We now put

$$X := (-i)\,(\widehat{X}_+ - \widehat{X}_-) \ .$$

Then we have

$$\overline{X} = i\,(\overline{\widehat{X}_+} - \overline{\widehat{X}_-}) \overset{(A.34)}{=} i\,(\widehat{X}_- - \widehat{X}_+) = X$$

and therefore $X \in \mathfrak{g}$. We have $H := \widehat{X} + iX = 2\,\widehat{X}_+ \in \mathfrak{g}_\lambda^{\mathbb{C}}$ and

$$\tau(H) = 2\,\tau(\widehat{X}_+) = 2\,\sigma_L^{\mathbb{C}}(\overline{\widehat{X}_+}) \overset{(A.34)}{=} 2\,\sigma_L^{\mathbb{C}}(\widehat{X}_-) \overset{(A.35)}{=} 2\,\widehat{X}_+ = H\ ,$$

and therefore $H \in \mathfrak{g}_\lambda^+$. Clearly, $\mathcal{R}(H) = \widehat{X}$ holds.

For (b). Let $X \in \mathfrak{m}_\lambda$ be given and put $\widehat{X} := \Phi_\lambda(X) \in \mathfrak{k}_\lambda$. We have $[X, \widehat{X}] \in \mathfrak{m}$. Also we have $H := \widehat{X} + i\,X \in \mathfrak{g}_\lambda^+ \subset \mathfrak{g}_\lambda^{\mathbb{C}}$ by (A.32) and therefore $\overline{H} = \widehat{X} - i\,X \in \mathfrak{g}_{-\lambda}^{\mathbb{C}}$ by Proposition A.9(a), hence we have $[H, \overline{H}] \in \mathfrak{g}_0^{\mathbb{C}}$ by Proposition A.9(c). Because we have

$$[H, \overline{H}] = [\widehat{X} + i\,X, \widehat{X} - i\,X] = 2i\,[X, \widehat{X}]$$

it follows that we have $[X, \widehat{X}] \in \mathfrak{g}_0^{\mathbb{C}}$. Thus we have shown $[X, \widehat{X}] \in \mathfrak{g}_0^{\mathbb{C}} \cap \mathfrak{m} = \mathfrak{a}$ (see Proposition A.9(b)). Now we have for every $Z \in \mathfrak{a}$ (because $\langle \cdot, \cdot \rangle$ is defined via the Killing form $\varkappa$)

$$\langle [X, \widehat{X}], Z\rangle = -\langle X, [Z, \widehat{X}]\rangle = -\langle X, -\lambda(Z)\,X\rangle = \langle\,\langle X, X\rangle \lambda^\sharp\,, Z\rangle\ .$$

Therefrom the statement follows. □

The Weyl group.

A.14 Definition. *For $\lambda \in \Delta$ we denote by $R_\lambda : \mathfrak{a} \to \mathfrak{a}$ the orthogonal reflection in the hyperplane $\lambda^{-1}(\{0\})$. Then the group of orthogonal transformations of $\mathfrak{a}$ generated by $\{\, R_\lambda \,|\, \lambda \in \Delta\,\}$ is called the* Weyl group W. *Its elements are called* Weyl transformations. *The Weyl group also acts on $\mathfrak{a}^*$ via the action $W \times \mathfrak{a}^* \to \mathfrak{a}^*,\ (g, \lambda) \mapsto \lambda \circ g^{-1}$.*

We note that we have

$$\forall\, \lambda \in \mathfrak{a}^*,\ g \in W \ :\ (\lambda \circ g)^\sharp = g(\lambda^\sharp)\ , \tag{A.36}$$

and therefore the actions of the Weyl group W on $\mathfrak{a}^*$ and $\mathfrak{a}$ correspond to each other under the map $\mathfrak{a}^* \to \mathfrak{a},\ \lambda \mapsto \lambda^\sharp$.

A.15 Proposition. *(a) Let us denote the isotropy group of the action of G on M at the origin point p_0 by K. Then for every $g \in W$ there exists $\widehat{g} \in K$ so that $g = \mathrm{Ad}(\widehat{g})|\mathfrak{a}$ holds.*

For $g = R_\lambda$ with $\lambda \in \Delta$ such a $\widehat{g} \in K$ can be given in the following way: Let $X \in \mathbb{S}(\mathfrak{m}_\lambda)$ be given and let $\widehat{X} \in \mathfrak{k}_\lambda \setminus \{0\}$ be related to X. Then

$$\widehat{g} := \mathrm{Exp}(\tfrac{\pi}{\|\lambda^\sharp\|} \cdot \widehat{X}\,) \ \in\ K \qquad\qquad (\pi = 3.14\ldots)$$

has the property $\mathrm{Ad}(\widehat{g})|\mathfrak{a} = R_\lambda$, where $\mathrm{Exp} : \mathfrak{k} \to K$ is the exponential map of K.

(b) For any $g \in W$ *we have*

$$\forall \lambda \in \mathfrak{a}^* \ : \big(\, \lambda \circ g^{-1} \in \Delta \Longleftrightarrow \lambda \in \Delta \quad \textit{and} \quad \mathfrak{m}_{\lambda \circ g^{-1}} = \mathrm{Ad}(\widehat{g})\mathfrak{m}_\lambda \, \big) \; ; \tag{A.37}$$

here $\widehat{g} \in K$ *is chosen so that* $g = \mathrm{Ad}(\widehat{g})|\mathfrak{a}$ *holds (see (a)).*

Therefore W *leaves* Δ *invariant.*

Proof. For (a). (See [Loo69]: Proposition VI.2.2, p. 67 and the definition of the Weyl group on p. 64; for the following proof, also see Lemma VI.1.5(c), p. 62.)

It suffices to prove the second part of (a). For this we let λ, X and $\widehat{X}$ be as in the proposition. Then we consider the 1-parameter subgroup $t \mapsto \mathrm{Exp}(t\widehat{X})$ of K tangential to $\widehat{X}$ and the induced 1-parameter subgroup $t \mapsto \mathrm{Ad}(\mathrm{Exp}(t\widehat{X}))|\mathfrak{m}$ of $\mathrm{O}(\mathfrak{m})$. We study the action of the latter 1-parameter subgroup on the linear space $\mathfrak{a}$; specifically we will show

$$\forall t \in \mathrm{I\!R} \ : \ \mathrm{Ad}(\,\mathrm{Exp}(t\widehat{X})\,)\lambda^\sharp = \cos(\,t\|\lambda^\sharp\|\,) \cdot \lambda^\sharp + \sin(\,t\|\lambda^\sharp\|\,)\,\|\lambda^\sharp\| \cdot X \tag{A.38}$$

and

$$\forall t \in \mathrm{I\!R} \ : \ \mathrm{Ad}(\,\mathrm{Exp}(t\widehat{X})\,)|(\ker\lambda) = \mathrm{id}_{\ker\lambda} \, . \tag{A.39}$$

By plugging $t := \frac{\pi}{\|\lambda^\sharp\|}$ into these equations, $\mathrm{Ad}(\widehat{g})|\mathfrak{a} = R_\lambda$ follows.

For the proof of Equations (A.38) and (A.39): Let $Z \in \mathfrak{a}$ be given. Then we have by Definition A.12

$$\mathrm{ad}(\widehat{X})Z = -\,\mathrm{ad}(Z)\widehat{X} = \lambda(Z) \cdot X \tag{A.40}$$

and by Proposition A.13(b)

$$\mathrm{ad}(\widehat{X})^2 Z \overset{(A.40)}{=} \lambda(Z) \cdot [\widehat{X}, X] = -\lambda(Z) \cdot \lambda^\sharp \, . \tag{A.41}$$

In particular we have

$$\mathrm{ad}(\widehat{X})\lambda^\sharp = \|\lambda^\sharp\|^2 \cdot X \quad \text{and} \quad \mathrm{ad}(\widehat{X})^2\lambda^\sharp = -\|\lambda^\sharp\|^2 \cdot \lambda^\sharp \, . \tag{A.42}$$

We therefore have for $n \geq 1$

$$\mathrm{ad}(\widehat{X})^{2n} Z = \mathrm{ad}(\widehat{X})^{2n-2}\,\mathrm{ad}(\widehat{X})^2 Z \overset{(A.41)}{=} -\lambda(Z) \cdot \mathrm{ad}(\widehat{X})^{2n-2}\lambda^\sharp \overset{(A.42)}{=} (-1)^n\,\lambda(Z)\,\|\lambda^\sharp\|^{2n-2} \cdot \lambda^\sharp \tag{A.43}$$

and

$$\mathrm{ad}(\widehat{X})^{2n+1} Z = \mathrm{ad}(\widehat{X})\,\mathrm{ad}(\widehat{X})^{2n} Z \overset{(A.43)}{=} (-1)^n\,\lambda(Z)\,\|\lambda^\sharp\|^{2n-2} \cdot \mathrm{ad}(\widehat{X})\lambda^\sharp \overset{(A.42)}{=} (-1)^n\,\lambda(Z)\,\|\lambda^\sharp\|^{2n} \cdot X \; ; \tag{A.44}$$

by comparison with (A.40) we see that Equation (A.44) is also true for $n = 0$.

If we now denote by $\exp : \mathrm{End}(\mathfrak{g}) \to \mathrm{GL}(\mathfrak{g})$ the usual exponential map of endomorphisms, we have for any $t \in \mathrm{I\!R}$

$$\begin{aligned}
\mathrm{Ad}(\,\mathrm{Exp}(t\widehat{X})\,)Z &= \exp(\,\mathrm{ad}(t\widehat{X})\,)Z = \sum_{n=0}^{\infty} \tfrac{1}{n!}\,\mathrm{ad}(t\widehat{X})^n Z = \sum_{n=0}^{\infty} \tfrac{t^n}{n!}\,\mathrm{ad}(\widehat{X})^n Z \\
&= \left(Z + \sum_{n=1}^{\infty} \tfrac{t^{2n}}{(2n)!}\,\mathrm{ad}(\widehat{X})^{2n} Z \right) + \left(\sum_{n=0}^{\infty} \tfrac{t^{2n+1}}{(2n+1)!}\,\mathrm{ad}(\widehat{X})^{2n+1} Z \right) \\
&\overset{(A.43)}{\underset{(A.44)}{=}} \left(Z + \tfrac{\lambda(Z)}{\|\lambda^\sharp\|^2} \sum_{n=1}^{\infty} \tfrac{(-1)^n\,(t\,\|\lambda^\sharp\|)^{2n}}{(2n)!}\,\lambda^\sharp \right) + \|\lambda^\sharp\| \left(\tfrac{\lambda(Z)}{\|\lambda^\sharp\|^2} \sum_{n=0}^{\infty} \tfrac{(-1)^n\,(t\,\|\lambda^\sharp\|)^{2n+1}}{(2n+1)!}\,X \right)
\end{aligned} \tag{A.45}$$

In the case $Z \in \ker\lambda$, Equation (A.45) shows that we have $\mathrm{Ad}(\,\mathrm{Exp}(t\widehat{X})\,)Z = Z$, and therefore Equation (A.39) holds. Moreover, we have $\lambda(\lambda^\sharp) = \|\lambda^\sharp\|^2$ and thus we obtain Equation (A.38) by plugging $Z = \lambda^\sharp$ into (A.45).

For (b). We have $\mathrm{Ad}(\widehat{g})\mathfrak{a} = \mathfrak{a}$, and we have $\mathrm{Ad}(\widehat{g})\mathfrak{m} = \mathfrak{m}$ by (A.8). For any $\lambda \in \mathfrak{a}^*$ we therefore have by Proposition A.10(d)

$$\begin{aligned}
\mathrm{Ad}(\widehat{g})\mathfrak{m}_\lambda &= \{\, \mathrm{Ad}(\widehat{g})X \mid X \in \mathfrak{m},\ \forall Z \in \mathfrak{a} : \mathrm{ad}(Z)^2 X = -\lambda(Z)^2\, X \,\} \\
&= \{\, X \in \mathrm{Ad}(\widehat{g})\mathfrak{m} \mid \forall Z \in \mathfrak{a} : \mathrm{ad}(Z)^2 (\mathrm{Ad}(\widehat{g})^{-1} X) = -\lambda(Z)^2\, (\mathrm{Ad}(\widehat{g})^{-1} X) \,\} \\
&= \{\, X \in \mathfrak{m} \mid \forall Z \in \mathfrak{a} : \mathrm{Ad}(\widehat{g})^{-1}\big(\ \mathrm{ad}(\mathrm{Ad}(\widehat{g})Z)^2 X\ \big) = \mathrm{Ad}(\widehat{g})^{-1}\big(\ -\lambda(Z)^2 X\ \big) \,\} \\
&= \{\, X \in \mathfrak{m} \mid \forall Z \in \mathfrak{a} : \mathrm{ad}(\mathrm{Ad}(\widehat{g})Z)^2 X = -\lambda(Z)^2 X \,\} \\
&= \{\, X \in \mathfrak{m} \mid \forall Z \in \mathrm{Ad}(\widehat{g})\mathfrak{a} : \mathrm{ad}(Z)^2 X = -(\lambda \circ \mathrm{Ad}(\widehat{g})^{-1})(Z)^2 X \,\} \\
&= \{\, X \in \mathfrak{m} \mid \forall Z \in \mathfrak{a} : \mathrm{ad}(Z)^2 X = -(\lambda \circ g^{-1})(Z)^2 X \,\} = \mathfrak{m}_{\lambda \circ g^{-1}} \ .
\end{aligned}$$

This also shows that $\lambda \in \Delta$ is equivalent to $\lambda \circ g^{-1} \in \Delta$, and therefore Δ is invariant under W. □

Appendix B

The Spin group, its representations and the Principle of Triality

For $n \geq 3$, the fundamental group of $\mathrm{SO}(n)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Consequently, there exists a two-fold Lie group covering map $\chi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$; the universal covering group $\mathrm{Spin}(n)$ is called the *spin group* of $\mathbb{R}^n$. χ is a linear representation of $\mathrm{Spin}(n)$ on $\mathbb{R}^n$, it is called the *vector representation* of $\mathrm{Spin}(n)$. It is clear that any representation of $\mathrm{SO}(n)$ gives rise to an representation of $\mathrm{Spin}(n)$ on the same space via χ. However, as was first noted by È. Cartan, there is a representation ρ of $\mathrm{Spin}(n)$ on a linear space S which does not arise in this way, and which Cartan called the *spin representation* of $\mathrm{Spin}(n)$; in this context S is called the *spinor space*. Analogous considerations can be made in the complex case, leading to the complex spin group $\mathrm{Spin}(n, \mathbb{C})$ and its vector and spin representations. These objects will be constructed in Sections B.4 and B.5 of the present appendix.

In the theory of the spin groups, the case $n = 8$ plays a special role. As will be described in Section B.6, there exists an automorphism of $\mathrm{Spin}(8, \mathbb{C})$ of order 3 which does not descend to an automorphism of $\mathrm{SO}(8, \mathbb{C})$ and which describes an "isomorphy" between the vector representation and the spin representation of $\mathrm{Spin}(8, \mathbb{C})$ in this case.[33] This phenomenon is called the *principle of triality*.

Together with the correspondence between certain spinors and maximal isotropic subspaces described in Section 8.5.2, the principle of triality provides the fundament of the construction of the isomorphism between Q^6 and $\mathrm{SO}(8)/\mathrm{U}(4)$ in Chapter 8.

The basis for the mentioned constructions are the *Clifford algebras*, which are introduced in Section B.3.

The principal sources on which this appendix is based are [LM89], [Che54] and [Rec04].

[33] Such an automorphism can also be constructed in the real case. For a treatment of the real case, see [Har90], Chapter 14, p. 271ff. or [Por95], Chapter 24, p. 256 (the former reference also treats the case where $\mathbb{R}^8$ is equipped with an inner product of signature $(4,4)$). Both references make use of the octonions (Cayley octaves) to describe the triality automorphism. For the positive definite real case, I know of no treatment of triality avoiding them.

Throughout this appendix, we presume all algebras to be associative $\mathbb{K}$-algebras with a unit element non-equal to zero, unless noted otherwise (for example, by the specification "Lie algebra").[34] If A is such an algebra, it contains a subalgebra isomorphic to the field $\mathbb{K}$ in such a way that $1 \in \mathbb{K}$ corresponds to the unit element of A. Therefore it is reasonable to denote the unit of A by 1_A or simply by 1. We also require that any homomorphism $\varphi : A \to A'$ between algebras maps 1_A onto $1_{A'}$, and when we speak of a subalgebra A_M of A generated by some set $M \subset A$, we require $1_A \in A_M$.

B.1 The tensor algebra and the exterior algebra

As a preparation for the introduction of Clifford algebras, we briefly remember fundamental facts about the tensor algebra and the exterior algebra of a linear space V.

The tensor algebra. (See [Lan93], §XVI.7, p. 632ff.) There exists an algebra T which contains $\mathbb{K} \oplus V$ as a linear subspace in such a way that $1 \in \mathbb{K}$ is the unit of T and which has the following universal property:

For every linear map $f : V \to A$ into another algebra A there exists one and only one algebra homomorphism $\psi : T \to A$ with $\psi|V = f$.

If $T, \tilde{T}$ are two such algebras, there exists an algebra isomorphism $\Phi : T \to \tilde{T}$ with $\Phi|(\mathbb{K}\oplus V) = \mathrm{id}_{\mathbb{K}\oplus V}$. Any such algebra is called *(a model of) the tensor algebra* of V. In the sequel, we denote by $\bigotimes V$ a model of the tensor algebra of V, and we denote the product of $x, y \in \bigotimes V$ by $x \otimes y$. As algebra $\bigotimes V$ is generated by V. $\bigotimes V$ is infinite-dimensional if $V \neq \{0\}$; in fact, if $(b_1, \ldots, b_n)$ is a basis of V then

$$\{\, b_{j_1} \otimes \ldots \otimes b_{j_k} \,|\, k \in \mathbb{N}_0,\ j_1, \ldots, j_k \in \{1, \ldots, n\} \,\}$$

is a basis of the linear space $\bigotimes V$ (here we define the "empty product" $b_{j_1} \otimes \ldots \otimes b_{j_0} := 1_{\bigotimes V}$).

The exterior algebra. There exists an algebra S which contains $\mathbb{K} \oplus V$ as a linear subspace in such a way that $1 \in \mathbb{K}$ is the unit of S and that $v \cdot v = 0$ holds for any $v \in V \subset S$, and which has the following universal property:

For every linear map $f : V \to A$ into another algebra A which satisfies $f(v) \cdot f(v) = 0$ for every $v \in V$, there exists one and only one algebra homomorphism $\psi : S \to A$ with $\psi|S = f$.

If $S, \tilde{S}$ are two such algebras, there exists an algebra isomorphism $\Phi : S \to \tilde{S}$ with $\Phi|(\mathbb{K}\oplus V) = \mathrm{id}_{\mathbb{K}\oplus V}$. Any such algebra is called *(a model of) the exterior algebra* of V. In the sequel, we denote by $\bigwedge V$ a model of the exterior algebra of V, and we denote the product of $x, y \in \bigwedge V$

[34] In Section B.6, we will study the triality algebra $(\mathfrak{T}, \diamond)$ which is not associative and does not have a unit element.

by $x \wedge y$. As algebra $\bigwedge V$ is generated by V. If $\dim V = n$ holds, then we have $\dim \bigwedge V = 2^n$; in fact if $(b_1, \ldots, b_n)$ is a basis of V then

$$\{ b_{j_1} \wedge \ldots \wedge b_{j_k} \mid 0 \le k \le n,\ 1 \le j_1 < \ldots < j_k \le n \}$$

is a basis of the linear space $\bigwedge V$ (here we define the "empty product" $b_{j_1} \wedge \ldots \wedge b_{j_0} := 1_{\bigwedge V}$).[35]

For $r \ge 0$, the linear subspace

$$\textstyle\bigwedge^r V := \operatorname{span}\{ v_1 \wedge \ldots \wedge v_r \mid v_1, \ldots, v_r \in V \}$$

of $\bigwedge V$ is called *(a model of) the r-fold exterior product* of V. The elements of $\bigwedge^r V$ are called *r-vectors*; the elements of $\bigwedge^2 V$ are also called *bi*vectors. Note that $\bigwedge^0 V = \mathbb{K}$ and $\bigwedge^1 V = V$ holds, and also that for $2 \le r \le n-1$ not every r-vector is of the form $v_1 \wedge \ldots \wedge v_r$ (the latter are called *decomposable* r-vectors). We have $\dim \bigwedge^r V = \binom{n}{r}$, in particular $\bigwedge^r V = \{0\}$ for $r > n$. To simplify the notation, we put $\bigwedge^{-r} V := \{0\}$. We have $\bigwedge V = \bigoplus_{r=0}^n \bigwedge^r V$, and if a given $\xi \in \bigwedge V$ is decomposed as $\xi = \sum_{r=0}^n \xi_r$ with $\xi_r \in \bigwedge^r V$, then ξ_r is called the *homogeneous component of degree r* of ξ. We note that the product $\wedge$ of $\bigwedge V$ observes the following permutation law:

$$\forall \xi \in \textstyle\bigwedge^r V, \eta \in \bigwedge^s V \ : \ \xi \wedge \eta = (-1)^{rs}\, \eta \wedge \xi \, . \tag{B.1}$$

We also define the linear subspace $\bigwedge^{\mathrm{odd}} V := \bigoplus_{r \in \{ 2r'-1 \mid r' \in \mathbb{N},\, 2r'-1 \le n \}} \bigwedge^r V$ and the subalgebra $\bigwedge^{\mathrm{even}} V := \bigoplus_{r \in \{ 2r' \mid r' \in \mathbb{N},\, 2r' \le n \}} \bigwedge^r V$ of $\bigwedge V$.

As a consequence of the universal property of $\bigwedge V$, we see that for any linear endomorphism $B : V \to V$ there exists one and only one algebra endomorphism $\widetilde{B} : \bigwedge V \to \bigwedge V$ with $\widetilde{B}|V = B$; it satisfies $\widetilde{B}(\bigwedge^r V) \subset \bigwedge^r V$ for all $r \ge 0$. We put $B^{(r)} := \widetilde{B}|\bigwedge^r V : \bigwedge^r V \to \bigwedge^r V$; the linear map $B^{(r)}$ is characterized by

$$\forall v_1, \ldots, v_r \in V \ : \ B^{(r)}(v_1 \wedge \ldots \wedge v_r) = Bv_1 \wedge \ldots \wedge Bv_r \, .$$

B.1 Proposition. (Theorem of Beez) *For any $B_1, B_2 \in \operatorname{End}(V)$ with $\operatorname{rk}(B_1) \ge 3$, $B_1^{(2)} = B_2^{(2)}$ already implies $B_1 = \pm B_2$.*

B.2 The Hodge operator

In this section only, V is an n-dimensional oriented[36] euclidean resp. unitary space. Its inner product $\langle \cdot, \cdot \rangle$ induces an inner product on $\bigwedge V$, which we also denote by $\langle \cdot, \cdot \rangle$ and which is characterized by the following two properties:

(i) $\forall v_1, \ldots, v_n,\ w_1, \ldots, w_n \in V \ : \ \langle v_1 \wedge \ldots \wedge v_n, w_1 \wedge \ldots \wedge w_n \rangle = \det(\langle v_j, w_k \rangle)_{j,k=1,\ldots,n}$.

(ii) $\bigwedge^k V$ and $\bigwedge^\ell V$ are orthogonal to each other for $k \neq \ell$.

[35] As we will see in Section B.3, $\bigwedge V$ is a Clifford algebra for V equipped with the quadratic form which vanishes identically on V. This observation provides an elegant way to derive the existence and uniqueness of the exterior algebra of V, as well as the preceding statements about $\bigwedge V$ from results of Section B.3.

[36] For the concept of an orientation on a complex linear space, see the Introduction.

For any positively oriented orthonormal resp. unitary basis $(a_1, \ldots, a_n)$ of V

$$\omega := a_1 \wedge \ldots \wedge a_n$$

is a unit vector in the one-dimensional space $\bigwedge^n V$, which does not depend on the choice of $(a_1, \ldots, a_n)$. ω is called the *positive unit n-vector* of V.

B.2 Proposition. *Let ω be the positive unit n-vector of V.*

(a) For every $k \in \{0, \ldots, n\}$ and $\eta \in \bigwedge^k V$ there is one and only one $*\eta \in \bigwedge^{n-k} V$ so that

$$\forall \xi \in \bigwedge^k V \;:\; \xi \wedge *\eta = \langle \xi, \eta \rangle \cdot \omega \tag{B.2}$$

holds. The map $* : \bigwedge V \to \bigwedge V$ so defined is bijective. It is linear in the case $\mathbb{K} = \mathbb{R}$, anti-linear[37] in the case $\mathbb{K} = \mathbb{C}$. $*$ is called the Hodge operator *of $\bigwedge V$.*

(b) If $(a_1, \ldots, a_n)$ is a positively oriented orthonormal resp. unitary basis of V, we have for any $k \in \{1, \ldots, n-1\}$

$$*(a_1 \wedge \ldots \wedge a_k) = a_{k+1} \wedge \ldots \wedge a_n \; . \tag{B.3}$$

Additionally, $*1 = \omega$ and $*\omega = 1$ holds.

(c) $\forall k \in \{0, \ldots, n\} \;:\; (* \circ *)|\bigwedge^k V = (-1)^{k(n-k)} \cdot \mathrm{id}_{\bigwedge^k V}$.

(d) $\forall k \in \{0, \ldots, n\}, \xi \in \bigwedge^k V, \eta \in \bigwedge V \;:\; \langle \eta, *\xi \rangle = (-1)^{k(n-k)} \cdot \langle \xi, *\eta \rangle$.

(e) $\forall \xi, \eta \in \bigwedge V \;:\; \langle *\xi, *\eta \rangle = \overline{\langle \xi, \eta \rangle}$. (The conjugation bar is void in the case $\mathbb{K} = \mathbb{R}$.) In particular, if $\mathbb{K} = \mathbb{R}$ holds, the Hodge operator is an orthogonal map, whereas if $\mathbb{K} = \mathbb{C}$ holds, the Hodge operator seen as an $\mathbb{R}$-linear map is orthogonal with respect to the real linear product $\mathrm{Re}\langle \cdot, \cdot \rangle$ on $\bigwedge V$ regarded as an $\mathbb{R}$-linear space.

Proof. For (a). Consider the bilinear form

$$\beta : \bigwedge^k V \times \bigwedge^{n-k} V \to \mathbb{K}, \; (\xi, \zeta) \mapsto \langle \xi \wedge \zeta, \omega \rangle \; . \tag{B.4}$$

β is non-degenerate, therefore the map $\beta^\sharp : \bigwedge^{n-k} V \to \bigwedge^k V$ characterized by

$$\forall \xi \in \bigwedge^k V, \zeta \in \bigwedge^{n-k} V \;:\; \langle \xi, \beta^\sharp(\zeta) \rangle = \beta(\xi, \zeta) \; , \tag{B.5}$$

which is $\mathbb{R}$-linear for $\mathbb{K} = \mathbb{R}$, anti-linear for $\mathbb{K} = \mathbb{C}$, is injective; it is in fact bijective because of $\dim \bigwedge^k V = \dim \bigwedge^{n-k} V$. It follows that for any given $\eta \in \bigwedge^k V$ there exists one and only one $*\eta \in \bigwedge^{n-k} V$ so that $\beta^\sharp(*\eta) = \eta$ holds. For any $\xi \in \bigwedge^k V$ we have $\xi \wedge *\eta \in \bigwedge^n V = \mathbb{K} \cdot \omega$ and thus

$$\xi \wedge *\eta = \langle \xi \wedge *\eta, \omega \rangle \cdot \omega \overset{(B.4)}{=} \beta(\xi, *\eta) \cdot \omega \overset{(B.5)}{=} \langle \xi, \beta^\sharp(*\eta) \rangle \cdot \omega = \langle \xi, \eta \rangle \cdot \omega \; .$$

Because we have $*|\bigwedge^k V = (\beta^\sharp)^{-1}$, it follows that $*$ is a bijective and $\mathbb{R}$-linear resp. anti-linear map.

[37] There are two ways to define the Hodge operator on a complex linear space V, namely such that it becomes $\mathbb{C}$-linear or such that it becomes anti-linear. In both cases it is the extension of the real Hodge operator on a maximal totally-real subspace of V. For our purposes, the anti-linear extension is preferable. For the $\mathbb{C}$-linear extension, see for example [Wei58], p. 18 or [Wel80], p. 155.

For (b). The equations $*1 = \omega$ and $*\omega = 1$ follow immediately from (B.2). For the proof of Equation (B.3), we fix a positively oriented orthonormal resp. unitary basis $(a_1, \ldots, a_n)$ of V and $k \in \{1, \ldots, n-1\}$. Equation (B.2) shows that (B.3) follows from the equation

$$\forall \xi \in \bigwedge^k V \;:\; \xi \wedge a_{k+1} \wedge \ldots \wedge a_n = \langle \xi, a_1 \wedge \ldots \wedge a_k \rangle \cdot \omega \,. \tag{B.6}$$

Because both sides of Equation (B.6) are linear in ξ, it suffices to show (B.6) for $\xi = a_{j_1} \wedge \ldots \wedge a_{j_k}$ with $1 \leq j_1 < \ldots < j_k \leq n$. If any j_ℓ is greater than k, then both sides of (B.6) vanish. Therefore, only the case $\xi = a_1 \wedge \ldots \wedge a_k$ remains, and then both sides of (B.6) are equal to ω.

For (c). It suffices now to show that $*(*\xi) = (-1)^{k(n-k)}\xi$ holds for $\xi = a_1 \wedge \ldots \wedge a_k$ with $1 \leq k \leq n-1$, where $(a_1, \ldots, a_n)$ is any positively oriented orthonormal resp. unitary basis of V. Then $(a_{k+1}, \ldots, a_n, (-1)^{k(n-k)}a_1, a_2, \ldots, a_k)$ is another positively oriented orthonormal resp. unitary basis of V and we have by twofold application of (b)

$$*(*\xi) = *(*(a_1 \wedge \ldots \wedge a_r)) = *(a_{r+1} \wedge \ldots \wedge a_n) = (-1)^{k(n-k)} a_1 \wedge \ldots \wedge a_k = (-1)^{k(n-k)} \cdot \xi \,.$$

For (d). Let $\xi \in \bigwedge^k V$ and $\eta \in \bigwedge V$ be given. Because the homogeneous components of η of degree unequal to $(n-k)$ do not contribute to either side of the equation (d), we may suppose without loss of generality that $\eta \in \bigwedge^{n-k} V$ holds. Then we have

$$\begin{aligned} \langle \eta, *\xi \rangle \cdot \omega &\overset{(B.2)}{=} \eta \wedge (* * \xi) \overset{(c)}{=} \eta \wedge ((-1)^{k(n-k)}\xi) = ((-1)^{(n-k)k}\eta) \wedge \xi \\ &\overset{(c)}{=} (* * \eta) \wedge \xi \overset{(B.1)}{=} (-1)^{(n-k)k} \xi \wedge (* * \eta) \overset{(B.2)}{=} (-1)^{k(n-k)} \langle \xi, *\eta \rangle \cdot \omega \end{aligned}$$

and therefore $\langle \eta, *\xi \rangle = (-1)^{k(n-k)} \cdot \langle \xi, *\eta \rangle$.

For (e). Because both sides of the equation in (e) are $\mathbb{R}$-linear resp. anti-linear in ξ, we may suppose without loss of generality that $\xi \in \bigwedge^k V$ holds for some $k \in \{0, \ldots, n\}$. Then we have

$$\langle *\xi, *\eta \rangle \overset{(d)}{=} (-1)^{k(n-k)} \cdot \langle \eta, * * \xi \rangle \overset{(c)}{=} \langle \eta, \xi \rangle = \overline{\langle \xi, \eta \rangle} \,.$$

The remaining statements of (e) are an obvious consequence. □

B.3 Example. We tabulate values of $*\xi$ for the case $\dim V = 4$. Let (a_1, a_2, a_3, a_4) be a positively oriented orthonormal resp. unitary basis of V. Then, we have as a consequence of Proposition B.2(b):

ξ	1	a_1	a_2	a_3	a_4
$*\xi$	$a_1 \wedge a_2 \wedge a_3 \wedge a_4$	$a_2 \wedge a_3 \wedge a_4$	$-a_1 \wedge a_3 \wedge a_4$	$a_1 \wedge a_2 \wedge a_4$	$-a_1 \wedge a_2 \wedge a_3$

ξ	$a_1 \wedge a_2$	$a_1 \wedge a_3$	$a_1 \wedge a_4$	$a_2 \wedge a_3$	$a_2 \wedge a_4$	$a_3 \wedge a_4$
$*\xi$	$a_3 \wedge a_4$	$-a_2 \wedge a_4$	$a_2 \wedge a_3$	$a_1 \wedge a_4$	$-a_1 \wedge a_3$	$a_1 \wedge a_2$

ξ	$a_1 \wedge a_2 \wedge a_3$	$a_1 \wedge a_2 \wedge a_4$	$a_1 \wedge a_3 \wedge a_4$	$a_2 \wedge a_3 \wedge a_4$	$a_1 \wedge a_2 \wedge a_3 \wedge a_4$
$*\xi$	a_4	$-a_3$	a_2	$-a_1$	1

B.3 Clifford Algebras

Let V be an n-dimensional linear space over the field $\mathbb{K}$ and $\beta : V \times V \to \mathbb{K}$ a symmetric bilinear form; β gives rise to the quadratic form

$$q : V \to \mathbb{K},\; v \mapsto \tfrac{1}{2}\beta(v, v) \,.$$

B.4 Definition. *A* Clifford algebra for (V,β) *is an algebra* C *with the following properties:*

(a) C *contains* $\mathbb{K}\oplus V$ *as a linear subspace in such a way that* $1\in\mathbb{K}$ *is the unit of* C *and*

$$\forall v\in V\subset C\;:\;v\cdot v=q(v)\cdot 1_C \tag{B.7}$$

holds.

(b) C *has the following universal property: Any linear map* $f:V\to A$ *into another algebra* A *which satisfies*

$$\forall v\in V\;:\;f(v)\cdot f(v)=q(v)\cdot 1_A \tag{B.8}$$

(we call such a map a Clifford map*) can be uniquely extended to an algebra homomorphism* $\varphi:C\to A$ *.*

B.5 Remark. The details of the definition of a Clifford algebra vary somewhat from author to author. In particular, Equation (B.7) is often replaced by the condition

$$\forall v\in V\;:\;v\cdot v=\varkappa\, q(v)\cdot 1_C\;,$$

where $\varkappa=2$ is common (then, $v\cdot v=\beta(v,v)\cdot 1$ holds for $v\in V$), but $\varkappa\in\{-1,-2\}$ can also be found. In these cases, also (B.8) has to be modified accordingly. The convention we follow here is that of Chevalley, see [Che54], p. 39; with this choice, the least number of factors is required in the treatment of triality in Section B.6

Also, some authors (but not Chevalley) replace the universal property of Definition B.4(b) by the weaker requirement that V generates C as algebra (see Theorem B.7 below). In this case, a Clifford algebra in our sense is called a *universal Clifford algebra.* For this use of terminology, and for results on non-universal Clifford algebras, see for example [Por95], Chapter 15, p. 123ff.

B.6 Proposition. *If* C *is a Clifford algebra for* (V,β)*, we have*

(a) $\forall v,w\in V\subset C\;:\;v\cdot w+w\cdot v=\beta(v,w)\cdot 1_C$,

(b) Denoting by $C^\times$ *the multiplicative group of invertible elements of* C*, we have* $C^\times\cap V=q^{-1}(\mathbb{K}^\times)$ *and* $\forall v\in q^{-1}(\mathbb{K}^\times):v^{-1}=\frac{1}{q(v)}\cdot v$ *.*

Proof. For (a). For $v,w\in C$, we have

$$\begin{aligned} v\cdot w+w\cdot v &= (v+w)\cdot(v+w)-v\cdot v-w\cdot w\\ &\overset{(B.7)}{=} \tfrac{1}{2}(\beta(v+w,v+w)-\beta(v,v)-\beta(w,w))\cdot 1_C=\beta(v,w)\cdot 1_C\;. \end{aligned}$$

For (b). This follows simply from the equation $v\cdot v=q(v)\cdot 1$. □

B.7 Theorem. *(a) There exists a Clifford algebra for* (V,β)*, and if* C*,* C' *are two such algebras, there exists an isomorphism of algebras* $\Phi:C\to C'$ *with* $\Phi|(\mathbb{K}\oplus V)=\mathrm{id}_{\mathbb{K}\oplus V}$*. In the sequel,* $C(V,\beta)$ *always denotes a Clifford algebra for* (V,β)*.*

(b) If $(b_1, \ldots, b_n)$ is a basis of V, then $C(V, \beta)$ is as algebra generated by $\{b_1, \ldots, b_n\}$.

For $\varnothing \neq S \subset \{1, \ldots, n\}$, say $S = \{\ell_1, \ldots, \ell_r\}$ with $1 \leq \ell_1 < \ldots < \ell_r \leq n$, we put

$$b_S := b_{\ell_1} \cdots b_{\ell_r} \; ; \quad \text{we also put } b_\varnothing := 1_C \; .$$

Then $(b_S)_{S \subset \{1,\ldots,n\}}$ is a basis of the linear space $C(V, \beta)$; in particular we have $\dim C(V, \beta) = 2^n$.

Proof. The uniqueness of $C(V, \beta)$ follows from the universal property by the canonical argument: Suppose C and C' are two Clifford algebras for (V, β). By the universal property, there exist algebra homomorphisms $\Phi : C \to C'$ and $\Phi' : C' \to C$ with $\Phi|V = \Phi'|V = \mathrm{id}_V$.

$$\begin{tikzcd} C \arrow[rr, shift right, "\Phi"'] & & C' \arrow[ll, shift right, "\Phi'"'] \\ & V \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Both $\Phi' \circ \Phi$ and id_C are extensions of id_V to algebra endomorphisms of C; the universal property of C therefore implies $\Phi' \circ \Phi = \mathrm{id}_C$. An analogous argument shows $\Phi \circ \Phi' = \mathrm{id}_{C'}$, and therefore Φ (and $\Phi' = \Phi^{-1}$) are algebra automorphisms. Remember, we have $\Phi|V = \mathrm{id}_V$ by the construction of Φ; also we have $\Phi(1_C) = 1_{C'}$, and therefore (because of $1_C = 1_{C'} = 1 \in \mathbb{K}$) $\Phi|\mathbb{K} = \mathrm{id}_{\mathbb{K}}$.

We only sketch the existence proof for $C(V, \beta)$ here, following the exposition [Rec04] by H. Reckziegel: Consider the tensor algebra $\bigotimes V$ of V, the two-sided ideal $\mathfrak{a}$ of $\bigotimes V$ generated by the set $\{\, v \otimes v - q(v) \cdot 1 \,|\, v \in V \,\}$, the algebra $C := (\bigotimes V)/\mathfrak{a}$ and the canonical projection $\pi : \bigotimes V \to C$. Because we interpreted V as a linear subspace of $\bigotimes V$, we also have the linear map $j := \pi|V : V \to C$ (note that it is not a priori clear that j is injective). j satisfies

$$\forall v \in V \; : \; j(v)^2 = q(v) \cdot 1_C \; . \tag{B.9}$$

We also note that if $(b_1, \ldots, b_n)$ is a basis of V, $\{b_1, \ldots, b_n\}$ generates $\bigotimes V$ as algebra, and therefore $\{j(b_1), \ldots, j(b_n)\}$ generates C as algebra.

Also, C solves the following universal problem:

> Whenever another algebra A and a Clifford map $f : V \to A$ are given, there exists one and only one algebra homomorphism $\varphi : C \to A$ with $\varphi \circ j = f$. (B.10)

For if f is as in (B.10), then the universal property of the tensor algebra $\bigotimes V$ shows that there exists one and only one algebra homomorphism $\psi : \bigotimes V \to A$ extending f and because f is a Clifford map, we see that $\mathfrak{a}$ is contained in the kernel of ψ, whence there exists an algebra homomorphism $\varphi : C \to A$ which satisfies $\psi = \varphi \circ \pi$ and therefore $f = \varphi \circ j$. The uniqueness of φ follows from the uniqueness of ψ.

In the sequel, we will call a pair (C, j) consisting of an algebra C and a linear map $j : V \to C$ a *pre-Clifford algebra* for (V, β), if it satisfies Equation (B.9) and solves the universal problem (B.10). By an argument analogous to the proof of the uniqueness of the Clifford algebra, one sees that the pre-Clifford algebra for (V, β) is unique in the following sense: If (C, j) and (C', j') are two pre-Clifford algebras for (V, β), there exists an algebra isomorphism $\Phi : C \to C'$ with $j' = \Phi \circ j$.

For any $v \in V$, we abbreviate $\hat{v} := j(v) \in C$, and for any $\varnothing \neq S \subset \{1, \ldots, n\}$, say $S = \{\ell_1, \ldots, \ell_r\}$ with $1 \leq \ell_1 < \ldots < \ell_r \leq n$, we put

$$\hat{b}_S := \hat{b}_{\ell_1} \cdots \hat{b}_{\ell_r} \; ; \quad \text{we also put } \hat{b}_\varnothing := 1_C \; .$$

One can show by induction on $\#S$ that for any $S \subset \{1, \dots, n\}$, the linear subspace $C(S) := \operatorname{span}\{\, \hat{b}_{S'} \,|\, S' \subset S\,\}$ of C in fact is a subalgebra of C. Because C is as algebra generated by $\{\hat{b}_1, \dots, \hat{b}_n\}$, it follows that $C = C(\{1, \dots, n\})$ holds, and therefore, we see that C is as a linear space spanned by $(\hat{b}_S)_{S\subset\{1,\dots,n\}}$.

Then, one shows by induction on $n = \dim V$ that $\dim C = 2^n$ holds. In the case $n = 1$, consider the polynomial ring $\mathbb{K}[X]$, the principal ideal $\mathfrak{b} \subset \mathbb{K}[X]$ generated by the polynomial $X^2 - q(b)$ with a fixed $b \in V \setminus \{0\}$, the algebra $C := \mathbb{K}[X]/\mathfrak{b}$ and the injective linear map $j : V \to C$ given by $j(b) = X + \mathfrak{b}$. Then (C, j) is a pre-Clifford algebra for (V, β) and $\dim C = 2$ holds.

For the induction step, let an n-dimensional linear space V (with $n \geq 2$) and a bilinear form $\beta : V \times V \to \mathbb{K}$ be given. Then it can be shown that there exists an $(n-1)$-dimensional subspace $U \subset V$ and $\varepsilon \in \mathbb{K}$ so that with $\tilde{V} := U \oplus \mathbb{K}$ and the bilinear form

$$\tilde{\beta} : \tilde{V} \times \tilde{V} \to \mathbb{K},\ ((v_1, t_1), (v_2, t_2)) \mapsto \beta(v_1, v_2) + \varepsilon\, t_1 t_2 ,$$

there exists an isomorphism of linear spaces $\Psi : V \to \tilde{V}$ with $\tilde{\beta}(\Psi(v), \Psi(w)) = \beta(v, w)$ for $v, w \in V$. One can further show that if (C_U, j_U) is a pre-Clifford algebra for (U, β_U) (with $\beta_U := \beta|(U \times U)$), then $C := C_U \oplus C_U$ becomes an algebra with unit $1_C = (1_{C_U}, 0)$ via the multiplication

$$\forall\, (\xi_1, \eta_1),\, (\xi_2, \eta_2) \in C \ :\ (\xi_1, \eta_1) \cdot (\xi_2, \eta_2) := (\xi_1\, \xi_2 + \varepsilon\, \alpha_U(\eta_1)\, \eta_2\,,\ \alpha_U(\xi_1)\, \eta_2 + \eta_1\, \xi_2)\ ;$$

here $\alpha_U : C_U \to C_U$ denotes the involutive algebra homomorphism uniquely determined by $\alpha_U \circ j_U = -j_U$ (which exists because of the universal property (B.10)). And the pair (C, j) with

$$j : \tilde{V} \to C,\ (v, t) \mapsto (j_U(v), t \cdot 1_{C_U})$$

is a pre-Clifford algebra for $(\tilde{V}, \tilde{\beta})$, whence the pair $(C, j \circ \Psi)$ is a pre-Clifford algebra for (V, β). We have $\dim C_U = 2^{n-1}$ by the induction hypothesis and therefore $\dim C = 2^n$. This completes the induction proof.

Because the system of generators $(\hat{b}_S)_{S\subset\{1,\dots,n\}}$ of the linear space C consists of 2^n elements, it follows that it is a basis of C, and as a consequence of this fact, one sees that $j : V \to C$ is injective. Therefore, we can identify V with the subset $j(V) \subset C$, then C becomes a Clifford algebra in the sense of Definition B.4, and the remaining statements of the theorem hold. □

B.8 Example. Let C be a Clifford algebra of V corresponding to the zero bilinear form on V. Then we see that $v \cdot v = 0$ holds in C for any $v \in V$, and the Clifford maps corresponding to C are exactly those linear maps $f : V \to A$ into another algebra A for which $f(v)^2 = 0$ holds for every $v \in V$. Therefore C is a model of the exterior algebra of V. It follows that with Theorem B.7 we have also proved the existence and uniqueness of the exterior algebra, and the statements made in Section B.1 about the bases and dimension of $\bigwedge V$.

B.9 Proposition. *Let $U \subset V$ be a linear subspace. Then the subalgebra C' of $C(V, \beta)$ generated by U is a Clifford algebra for (U, β_U) with $\beta_U := \beta|U \times U$.*

Proof. The inclusion map $U \hookrightarrow C'$ is a Clifford map for the "abstract" Clifford algebra $C(U, \beta_U)$ and consequently can be extended to an algebra homomorphism $\varphi : C(U, \beta_U) \to C'$. If $(b_1, \dots, b_k)$ is a basis of U then φ maps $b_{j_1} \cdots b_{j_\ell} \in C(U, \beta_U)$ onto $b_{j_1} \cdots b_{j_\ell} \in C'$ and is therefore surjective. Because we have $\dim C' = 2^k = \dim C(U, \beta_U)$, it follows that φ is an algebra isomorphism with $\varphi|(\mathbb{K} \oplus U) = \mathrm{id}_{\mathbb{K}\oplus U}$. Therefore C' also is a Clifford algebra for (U, β_U). □

B.10 Proposition. *(a) There is one and only one algebra homomorphism $\alpha : C(V, \beta) \to C(V, \beta)$ with $\alpha(v) = -v$ for every $v \in V$. α is an involutive algebra automorphism; it is called the* canonical involution *of $C(V, \beta)$.*

(b) $C^+(V,\beta) := \mathrm{Eig}(\alpha,1)$ *is a subalgebra of* $C(V,\beta)$ *called the* even subalgebra *of* $C(V,\beta)$*; if* $(b_1,\dots,b_n)$ *is a basis of* V*, then* $C^+(V,\beta)$ *is as algebra generated by the set*

$$\{\, b_{\ell_1}\cdot b_{\ell_2} \,|\, 1 \le \ell_1 < \ell_2 \le n \,\} .$$

On the other hand, $C^-(V,\beta) := \mathrm{Eig}(\alpha,-1)$ *only is a linear subspace of* $C(V,\beta)$ *called the* odd subspace *of* $C(V,\beta)$*; it is spanned by the set*

$$\{\, b_S \,|\, S \subset \{1,\dots,n\},\ \#S \ \textit{odd} \}$$

(for the notation b_S *see Theorem B.7). If we abbreviate* $C^\pm := C^\pm(V,\beta)$*, we have* $C(V,\beta) = C^+ \oplus C^-$ *and*

$$C^+\cdot C^+ \subset C^+\ ,\quad C^+\cdot C^- \subset C^-\ ,\quad C^-\cdot C^+ \subset C^- \quad \textit{and} \quad C^-\cdot C^- \subset C^+\ . \tag{B.11}$$

Proof. For (a). $V \to C(V,\beta),\ v \mapsto -v$ is a Clifford map, therefore the existence and uniqueness of α follows from the universal property of $C(V,\beta)$. Both $\alpha\circ\alpha$ and $\mathrm{id}_{C(V,\beta)}$ are algebra homomorphisms which extend the Clifford map $V \to C(V,\beta),\ v \mapsto v$, and therefore $\alpha\circ\alpha = \mathrm{id}_{C(V,\beta)}$ holds. This shows α to be an involutive algebra automorphism.

For (b). Because α is an involutive linear map, $C(V,\beta) = C^+\oplus C^-$ holds, and the inclusions (B.11) follow from the fact that α is an algebra homomorphism. In particular, we see that C^+ is a subalgebra of $C(V,\beta)$. The remaining statements now follow from Theorem B.7(b). □

B.11 Proposition. *There is one and only one algebra anti-homomorphism* $\gamma : C(V,\beta) \to C(V,\beta)$ *with* $\gamma(v) = -v$ *for every* $v \in V$*.* γ *is an involutive algebra anti-automorphism and* $\alpha\circ\gamma = \gamma\circ\alpha$ *holds.* γ *is called the* conjugation *of* $C(V,\beta)$*.*[38]

Proof. We abbreviate $C := C(V,\beta)$ and denote by $\mu : C\times C \to C$ the multiplication map of this algebra; we also consider the algebra C^{op} which as linear space is identical to C but whose multiplication is given by

$$\mu^{op} : C^{op}\times C^{op} \to C^{op},\ (\xi,\eta) \mapsto \mu(\eta,\xi) .$$

Note that $1_{C^{op}} = 1_C$ holds.

$f : V \to C^{op},\ v \mapsto -v$ is a Clifford map. By virtue of the universal property of C, there is one and only one algebra homomorphism $\gamma : C \to C^{op}$ which extends f. If we now regard γ as a map into C, it is an algebra anti-homomorphism. Both $\gamma\circ\gamma : C \to C$ and id_C are algebra homomorphisms which extend the Clifford map $V \to C,\ v \mapsto v$. By the universal property of C, $\gamma\circ\gamma = \mathrm{id}_C$ follows; in particular, γ is an algebra automorphism. It remains to show $\alpha\circ\gamma = \gamma\circ\alpha$. For this, we note that $\gamma\circ\alpha\circ\gamma\circ\alpha$ and id_C are algebra homomorphisms $C \to C$ which both extend the Clifford map $V \to C,\ v \mapsto v$. Consequently, the universal property of C shows that $\gamma\circ\alpha\circ\gamma\circ\alpha = \mathrm{id}_C$ holds, and therefore we have $\alpha\circ\gamma = \gamma\circ\alpha$ because α and γ are involutions. □

[38] $\gamma(\xi)$ is often also denoted by $\overline{\xi}$. We do not use this notation here to prevent confusion with the complex conjugate $\overline{\lambda}$ of a complex number λ.

B.4 The Clifford and Spin groups, and their representations

In the situation of the previous section we now suppose that $n = \dim V \geq 2$ holds and that β is non-degenerate. We consider the *orthogonal group*

$$\begin{aligned} \mathrm{O}(V,\beta) &:= \{ B \in \mathrm{GL}(V) \,|\, \forall v, w \in V : \beta(Bv, Bw) = \beta(v,w) \} \\ &= \{ B \in \mathrm{GL}(V) \,|\, q \circ B = q \} \end{aligned}$$

and the *special orthogonal group*

$$\mathrm{SO}(V,\beta) := \{ B \in \mathrm{O}(V,\beta) \,|\, \det(B) = 1 \}$$

with respect to β. Because β is non-degenerate, $\mathrm{O}(V,\beta)$ is a subgroup of $\mathrm{GL}(V)$, and $\mathrm{SO}(V,\beta)$ is a subgroup of $\mathrm{O}(V,\beta)$. We have $\det(\mathrm{O}(V,\beta)) = \{\pm 1\}$, and therefore $\mathrm{SO}(V,\beta)$ is a subgroup of $\mathrm{O}(V,\beta)$ of index 2.

Proof for $\det(\mathrm{O}(V,\beta)) = \{\pm 1\}$. It is clear that $\{\pm 1\} \subset \det(\mathrm{O}(V,\beta))$ holds. Now, let $A \in \mathrm{O}(V,\beta)$ be given. We fix an β-orthonormal basis $(b_1, \ldots, b_n)$ of V, so $\beta(b_j, b_k) = \varepsilon_j \cdot \delta_{jk}$ holds with $\varepsilon_j \in \{\pm 1\}$. Then there exist $a_{jk} \in \mathbb{K}$ so that $Ab_k = \sum_{j=1}^n a_{jk}\, b_j$ holds. We then have for any $j, k \in \{1, \ldots, n\}$

$$\varepsilon_j \cdot \delta_{jk} = \beta(b_j, b_k) = \beta(Ab_j, Ab_k) = \beta(\textstyle\sum_\nu a_{\nu j} b_\nu, \sum_\mu a_{\mu k} b_\mu) = \sum_\nu \varepsilon_\nu a_{\nu j} a_{\nu k} \,. \tag{B.12}$$

If we now denote by $M := (a_{jk})_{j,k}$ the matrix corresponding to A with respect to the basis $(b_1, \ldots, b_n)$, by M^T its transpose and put $\widetilde{M} := (\varepsilon_j \cdot a_{jk})_{j,k}$, we obtain by (B.12):

$$\textstyle\prod_j \varepsilon_j = \det(M^T \cdot \widetilde{M}) = \det(M^T) \cdot \det(\widetilde{M}) = \det(M) \cdot \prod_j \varepsilon_j \cdot \det(M)$$

and thus $\det(M)^2 = 1$, hence $\det(A) = \det(M) \in \{\pm 1\}$. □

In the sequel, we denote by

$$C(V,\beta)^\times := \{ \xi \in C(V,\beta) \,|\, \exists\, \eta \in C(V,\beta) : \xi\eta = \eta\xi = 1_C \}$$

the multiplicative group of invertible elements of $C(V,\beta)$; for any $\xi \in C(V,\beta)^\times$, we denote by $\xi^{-1} \in C(V,\beta)^\times$ the inverse of ξ.

The group

$$\Gamma(V,\beta) := \{ g \in C(V,\beta)^\times \,|\, \forall v \in V : \alpha(g) v g^{-1} \in V \}$$

is called the *Clifford group* of $C(V,\beta)$ and the map

$$\chi : \Gamma(V,\beta) \to \mathrm{End}(V),\ g \mapsto (v \mapsto \alpha(g) v g^{-1})$$

is called the *vector representation* of $\Gamma(V,\beta)$.

B.12 Proposition. *(a)* $\Gamma(V,\beta)$ *is a subgroup of* $C(V,\beta)^\times$ *and* $\chi : \Gamma(V,\beta) \to \mathrm{GL}(V)$ *is a linear representation of* $\Gamma(V,\beta)$.

(b) For any $g \in \Gamma(V,\beta)$ *and* $t \in \mathbb{K}^\times$, $tg \in \Gamma(V,\beta)$ *and* $\chi(tg) = \chi(g)$ *holds.*
$\alpha(\Gamma(V,\beta)) = \Gamma(V,\beta)$, *and for any* $g \in \Gamma(V,\beta)$, $\chi(\alpha(g)) = \chi(g)$ *holds.*
$\gamma(\Gamma(V,\beta)) = \Gamma(V,\beta)$, *and for any* $g \in \Gamma(V,\beta)$, $\chi(\gamma(g)) = \chi(g^{-1})$ *holds.*

(c) *We have* $\Gamma(V,\beta) \cap V = q^{-1}(\mathbb{K}^{\times})$ *and*

$$\forall w \in q^{-1}(\mathbb{K}^{\times}),\ v \in V\ :\ \chi(w)v = v - 2\frac{\beta(v,w)}{\beta(w,w)} \cdot w\ .$$

For $w \in q^{-1}(\mathbb{K}^{\times})$, $\chi(w) \in \mathrm{O}(V,\beta)$ *holds. More precisely,* $\chi(w) : V \to V$ *is the* β*-orthogonal reflection in the hyperplane* $(\mathbb{K}w)^{\perp,\beta}$.

(d) $\mathbb{K}^{\times} \subset \Gamma(V,\beta)$ *and* $\ker\chi = \mathbb{K}^{\times}$.

(e) $\chi(\Gamma(V,\beta)) = \mathrm{O}(V,\beta)$.

(f) $\Gamma(V,\beta) = \{\, w_1 \cdots w_r \mid 1 \le r \le n,\ w_1,\dots,w_r \in q^{-1}(\mathbb{K}^{\times}) \,\}$, [39]
in particular $\Gamma(V,\beta) \subset C^+(V,\beta) \cup C^-(V,\beta)$,
and for any $g \in \Gamma(V,\beta)$ *we have* $\varepsilon(g) := \alpha(g)\cdot g^{-1} \in \{\pm 1\}$.

Proof. For (a). Let us abbreviate $C := C(V,\beta)$ and consider the map

$$\widetilde{\chi} : C^{\times} \to \mathrm{End}(C),\ g \mapsto (\xi \mapsto \alpha(g)\xi g^{-1})\ .$$

Then elementary calculations show for any $g_1, g_2 \in C^{\times}$:

$$\widetilde{\chi}(g_1 \cdot g_2^{-1}) = \widetilde{\chi}(g_1) \circ \widetilde{\chi}(g_2)^{-1} \quad \text{and} \quad \widetilde{\chi}(1_C) = \mathrm{id}_C\ .$$

Consequently, we see that $1_C \in \Gamma(V,\beta)$ holds, that $g_1, g_2 \in \Gamma(V,\beta)$ implies $g_1 \cdot g_2^{-1} \in \Gamma(V,\beta)$ and that $\chi : \Gamma(V,\beta) \to \mathrm{GL}(V)$ is a group homomorphism.

For (b). The statement on tg (with $g \in \Gamma(V,\beta),\ t \in \mathbb{K}^{\times}$) is obvious and the statement on α can be straightforwardly checked by a direct calculation. For the statement on γ one first verifies for $g \in \Gamma(V,\beta)$

$$\widetilde{\chi}(\gamma(g)) = \gamma \circ \widetilde{\chi}(\alpha(g^{-1})) \circ \gamma\ .$$

By use of (a) and the previous result on α, one sees that $\gamma(g) \in \Gamma(V,\beta)$ and $\chi(\gamma(g)) = \chi(g^{-1})$ holds.

For (c). For any $w \in \Gamma(V,\beta) \cap V$ we have $q(w) \neq 0$ by Proposition B.6(b). Conversely, let $w \in q^{-1}(\mathbb{K}^{\times})$ be given. Then $w \in C(V,\beta)^{\times}$ and $w^{-1} = \frac{1}{q(w)} \cdot w$ holds by Proposition B.6(b), and therefore we have for any $v \in V$

$$\begin{aligned}
\widetilde{\chi}(w)v = \alpha(w)\cdot v \cdot w^{-1} &= -\tfrac{1}{q(w)} \cdot w \cdot v \cdot w \\
&= -\tfrac{1}{q(w)}(-v \cdot w + \beta(w,v)) \cdot w \\
&= \tfrac{1}{q(w)} \cdot v \cdot w \cdot w - \tfrac{\beta(v,w)}{q(w)} \cdot w \\
&= v - \tfrac{2\beta(v,w)}{\beta(w,w)} \cdot w \ \in\ V\ .
\end{aligned}$$

This shows that $w \in \Gamma(V,\beta)$ holds and that $\chi(w)$ is as given in the proposition. We have

$$q(\chi(w)v) = \tfrac{1}{2}\beta\left(v - \tfrac{2\beta(v,w)}{\beta(w,w)} w\,,\ v - \tfrac{2\beta(v,w)}{\beta(w,w)} w\right) = q(v)$$

and therefore $\chi(w) \in \mathrm{O}(V,\beta)$. We have $\chi(w)w = -w$ and $\chi(w)v = v$ for any $v \in V$ with $\beta(w,v) = 0$. Hence, $\chi(w)$ is the β-orthogonal reflection in the hyperplane $(\mathbb{K}w)^{\perp,\beta}$.

For (d). See [LM89], Proposition I.2.4, p. 14.

For (e) and (f). (a) and (c) show that $\widetilde{\Gamma} := \{\, w_1 \cdots w_r \mid 1 \le r \le n,\ w_1,\dots,w_r \in q^{-1}(\mathbb{K}^{\times}) \,\} \subset \Gamma(V,\beta)$ and $\chi(\widetilde{\Gamma}) \subset \mathrm{O}(V,\beta)$ holds. Conversely, any given $B \in \mathrm{O}(V,\beta)$ can by the Theorem of Cartan/Dieudonné (see

[39] In particular, the elements of $\mathbb{K}^{\times} \subset \Gamma(V,\beta)$ can be represented as such a product, namely we have $t = (t\,w)\cdot w$ for any $t \in \mathbb{K}^{\times}$ and $w \in q^{-1}(\{1\})$. Note that the hypothesis $n \ge 2$ is of importance here.

[Art57], Theorem III.3.20, p. 129) be written in the form $B = \chi(w_1) \circ \ldots \circ \chi(w_r) = \chi(w_1 \cdots w_r)$, where $r \leq n$ and $w_1, \ldots, w_r \in q^{-1}(\{\pm 1\})$ are suitably chosen. This shows that $\mathrm{O}(V,\beta) \subset \chi(\widetilde{\Gamma})$ holds.

We now prove $\Gamma(V,\beta) \subset \widetilde{\Gamma}$. Let $g \in \Gamma(V,\beta)$ be given. By (a) and (b), we have $\chi(\alpha(g) \cdot g^{-1}) = \mathrm{id}_V$ and therefore by (d) $\varepsilon(g) = \alpha(g) \cdot g^{-1} \in \mathbb{K}^\times$. Hence, $g \neq 0$ is an eigenvector of α for the eigenvalue $\varepsilon(g)$. Because α is involutive, we have $\varepsilon(g)^2 = 1$ (whence $\varepsilon(g) \in \{\pm 1\}$ and $g \in C^+(V,\beta) \cup C^-(V,\beta)$ follows) and therefore for any $v \in V$:

$$\begin{aligned} q(\chi(g)v) = (\chi(g)v)^2 &= \alpha(g)vg^{-1} \cdot \alpha(g)vg^{-1} = \varepsilon(g)^2 \cdot gvg^{-1} \cdot gvg^{-1} \\ &= g \cdot v^2 \cdot g^{-1} = q(v) \cdot gg^{-1} = q(v) \,. \end{aligned}$$

This shows $\chi(g) \in \mathrm{O}(V,\beta) \subset \chi(\widetilde{\Gamma})$ (see above). Therefore there exists $\widetilde{g} \in \widetilde{\Gamma}$ with $\chi(\widetilde{g}) = \chi(g)$, say $\widetilde{g} = w_1 \cdots w_r$ with $w_1, \ldots, w_r \in q^{-1}(\mathbb{K}^\times)$. By (d), we have $t := g \cdot \widetilde{g}^{-1} \in \mathbb{K}^\times$ and therefore $g = t\widetilde{g} = (tw_1) \cdot w_2 \cdots w_r \in \widetilde{\Gamma}$. □

We briefly mention the *special Clifford group*

$$\Gamma^+(V,\beta) := \Gamma(V,\beta) \cap C^+(V,\beta) \,,$$

which is the kernel of the surjective group homomorphism $\Gamma(V,\beta) \to \{\pm 1\},\ g \mapsto \alpha(g) \cdot g^{-1}$ and therefore a subgroup of index 2 of $\Gamma(V,\beta)$.

B.13 Remark. Some authors, for example Chevalley (see [Che54], p. 49), rather define the Clifford group as $\{\, g \in C(V,\beta)^\times \mid \forall v \in V : gvg^{-1} \in V \,\}$ and the vector representation by $\chi(g) = (v \mapsto gvg^{-1})$. In that terminology, our χ is called the *twisted vector representation.* However, because we have $\alpha(\xi) = \xi$ for $\xi \in C^+(V,\beta)$, the special Clifford group induced by Chevalley's definition is identical to our special Clifford group, and Chevalley's vector representation coincides on the special Clifford group with our vector representation.

Because $\ker \chi = \mathbb{K}^\times$ is not a discrete subgroup of $\Gamma(V,\beta)$, the Clifford group is too large to be a covering group over $\mathrm{O}(V,\beta)$. To reduce the size of $\Gamma(V,\beta)$, we now introduce the *norm function*[40]

$$\lambda : C(V,\beta) \to C(V,\beta),\ \xi \mapsto \xi \cdot \gamma(\xi) \,.$$

B.14 Proposition. *(a)* $\lambda(1_{C(V,\beta)}) = 1_{C(V,\beta)}$ *and* $\forall t \in \mathbb{K},\ \xi \in C(V,\beta) \ :\ \lambda(t\xi) = t^2 \cdot \lambda(\xi)$.

(b) $\forall v, w \in V \ : \big(\ \lambda(v) = -q(v)$ *and* $\lambda(v \cdot w) = \lambda(v) \cdot \lambda(w)\ \big)$.

(c) $\lambda(\Gamma(V,\beta)) \subset \mathbb{K}^\times$ *and* $\lambda|\Gamma(V,\beta) : \Gamma(V,\beta) \to \mathbb{K}^\times$ *is a group homomorphism.*

Proof. (a) and the first part of (b) are obvious. For the second part of (b), let $v, w \in V$ be given. Then we have

$$\lambda(v \cdot w) = v \cdot w \cdot \gamma(v \cdot w) = v \cdot w \cdot \gamma(w) \cdot \gamma(v) = v \cdot \underbrace{\lambda(w)}_{\in \mathbb{K}} \cdot \gamma(v) = v \cdot \gamma(v) \cdot \lambda(w) = \lambda(v) \cdot \lambda(w) \,.$$

For (c). Let $g \in \Gamma(V,\beta)$ be given. Then we have for every $v \in V$ by Proposition B.12(a),(b)

$$\chi(\lambda(g))v = \chi(g \cdot \gamma(g))v = \chi(g)(\chi(g)^{-1}v) = v$$

[40] It should be noted that in spite of the name "norm function", λ is not a norm in the usual sense. In particular, for $t \in \mathbb{K}$ it does not satisfy $\lambda(t\xi) = |t| \cdot \lambda(\xi)$, but rather $\lambda(t\xi) = t^2 \cdot \lambda(\xi)$, see Proposition B.14(a).

and therefore $\lambda(g) \in \ker \chi = \mathbb{K}^\times$, see Proposition B.12(d). An analogous calculation as in the proof of (b) therefore shows that $\lambda|\Gamma(V,\beta) : \Gamma(V,\beta) \to \mathbb{K}^\times$ is a group homomorphism. □

In the sequel, we restrict our considerations to the following situation:

> In the case $\mathbb{K} = \mathbb{R}$ we suppose β to be negative definite, so that $-\beta$ is a positive definite inner product on V. (In the case $\mathbb{K} = \mathbb{C}$ we impose no further restriction on β.) (B.13)

Then we define the *Pin group* (also called the *reduced Clifford group*) of (V,β) by

$$\mathrm{Pin}(V,\beta) := \{ g \in \Gamma(V,\beta) \,|\, \lambda(g) = 1 \} \tag{B.14}$$

and the *Spin group* (also called the *special reduced Clifford group*) of (V,β) by

$$\mathrm{Spin}(V,\beta) := \{ g \in \mathrm{Pin}(V,\beta) \,|\, \chi(g) \in \mathrm{SO}(V,\beta) \} \,. \tag{B.15}$$

B.15 Proposition. *(a)* $\mathrm{Pin}(V,\beta)$ *and* $\mathrm{Spin}(V,\beta)$ *are subgroups of* $\Gamma(V,\beta)$.

(b) $\mathrm{Pin}(V,\beta) \cap V = q^{-1}(\{-1\})$.

(c) *(i)* $\mathrm{Pin}(V,\beta) = \{ w_1 \cdots w_r \,|\, 1 \le r \le n,\ w_1, \ldots, w_r \in q^{-1}(\{-1\}) \}$

(ii) $\mathrm{Spin}(V,\beta) = \{ w_1 \cdots w_r \,|\, 2 \le r \le n,\ r\ \textit{even},\ w_1, \ldots, w_r \in q^{-1}(\{-1\}) \}$
$= \mathrm{Pin}(V,\beta) \cap \Gamma^+(V,\beta)$

(d) $\ker(\chi|\mathrm{Pin}(V,\beta)) = \ker(\chi|\mathrm{Spin}(V,\beta)) = \{\pm 1\}$.

(e) $\chi(\mathrm{Pin}(V,\beta)) = \mathrm{O}(V,\beta)$ *and* $\chi(\mathrm{Spin}(V,\beta)) = \mathrm{SO}(V,\beta)$.

Proof. For (a) and (b). (a) is an immediate consequence from the definitions of $\mathrm{Pin}(V,\beta)$ and $\mathrm{Spin}(V,\beta)$ because $\lambda : \Gamma(V,\beta) \to \mathbb{K}^\times$ and $\chi : \Gamma(V,\beta) \to \mathrm{O}(V,\beta)$ are group homomorphisms. (b) follows from Proposition B.12(c) and Proposition B.14(b).

For (c)(i). First, suppose that $g = w_1 \cdots w_r$ with $w_1, \ldots, w_r \in q^{-1}(\{-1\})$ and $r \le n$ is given. We then have $w_j \in \mathrm{Pin}(V,\beta)$ by (b) and therefore $g \in \mathrm{Pin}(V,\beta)$ by (a). Conversely, let $g \in \mathrm{Pin}(V,\beta)$ be given. Proposition B.12(f) shows that there exist $1 \le r \le n$ and $w_1', \ldots, w_r' \in q^{-1}(\mathbb{K}^\times)$ so that $g = w_1' \cdots w_r'$ holds. In the case $\mathbb{K} = \mathbb{R}$, we have for each $j \in \{1, \ldots, r\}$: $\lambda(w_j') = -q(w_j') > 0$ by (B.13); therefore both in the case $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ there exist $t_1, \ldots, t_r \in \mathbb{K}^\times$ with $t_j^2 = \lambda(w_j')$. We then have by Proposition B.14(b)

$$\left(\prod_j t_j\right)^2 = \prod_j t_j^2 = \prod_j \lambda(w_j') = \lambda(g) = 1$$

and therefore $\prod_j t_j \in \{\pm 1\}$. By replacing t_1 with $-t_1$ if necessary we can ensure $\prod_j t_j = 1$. Now put $w_j := w_j'/t_j$ for each j; we then have $-q(w_j) = \lambda(w_j) = \lambda(w_j')/t_j^2 = 1$ and therefore $w_j \in q^{-1}(\{-1\})$, and also $w_1 \cdots w_r = (1/\prod_j t_j) \cdot w_1' \cdots w_r' = g$.

For (c)(ii). Let $g \in \mathrm{Pin}(V,\beta)$ be given; by (c)(i) there exist $w_1, \ldots, w_r \in q^{-1}(\{-1\})$ with $r \le n$ so that $g = w_1 \cdots w_r$ holds. Then we have

$$\chi(g) = \chi(w_1) \circ \ldots \circ \chi(w_r) \,;$$

because each $\chi(w_j)$ is a reflection in a hyperplane by Proposition B.12(c), we have $\det \chi(w_j) = -1$ and therefore

$$\det \chi(g) = (-1)^r \,.$$

It follows that $g \in \mathrm{Spin}(V,\beta)$ holds if and only if r is even, which proves the first equality sign in (c)(ii). The second equality sign now follows from (i) and the definition of $\Gamma^+(V,\beta)$.

For (d). By Proposition B.12(d), we have $\ker(\chi|\mathrm{Pin}(V,\beta)) = \ker(\chi) \cap \mathrm{Pin}(V,\beta) = \mathbb{K}^\times \cap \mathrm{Pin}(V,\beta) = \{\, t \in \mathbb{K}^\times \,|\, t^2 = 1\,\} = \{\pm 1\}$. Because we have $\{\pm 1\} \subset \mathrm{Spin}(V,\beta)$ by (c)(ii), $\ker(\chi|\mathrm{Spin}(V,\beta)) = \{\pm 1\}$ follows.

For (e). We have $\chi(\mathrm{Pin}(V,\beta)) \subset \mathrm{O}(V,\beta)$ by Proposition B.12(e). For the converse direction, let $B \in \mathrm{O}(V,\beta)$ be given. By Proposition B.12(e) there exists $\widetilde{g} \in \Gamma(V,\beta)$ with $\chi(\widetilde{g}) = B$. In the case $\mathbb{K} = \mathbb{R}$, (B.13) together with Proposition B.12(f) shows that $\lambda(\widetilde{g}) > 0$ holds, and therefore both for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ there exists $t \in \mathbb{K}$ with $t^2 = \lambda(\widetilde{g})$. We then have $g := \frac{1}{t} \cdot \widetilde{g} \in \mathrm{Pin}(V,\beta)$ and $\chi(g) = \chi(\widetilde{g}) = B$.

We have $\chi(\mathrm{Spin}(V,\beta)) \subset \mathrm{SO}(V,\beta)$ by definition. Conversely, let $B \in \mathrm{SO}(V,\beta)$ be given. By the previous argument there exists $g \in \mathrm{Pin}(V,\beta)$ with $\chi(g) = B$ and the proof of (c)(ii) shows that $g \in C^+(V,\beta)$ holds. Consequently, we have $g \in \mathrm{Pin}(V,\beta) \cap C^+(V,\beta) = \mathrm{Spin}(V,\beta)$. □

B.16 Proposition. *There is one and only one structure of a Lie group on* $\mathrm{Spin}(V,\beta)$ *so that* $\tau := \chi|\mathrm{Spin}(V,\beta) : \mathrm{Spin}(V,\beta) \to \mathrm{SO}(V,\beta)$ *becomes a two-fold Lie group covering map. Regarded in this way,* $\mathrm{Spin}(V,\beta)$ *is connected. For* $\mathbb{K} = \mathbb{R}$ *it is compact, and for* $\mathbb{K} = \mathbb{R}$, $n \geq 3$ *it is simply connected, so that* $\mathrm{Spin}(V,\beta)$ *is the universal covering Lie group of* $\mathrm{SO}(V,\beta)$.[41]

Proof. See [Rec04], Satz 3 and Satz 4. □

Besides the vector representation χ of $\Gamma(V,\beta)$, there are different linear representations ρ of $\Gamma(V,\beta)$ on some linear space S, which are not induced via χ by a representation of $\mathrm{O}(V,\beta)$. It turns out that they stem from algebra representations of $C(V,\beta)$ on S; they are called *spin representations.*

To prepare the introduction of spin representations, we review some definitions and facts on representations of algebras.

B.17 Definition. *Let A be an algebra and V be a linear space.*

(a) A subalgebra $\mathfrak{a}$ *of A is called a* (two-sided) ideal *of A if $xy, yx \in \mathfrak{a}$ holds for every $x \in \mathfrak{a}$ and $y \in A$.*

(b) A is called simple, *if it contains no two-sided ideals besides $\{0\}$ and A itself.*

(c) A representation *of A on V is a homomorphism of algebras* $\rho : A \to \mathrm{End}(V)$. *Here, we regard* $\mathrm{End}(V)$ *as algebra via the multiplication* $(X_1, X_2) \mapsto X_1 \circ X_2$.

Let $\rho : A \to \mathrm{End}(V)$ *and* $\rho' : A \to \mathrm{End}(V')$ *be representations of A.*

(d) A linear subspace $U \subset V$ is called ρ-invariant *if $\rho(x)U \subset U$ holds for every $x \in A$.*

(e) $\rho \neq 0$ *is called* irreducible *if $\{0\}$ and V are the only ρ-invariant subspaces of V.*

[41] One can find in [Tit67] that for $\mathbb{K} = \mathbb{C}$, $n \in \{5\} \cup \mathbb{N}_{\geq 7}$ $\mathrm{Spin}(V,\beta)$ also is the universal covering Lie group of $\mathrm{SO}(V,\beta)$. (See the following locations in [Tit67]: p. 30 for $n \geq 5$ odd, p. 35 for $n \geq 8$ even.)

(f) ρ *and* ρ' *are called* similar *if there is an isomorphism of linear spaces* $\Phi : V \to V'$ *so that*

$$\forall x \in A \ : \ \rho'(x) = \Phi \circ \rho(x) \circ \Phi^{-1}$$

holds.

B.18 Proposition. *Let* V *be a linear space.*

(a) $\mathrm{End}(V)$ *is a simple algebra.*

(b) The trivial representation $\mathrm{id}_{\mathrm{End}(V)}$ *of* $\mathrm{End}(V)$ *on* V *is irreducible.*

Proof. For (a). See [Jac64], Proposition III.2.2, p. 40. *For (b).* This follows from the fact that for any $v_1, v_2 \in V \setminus \{0\}$, there exists $A \in \mathrm{End}(V)$ with $Av_1 = v_2$. □

B.19 Proposition. *Let* A *be a simple algebra. Then any two irreducible representations of* A *on non-zero linear spaces are similar.*

Proof. See [Jac43], Theorem 5.1, p. 93. □

B.20 Definition. *An irreducible representation* $\rho : C(V,\beta) \to \mathrm{End}(S)$ *of the Clifford algebra* $C(V,\beta)$ *on some linear space* $S \neq \{0\}$ *is called a* spin representation *of* $C(V,\beta)$. *In this context,* S *is called the* spinor space *of* ρ *and its elements are called* spinors *of* ρ.

In any case there are at most two spin representations of $C(V,\beta)$ which are not similar to each other (see [LM89], Section I.5, p. 30ff.), but whether there are one or two such representations depends on the base field $\mathbb{K}$, the dimension of V and the index of β. In the following section, we will study one specific case.

B.5 Spin representations for complex linear spaces of even dimension

Let $\mathbb{K} = \mathbb{C}$, V be a $\mathbb{C}$-linear space of even dimension $n = 2r$, and $\beta : V \times V \to \mathbb{C}$ be a non-degenerate symmetric bilinear form on V. It is the object of the present section to show that there is (up to similarity) only one spin representation of the Clifford algebra $C := C(V,\beta)$ and to give an explicit description of this spin representation.

In generalization of Definition 2.19 we call $v \in V$ *isotropic* if $q(v) = 0$ holds; we call a linear subspace $W \subset V$ isotropic if every $v \in W$ is isotropic; in this case we already have $\beta|(W \times W) = 0$. All maximal isotropic subspaces of V have the same dimension ([Che54], I.4.3, p. 17); following the terminology of Chevalley, we call their dimension the *index* of β.

The following relationship between the concept of isotropy defined here and the concept of isotropy in a $\mathbb{C}Q$-space (see Section 2.3) should be noted:

B.21 Lemma. *There exists a complex inner product $\langle\cdot,\cdot\rangle$ on V and a conjugation A on $(V,\langle\cdot,\cdot\rangle)$ so that a linear subspace $W \subset V$ is isotropic in the sense of the previous definition if and only it is $(\mathbb{S}^1 \cdot A)$-isotropic in the sense of Definition 2.19. Moreover, A can be chosen so that β and $\langle\cdot,\cdot\rangle$ coincide on $V(A) \times V(A)$.*

Proof. Let $(b_1,\ldots,b_n)$ be an adapted basis for β (see Proposition 1.4). Then the statement of the lemma is fulfilled with the inner product $\langle\cdot,\cdot\rangle$ characterized by $(b_1,\ldots,b_n)$ being a unitary basis and the anti-linear map $A : V \to V$ characterized by

$$\forall u, v \in V \;:\; \beta(u,v) = \langle u, Av\rangle\,,$$

which is a conjugation on $(V,\langle\cdot,\cdot\rangle)$ by Proposition 1.7. □

B.22 Proposition. *In the present situation, β is necessarily of index r.*

Proof. We regard V as a $\mathbb{C}$Q-space in the way described in Lemma B.21. Then Corollary 2.22 shows that there exist isotropic subspaces of V of any complex dimension $\leq r$, but none of dimension $> r$. Hence the index of β is r. □

In the present situation the structure of the bilinear form β is known completely, as is shown by the following proposition:

B.23 Proposition. *For every r-dimensional isotropic subspace W of V, there exists an r-dimensional isotropic subspace W' of V so that $W \oplus W' = V$ holds. Moreover, for every such subspace W' and every basis $(w_1,\ldots,w_r)$ of W there exists a basis $(w'_1,\ldots,w'_r)$ of W' so that*

$$\forall j,\ell \in \{1,\ldots,r\} \;:\; \beta(w_j, w'_\ell) = \delta_{j\ell} \tag{B.16}$$

holds, and in this situation, we have the following analogue to the Fourier representation:

$$\forall w \in W \;:\; w = \sum_{j=1}^{r} \beta(w, w'_j)\, w_j \quad \textit{and} \quad \forall w' \in W' \;:\; w' = \sum_{j=1}^{r} \beta(w', w_j)\, w'_j\,. \tag{B.17}$$

Proof. For the existence of W' and $(w'_1,\ldots,w'_r)$, see [Che54], I.3.2, p. 13. (When reading that proof, note that Chevalley's notions of totally isotropic subspaces and of singular subspaces coincide in fields of characteristic $\neq 2$ (see [Che54], I.2.1, p. 11), and they correspond to our notion of isotropic subspaces.) Equations (B.17) are an obvious consequence of the isotropy of W and W', and Equation (B.16). □

B.24 Proposition. *Let W be an isotropic subspace of V. Then the subalgebra C^W of C generated by W is a model of the exterior algebra over W.*

Proof. We have $\beta|(W \times W) = 0$, and hence C^W is a Clifford algebra for W equipped with the zero bilinear form by Proposition B.9. Therefore Example B.8 shows that C^W is a model of the exterior algebra over W. □

We now fix an r-dimensional isotropic subspace W of V – the existence of such a subspace follows from Proposition B.22 – and choose via Proposition B.23 another r-dimensional isotropic subspace W' of V which is complementary to W. Moreover, we fix a "unit volume" $\omega \in \bigwedge^r W \setminus \{0\}$. By Proposition B.24, the subalgebra of C generated by W is a model of the exterior algebra of W; for this reason we denote this subalgebra by $\bigwedge W$ in the sequel and define $\bigwedge^k W$ for $k \in \mathbb{Z}$ as in Section B.1. In particular we have $w \cdot \widetilde{w} = -\widetilde{w} \cdot w$ for any $w, \widetilde{w} \in W$; to remind of this rule, we will denote the product $w \cdot \widetilde{w}$ also by $w \wedge \widetilde{w}$. We apply analogous conventions to the subalgebra $\bigwedge W'$ of C generated by W'. It should be noted that the elements of W do not generally anti-commute with the elements of W', rather we have the equation

$$\forall w \in W,\ w' \in W' \ :\ w \cdot w' = \beta(w, w') - w' \cdot w \ .$$

The following proposition gives a first example how the fixation of W and W' gives insight into the structure of the Clifford algebra $C(V, \beta)$. It provides two methods to construct elements of $\Gamma(V, \beta)$ explicitly.

B.25 Proposition. *(a) For every pair $(w, w') \in W \times W'$, we have $q(w + w') = \beta(w, w')$. If $\beta(w, w') \neq 0$ holds, we have $w + w' \in \Gamma(V, \beta)$ and $(w + w')^{-1} = \frac{1}{\beta(w,w')}(w + w')$.*

(b) For any $w, \widetilde{w} \in W$ we have $(1 + w \cdot \widetilde{w}),\ (1 - w \cdot \widetilde{w}) \in \Gamma(V, \beta)$, and these two elements are inverse to each other.

Proof. For (a). We have $q(w+w') = \frac{1}{2}\,\beta(w+w', w+w') = \frac{1}{2}(\beta(w, w)+\beta(w, w')+\beta(w', w)+\beta(w', w')) = \beta(w, w')$. In the case $\beta(w, w') \neq 0$, this equation implies $w + w' \in \Gamma(V, \beta)$ by Proposition B.12(c). The statement on $(w + w')^{-1}$ follows from Proposition B.6(b).

For (b). We have

$$(1 + w \cdot \widetilde{w}) \cdot (1 - w \cdot \widetilde{w}) = 1 - w \cdot \underbrace{\widetilde{w} \cdot w}_{=-w\cdot\widetilde{w}} \cdot \widetilde{w} = 1 + \underbrace{w \cdot w}_{=0} \cdot \widetilde{w} \cdot \widetilde{w} = 1$$

and therefore $(1 + w \cdot \widetilde{w})$ and $(1 - w \cdot \widetilde{w})$ are invertible and inverse to each other. Moreover, for any $v \in V$ we have

$$w \cdot \widetilde{w} \cdot v \cdot w \cdot \widetilde{w} = w \cdot \widetilde{w} \cdot (\beta(v, w) - w \cdot v) \cdot \widetilde{w} = \beta(v, w)\, w \cdot \underbrace{\widetilde{w} \cdot \widetilde{w}}_{=0} - \underbrace{w \cdot \widetilde{w} \cdot w}_{=-w\cdot w\cdot\widetilde{w}=0} \cdot v \cdot \widetilde{w} = 0 \qquad \text{(B.18)}$$

and therefore $g := (1 + w \cdot \widetilde{w})$ satisfies

$$\begin{aligned} \alpha(g) \cdot v \cdot g^{-1} &= (1 + w\widetilde{w}) \cdot v \cdot (1 - w\widetilde{w}) \\ &= v - vw\widetilde{w} + w\widetilde{w}v - w\widetilde{w}vw\widetilde{w} \overset{\text{(B.18)}}{=} v - vw\widetilde{w} + w\widetilde{w}v \\ &= v - (\beta(v, w) - wv)\widetilde{w} + w(\beta(\widetilde{w}, v) - v\widetilde{w}) = v - \beta(v, w)\,\widetilde{w} + \beta(\widetilde{w}, v)\, w \ \in\ V \ , \end{aligned}$$

whence $g \in \Gamma(V, \beta)$ follows. Then we also have $(1 - w \cdot \widetilde{w}) = g^{-1} \in \Gamma(V, \beta)$. □

A linear map $\nu : \bigwedge W \to \bigwedge W$ is called an *anti-derivation of degree* -1, if $\nu(\bigwedge^k W) \subset \bigwedge^{k-1} W$ and

$$\forall \xi \in {\textstyle\bigwedge^k} W,\ \eta \in {\textstyle\bigwedge} W \ :\ \nu(\xi \wedge \eta) = \nu(\xi) \wedge \eta + (-1)^k\, \xi \wedge \nu(\eta) \qquad \text{(B.19)}$$

holds for $k \in \{0, \dots, r\}$. If ν is an anti-derivation of degree -1, then we have $\nu \circ \nu = 0$, as an induction argument based on Equation (B.19) shows.

If a linear form $\delta \in W^*$ is given, there is one and only one anti-derivation $\nu_\delta : \bigwedge W \to \bigwedge W$ of degree -1 which extends δ. ν_δ is the linear map characterized by

$$\forall k \in \mathbb{N},\ w_1, \dots, w_k \in W \ :\ \nu_\delta(w_1 \wedge \dots \wedge w_k) = \sum_{i=1}^{k} (-1)^{i+1} \cdot \delta(w_i) \cdot w_1 \wedge \dots \widehat{w_i} \dots \wedge w_k \ , \quad \text{(B.20)}$$

where $\widehat{w_i}$ denotes the omission of w_i from the product. Moreover, every anti-derivation ν of degree -1 on $\bigwedge W$ is obtained in this way: we have $\nu = \nu_\delta$ with $\delta = \nu|W : W \to \mathbb{K}$. In this way, the space of anti-derivations on $\bigwedge W$ is isomorphic to W^*.

We are now ready to describe the spin representation of C:

B.26 Theorem. *Let us put* $S := \bigwedge W$ *and consider the linear map* $f : V \to \mathrm{End}(S)$ *given by*

$$\forall w \in W \ : \ f(w) = (\xi \mapsto w \wedge \xi) \quad \textit{and} \quad \forall w' \in W' \ : \ f(w') = \nu_{\beta(\cdot,w')} ,$$

where $\nu_{\beta(\cdot,w')}$ *is defined as in Equation* (B.20).

Then f *is a Clifford map and the algebra homomorphism* $\rho : C(V,\beta) \to \mathrm{End}(S)$ *uniquely determined by* $\rho|V = f$ *is in fact an algebra isomorphism.* ρ *is a spin representation of* $C(V,\beta)$*, and any other spin representation of* $C(V,\beta)$ *is similar to* ρ*.*

For the proof of this theorem, we introduce the following notation, which will also be used on several other occasions: Whenever $(w_1,\dots,w_r)$ is a basis of W and $\varnothing \neq N \subset \{1,\dots,r\}$ holds, say $N = \{j_1,\dots,j_k\}$ with $1 \leq j_1 < \dots < j_k \leq r$, we put

$$w_N := w_{j_1} \wedge \dots \wedge w_{j_k} \ ; \quad \text{we also put } w_\varnothing := 1_{\bigwedge W} . \tag{B.21}$$

Then $(w_N)_{N\subset\{1,\dots,r\}}$ is a basis of $\bigwedge W$, see Section B.1.

Proof of Theorem B.26. To prove that f is a Clifford map, one has to show for any $w \in W$, $w' \in W'$:

$$f(w) \circ f(w) = 0 \ , \quad f(w') \circ f(w') = 0 \tag{B.22}$$

and

$$f(w) \circ f(w') + f(w') \circ f(w) = \beta(w,w') \cdot \mathrm{id}_S \ . \tag{B.23}$$

The first equation of (B.22) is obvious and the second equation of (B.22) follows from the fact that $\nu \circ \nu = 0$ holds for any anti-derivation ν. For (B.23): Let $\xi \in S$ be given. Using the fact that $f(w')$ is an anti-derivation of degree -1 on S, we obtain

$$\begin{aligned} (\, f(w) \circ f(w') + f(w') \circ f(w)\,)\xi \ &= \ w \wedge (f(w')\xi) + f(w')(w \wedge \xi) \\ &\overset{(B.19)}{=} w \wedge (f(w')\xi) + (f(w')w) \wedge \xi - w \wedge (f(w')\xi) = \beta(w',w) \cdot \xi \end{aligned}$$

and therefore Equation (B.23). This shows that f is a Clifford map and therefore the existence and uniqueness of the algebra homomorphism ρ.

We next show that ρ is in fact an isomorphism of algebras. For this purpose, we fix a basis $(w_1,\dots,w_r)$ of W and use the notation w_N of (B.21) with respect to this basis. Below, we will show

$$\forall\, N, N' \subset \{1,\dots,r\}\ \exists\, \xi \in C(V,\beta)\ \forall\, M \subset \{1,\dots,r\} \ : \ \rho(\xi) w_M = \begin{cases} w_{N'} & \text{if } M = N \\ 0 & \text{if } M \neq N \end{cases} . \tag{B.24}$$

Because $(w_N)_{N\subset\{1,\dots,r\}}$ is a basis of S, (B.24) shows that when N and N' run through all subsets of $\{1,\dots,r\}$, the corresponding endomorphisms $\rho(\xi)$ run through a basis of $\mathrm{End}(S)$. Therefore $\rho : C(V,\beta) \to \mathrm{End}(S)$ is surjective. Because we have $\dim \mathrm{End}(S) = (2^r)^2 = 2^n = \dim C(V,\beta)$, ρ is an isomorphism of algebras.

For the proof of (B.24): By Proposition B.23 there exists a basis $(w_1',\dots,w_r')$ of W' so that $\beta(w_k,w_\ell') = \delta_{k\ell}$ holds for any k,ℓ. Now let $N,N' \subset \{1,\dots,r\}$ be given, say $N = \{j_1,\dots,j_k\}$ with $1 \leq j_1 < \dots < j_k \leq r$ and analogously $N' = \{j_1',\dots,j_{k'}'\}$ with $1 \leq j_1' < \dots < j_{k'}' \leq r$. ($k = 0$ or $k' = 0$ is permitted, then N resp. N'

is empty.) We also consider $1 \le \widehat{j}_1 < \ldots < \widehat{j}_{r-k} \le r$ so that $\{\widehat{j}_1, \ldots, \widehat{j}_{r-k}\} = \{1, \ldots, r\} \setminus N$ holds. Then we put $\xi := \xi_3\, \xi_2\, \xi_1 \in C(V, \beta)$ with

$$\xi_1 := w'_{j_k} \cdots w'_{j_1} \;, \quad \xi_2 := (w'_{\widehat{j}_{r-k}} \cdot w_{\widehat{j}_{r-k}}) \cdots (w'_{\widehat{j}_1} \cdot w_{\widehat{j}_1}) \quad \text{and} \quad \xi_3 := w_{j'_1} \cdots w_{j'_{k'}} \;.$$

Now let $M \subset \{1, \ldots, r\}$ be given. Then we have

$$\rho(\xi_1) w_M = \begin{cases} 1 & \text{if } N = M \\ \pm w_{M \setminus N} & \text{if } N \subsetneq M \\ 0 & \text{if } N \not\subset M \end{cases} ,$$

therefore

$$\rho(\xi_2\, \xi_1) w_M = \begin{cases} 1 & \text{if } N = M \\ 0 & \text{if } N \neq M \end{cases}$$

and hence

$$\rho(\xi) w_M = \rho(\xi_3\, \xi_2\, \xi_1) w_M = \begin{cases} w_{N'} & \text{if } N = M \\ 0 & \text{if } N \neq M \end{cases} .$$

Thus, Equation (B.24) is shown with this choice of ξ.

The representation ρ is irreducible because $\rho(C) = \mathrm{End}(S)$ acts irreducibly on S by Proposition B.18(b), and therefore ρ is a spin representation. Because C is via ρ isomorphic to the algebra $\mathrm{End}(S)$, it is simple by Proposition B.18(a), and therefore Proposition B.19 shows that any other spin representation of C is similar to ρ. □

As Theorem B.26 shows, any two spin representations of C are similar. From here on, we therefore denote by $\rho : C \to \mathrm{End}(S)$ always the spin representation described in Theorem B.26, and by $S = \bigwedge W$ the corresponding spinor space.

We note some elementary properties of ρ:

B.27 Proposition. *(a) $\Gamma(V, \beta) \times S \to S,\ (g, s) \mapsto \rho(g)s$ is a linear Lie group action.*

(b) (i) For any $k \in \mathbb{N}$ and $w \in W$ we have $\rho(w) \bigwedge^k W \subset \bigwedge^{k+1} W$, also for $w' \in W'$ we have $\rho(w') \bigwedge^k W \subset \bigwedge^{k-1} W$.

(ii) For any $v \in V$, $\rho(v)$ maps $\bigwedge^{\mathrm{even}} W$ into $\bigwedge^{\mathrm{odd}} W$ and conversely.

(iii) For any $\xi \in C^+(V, \beta)$, $\rho(\xi)$ leaves $\bigwedge^{\mathrm{even}} W$ and $\bigwedge^{\mathrm{odd}} W$ invariant.

(c) For any $s \in S$, we have $\rho(s) 1_S = s$; for any $\xi \in \bigwedge W'$, we have $\rho(\xi) 1_S = 0$.

Proof. (a) is obvious. *For (b).* For (i), we have for any $k \in \mathbb{N}$ and any $w \in W$, $w' \in W'$ and $s \in \bigwedge^k W$

$$\rho(w)s = w \wedge s \in \bigwedge\nolimits^{k+1} W \quad \text{and} \quad \rho(w')s = \nu_{\beta(\cdot, w')} s \in \bigwedge\nolimits^{k-1} W \,.$$

For (ii), let $v \in V$ be given, say $v = w + w'$ with $w \in W$ and $w' \in W'$. Then we have for any $s \in \bigwedge^k W$ by (b)(i)

$$\rho(v)s = \rho(w)s + \rho(w')s \in \bigwedge\nolimits^{k+1} W \oplus \bigwedge\nolimits^{k-1} W \,,$$

whence (ii) follows. As a consequence, we see that for any $\ell \in \mathbb{N}$, $v_1, \ldots v_{2\ell} \in V$ and $\eta := v_1 \cdots v_{2\ell} \in C^+(V, \beta)$, $\rho(\eta)$ leaves $\bigwedge^{\mathrm{even}} W$ and $\bigwedge^{\mathrm{odd}} W$ invariant. Because $C^+(V, \beta)$ is spanned by the elements of the form of η, (iii) follows.

For (c). Because of the linearity of ρ it suffices to prove the first part of (c) for the case where $s = w_1 \wedge \ldots \wedge w_k \in S$ is a decomposable spinor, and then we have

$$\rho(s) 1_S = \rho(w_1 \cdots w_k) 1_S = \rho(w_1 \cdots w_{k-1}) w_k = \ldots = \rho(1)(w_1 \wedge \ldots \wedge w_k) = s \,.$$

The second part of (c) follows from (b)(i). □

The restriction of ρ to $C^+(V,\beta)$ is no longer irreducible, because it leaves the spaces $S_+ := \bigwedge^{\text{even}} W$ and $S_- := \bigwedge^{\text{odd}} W$ invariant by Proposition B.27(b)(iii). S_+ resp. S_- is called the *space of even* resp. *odd half-spinors*. It can be shown that the representations $\widetilde{\rho}_\pm : C^+(V,\beta) \to \operatorname{End}(S_\pm)$, $g \mapsto \rho(g)|S_\pm$ are irreducible, and that $\rho_\pm := \widetilde{\rho}_\pm|\operatorname{Spin}(V,\beta)$ is an irreducible linear Lie group action of the Lie group $\operatorname{Spin}(V,\beta)$ on $S_\pm$ (see [LM89], Proposition I.5.15, p. 36).

B.28 Proposition. *There exists no group representation $\sigma_\pm : \operatorname{SO}(V,\beta) \to \operatorname{GL}(S_\pm)$ so that $\rho_\pm = \sigma_\pm \circ (\chi|\operatorname{Spin}(V,\beta))$ holds. In particular, neither ρ_+ nor ρ_- is similar to $\chi|\operatorname{Spin}(V,\beta)$.*

Proof. If such a representation $\sigma_\pm$ existed, we would have $\ker(\chi|\operatorname{Spin}(V,\beta)) \subset \ker(\rho_\pm)$. But we have $\ker(\chi|\operatorname{Spin}(V,\beta)) = \{\pm 1\}$ by Proposition B.15(d), whereas $\rho(-1) = -\operatorname{id}_S$ and therefore $-1 \notin \ker(\rho_\pm)$ holds. □

B.29 Proposition. $\dim S = 2^r$ *and* $\dim S_+ = \dim S_- = 2^{r-1}$.

Proof. We note that $\dim \bigwedge^k W = \binom{r}{k}$ holds. If we set $x=1$ in the binomial equation $(1+x)^r = \sum_{k=0}^r \binom{r}{k} x^k$, we obtain $2^r = \sum_{k=0}^r \dim \bigwedge^k W = \dim S$. If we set $x=-1$ in the binomial equation, we obtain $0 = \sum_{k=0}^r (-1)^k \binom{r}{k}$, and hence $\dim S_+ = \dim S_-$. □

As we saw in Section B.4, the action of $\operatorname{Spin}(V,\beta)$ on V via the vector representation χ leaves the bilinear form β invariant. We now introduce a bilinear form β_S on S which is invariant under the action of $\operatorname{Spin}(V,\beta)$ on S via the spin representation ρ. For the study of the spinor space S, β_S will play a similar role as β does for the study of V.

For this, we consider the involutive algebra anti-automorphism $\kappa := \alpha \circ \gamma = \gamma \circ \alpha : C \to C$, where α is the canonical involution of C (see Proposition B.10) and γ is the conjugation of C (see Proposition B.11). κ is called the *main anti-automorphism* of C. We have $\kappa|V = \operatorname{id}_V$ and therefore

$$\forall v_1, \dots, v_k \in V \ : \ \kappa(v_1 \cdots v_k) = v_k \cdots v_1 \ ; \tag{B.25}$$

as a consequence of this equation we see that κ leaves $\bigwedge W = S$ and $\bigwedge W'$ invariant. It also follows that we have

$$\forall k \in \{0, \dots, r\},\ \xi \in \textstyle\bigwedge^k W \cup \bigwedge^k W' \ : \ \kappa(\xi) = (-1)^{k(k-1)/2}\, \xi \, . \tag{B.26}$$

B.30 Proposition. *Via the fixed "unit volume" $\omega \in \bigwedge^r W \setminus \{0\}$ we define a linear form $\varphi : S \to \mathbb{C}$ by*

$$\varphi(\omega) = 1 \quad \textit{and} \quad \forall k < r \ : \ \varphi|\textstyle\bigwedge^k W = 0 \, .$$

(a) The map

$$\beta_S : S \times S \to \mathbb{C},\ (s_1, s_2) \mapsto \varphi(\kappa(s_1) \wedge s_2)$$

is bilinear and non-degenerate.

(b) For $g \in \Gamma(V,\beta)$, we put $\varepsilon(g) := \alpha(g)\, g^{-1} \in \{\pm 1\}$ (see Proposition B.12(f)). For $s_1, s_2 \in S$ and $v \in V$, we then have

(i) $\beta_S(\rho(v)s_1, \rho(v)s_2) = q(v) \cdot \beta_S(s_1, s_2)$

(ii) $\beta_S(\rho(g)s_1, \rho(g)s_2) = \varepsilon(g) \cdot \lambda(g) \cdot \beta_S(s_1, s_2)$

(iii) $\beta_S(\rho(g)s_1, \rho(g)s_2) = \beta_S(s_1, s_2)$ *for* $g \in \mathrm{Spin}(V, \beta)$

(iv) $\beta_S(\rho(v)s_1, s_2) = \beta_S(s_1, \rho(v)s_2)$

(v) $\beta_S(s_2, s_1) = (-1)^{r(r-1)/2} \cdot \beta_S(s_1, s_2)$

(c) *For* $s_1 \in \bigwedge^{k_1} W$ *and* $s_2 \in \bigwedge^{k_2} W$, *we have* $\beta_S(s_1, s_2) = 0$ *whenever* $k_1 + k_2 \neq r$. *Consequently:*

If r *is even, we have* $\beta_S|(S_+ \times S_-) = 0$ *and* $\beta_S|(S_- \times S_+) = 0$; *if* r *is odd, we have* $\beta_S|(S_+ \times S_+) = 0$ *and* $\beta_S|(S_- \times S_-) = 0$.

(d) *Suppose that* W *is endowed with the structure of an oriented unitary space so that* ω *is the positive unit* r*-vector of* W *(see Section B.2). Then we have*

$$\forall s_1 \in \textstyle\bigwedge^k W, s_2 \in \bigwedge^{r-k} W \ : \ \beta_S(s_1, s_2) = (-1)^{rk} \cdot (-1)^{k(k+1)/2} \cdot \langle s_1, *s_2 \rangle \, .$$

Here, $*$ *denotes the Hodge operator of* $\bigwedge W$, *see Proposition B.2.*

Proof. For (a). It is obvious that β_S is bilinear. For the proof of the non-degeneracy of β_S, we let $s \in S$ be given so that $\beta(s, \cdot) = 0$ holds. We fix a basis $(w_1, \ldots, w_r)$ of W and use the notation w_N of (B.21) with respect to this basis. Because $(w_N)_{N \subset \{1,\ldots,r\}}$ is a basis of S, there exist numbers $c_N \in \mathbb{C}$ so that $s = \sum c_N \, w_N$ holds. Let $N \subset \{1, \ldots, r\}$ be given, then we have

$$0 = \beta_S(s, w_{\{1,\ldots,r\}\setminus N}) = c_N \, \beta_S(w_N, w_{\{1,\ldots,r\}\setminus N}) = \pm c_N \underbrace{\varphi(w_{\{1,\ldots,r\}})}_{\neq 0}$$

and therefore $c_N = 0$. Thus we have shown $s = 0$.

For (b). (See also [Che54], p. 77f.) There exists a basis $(w_1, \ldots, w_r)$ of W so that $\omega = w_1 \wedge \ldots \wedge w_r$ holds, and a basis $(w'_1, \ldots, w'_r)$ of W' so that

$$\forall j, k \leq r \ : \ \beta(w_j, w'_k) = \delta_{jk} \tag{B.27}$$

holds (see Proposition B.23). By Proposition B.6(a) we have $w \cdot w' = \beta(w, w') - w' \cdot w$ for any $w \in W$, $w' \in W'$ and therefore

$$\forall j, k \leq r \ : \ w_j \cdot w'_k = \delta_{jk} - w'_k \cdot w_j \, . \tag{B.28}$$

We put $\omega' := w'_1 \wedge \ldots \wedge w'_r \in \bigwedge^r W'$ and note that we have by Equation (B.26)

$$\kappa(\omega') = \varepsilon \, \omega' \quad \text{with} \quad \varepsilon := (-1)^{r(r-1)/2} \, . \tag{B.29}$$

The most important objects of the present situation can be expressed using the multiplication of the Clifford algebra C and its main anti-automorphism κ, as the following equations show.

$$\forall s \in S \ : \ \varphi(s) \cdot \omega' = \varepsilon \, \omega' \cdot s \cdot \omega' \tag{B.30}$$

$$\forall s_1, s_2 \in S \ : \ \beta_S(s_1, s_2) \cdot \omega' = \varepsilon \, \omega' \cdot \kappa(s_1) \cdot s_2 \cdot \omega' \tag{B.31}$$

$$\forall v \in V, \, s \in S \ : \ (\rho(v)s) \cdot \omega' = v \cdot s \cdot \omega' \tag{B.32}$$

$$\forall v \in V, \, s \in S \ : \ \omega' \cdot \kappa(\rho(v)s) = \omega' \cdot \kappa(s) \cdot v \, . \tag{B.33}$$

For (B.30): Because both sides of Equation (B.30) are linear in s, it suffices to prove that equation for $s = w_N$ with $N \subset \{1, \ldots, r\}$. In the case $k < r$ there exists $j \in \{1, \ldots, r\} \setminus N$, and (B.28) shows that we have $w'_j \cdot s = (-1)^k s \cdot w'_j$. From this fact we obtain

$$\omega' \cdot s \cdot \omega' = (-1)^{(r-j)+k+(j-1)} \cdot (w'_1 \wedge \ldots \wedge w'_{j-1} \wedge w'_{j+1} \wedge \ldots \wedge w'_r) \cdot s \cdot (w'_1 \wedge \ldots \wedge w'_{j-1} \wedge \underbrace{w'_j \wedge w'_j}_{=0} \wedge w'_{j+1} \wedge \ldots \wedge w'_r)$$

$$= 0 = \varepsilon \varphi(s) \cdot \omega' \, .$$

On the other hand, in the case $k = r$ we have $s = \omega$ and therefore $\varphi(s) = 1$. Hence we have to show the equality

$$\omega' \cdot \omega \cdot \omega' = (-1)^{r(r-1)/2} \cdot \omega' ,$$

which is verified by a direct calculation using Equation (B.28).

For (B.31): For given $s_1, s_2 \in S$ we have by Equation (B.30): $\beta_S(s_1,s_2)\cdot\omega' = \varphi(\kappa(s_1)\cdot s_2)\cdot\omega' = \varepsilon\omega'\cdot\kappa(s_1)\cdot s_2\cdot\omega'$.

For (B.32): Both sides of Equation (B.32) are linear in v, therefore it suffices to show that equation for the elements of the basis $(w_1,\dots,w_r,w_1',\dots,w_r')$ of V. If we have $v = w_j \in W$, we have for any $s \in S$ by the definition of ρ: $(\rho(w_k)s)\cdot\omega' = (w_k \cdot s)\cdot\omega'$. Let us now consider the case $v = w_j'$. Because both sides of (B.32) are also linear in s, we may restrict our considerations to $s = w_N$ with $N \subset \{1,\dots,r\}$. We further distinguish the cases $j \in N$ and $j \notin N$. In the case $j \in N$ we put $\ell := \#\{\, j' \in N \,|\, j' < \ell \,\}$, then we have

$$\begin{aligned} v\cdot s\cdot\omega' &= w_j'\cdot w_N\cdot\omega' = (-1)^\ell\, w_j'\cdot w_j\cdot w_{N\setminus\{j\}}\cdot\omega' \\ &\overset{(B.28)}{=} (-1)^\ell\,(1 - w_j\cdot w_j')\cdot w_{N\setminus\{j\}}\cdot\omega' \\ &= (-1)^\ell\, w_{N\setminus\{j\}}\cdot\omega' - (-1)^{\ell+\#N-1}\, w_j\cdot w_{N\setminus\{j\}}\cdot\underbrace{w_j'\cdot\omega'}_{=0} = \rho(w_j')s\cdot\omega' . \end{aligned}$$

On the other hand, if $j \notin N$ holds, we have

$$v\cdot s\cdot\omega' = w_j'\cdot w_N\cdot\omega' \overset{(B.28)}{=} (-1)^{\#N}\, w_N\cdot w_j'\cdot\omega' = 0 = \nu_{\beta(\cdot,w_j')}w_N\cdot\omega' = \rho(v)s\cdot\omega' .$$

For (B.33): We first note that we have for any $s \in S$

$$\omega'\cdot\kappa(s) \overset{(B.29)}{=} \varepsilon\,\kappa(\omega')\cdot\kappa(s) = \varepsilon\,\kappa(s\cdot\omega') . \tag{B.34}$$

Now we obtain

$$\omega'\cdot\kappa(\rho(v)s) \overset{(B.34)}{=} \varepsilon\,\kappa(\rho(v)s\cdot\omega') \overset{(B.32)}{=} \varepsilon\,\kappa(v\cdot s\cdot\omega') = \underbrace{\varepsilon\,\kappa(\omega')}_{\overset{(B.29)}{=}\omega'}\cdot\kappa(s)\cdot\underbrace{\kappa(v)}_{=v} = \omega'\cdot\kappa(s)\cdot v .$$

For (b)(i). We have

$$\begin{aligned} \beta_S(\rho(v)s_1,\rho(v)s_2)\cdot\omega' &\overset{(B.31)}{=} \varepsilon\,\omega'\cdot\kappa(\rho(v)s_1)\cdot\rho(v)s_2\cdot\omega' \overset{\substack{(B.32)\\(B.33)}}{=} \varepsilon\,\omega'\cdot\kappa(s_1)\cdot\underbrace{v\cdot v}_{=q(v)\cdot 1_C}\cdot s_2\cdot\omega' \\ &= q(v)\,\varepsilon\,\omega'\cdot\kappa(s_1)\cdot s_2\cdot\omega' \overset{(B.31)}{=} q(v)\,\beta_S(s_1,s_2)\cdot\omega' , \end{aligned}$$

whence (b)(i) follows.

For (b)(ii). Let $g \in \Gamma(V,\beta)$ be given. By Proposition B.12(f), there exist $v_1,\dots,v_k \in V$ with $g = v_1\cdots v_k$. By (b)(i) and Proposition B.14(b), we have

$$\begin{aligned} \beta_S(\rho(g)s_1,\rho(g)s_2) &= \beta_S(\rho(v_1)\cdots\rho(v_k)s_1,\rho(v_1)\cdots\rho(v_k)s_2) \\ &= q(v_1)\cdots q(v_k)\cdot\beta_S(s_1,s_2) = (-\lambda(v_1))\cdots(-\lambda(v_k))\cdot\beta_S(s_1,s_2) \\ &= \varepsilon(g)\cdot\lambda(g)\cdot\beta_S(s_1,s_2) . \end{aligned}$$

For (b)(iii). This is an immediate consequence of (b)(ii) and Proposition B.15(c)(ii).

For (b)(iv). We have

$$\beta_S(\rho(v)s_1,s_2)\cdot\omega' \overset{(B.31)}{=} \varepsilon\,\omega'\cdot\kappa(\rho(v)s_1)\cdot s_2\cdot\omega' \overset{(B.33)}{=} \varepsilon\,\omega'\cdot\kappa(s_1)\cdot v\cdot s_2\cdot\omega' \overset{(B.32)}{=} \varepsilon\,\omega'\cdot\kappa(s_1)\cdot\rho(v)s_2\cdot\omega' \overset{(B.31)}{=} \beta_S(s_1,\rho(v)s_2)\cdot\omega' .$$

For (b)(v). We have

$$\begin{aligned} \beta_S(s_2,s_1)\cdot\omega' &\overset{(B.31)}{=} \varepsilon\,\omega'\cdot\kappa(s_2)\cdot s_1\cdot\omega' \overset{(B.29)}{=} \kappa(\omega')\cdot\kappa(s_2)\cdot\kappa(\kappa(s_1))\cdot\kappa(\kappa(\omega')) = \kappa(\kappa(\omega')\cdot\kappa(s_1)\cdot s_2\cdot\omega') \\ &\overset{(B.29)}{=} \kappa(\varepsilon\,\omega'\cdot\kappa(s_1)\cdot s_2\cdot\omega') \overset{(B.31)}{=} \kappa(\beta_S(s_1,s_2)\cdot\omega') = \beta_S(s_1,s_2)\cdot\kappa(\omega') \overset{(B.29)}{=} \varepsilon\,\beta_S(s_1,s_2)\cdot\omega' . \end{aligned}$$

For (c). This is a direct consequence of the definition of β_S .

For (d). Let $s_1 \in \bigwedge^k W$ and $s_2 \in \bigwedge^{r-k} W$ be given. We have

$$\kappa(s_1) = (-1)^{k(k-1)/2} \cdot s_1 \ , \tag{B.35}$$

and from Proposition B.2(c) we see that

$$s_2 = (-1)^{(r-k)k} \cdot (* * s_2) \tag{B.36}$$

holds. Using these equations, we obtain

$$\begin{aligned} \beta_S(s_1, s_2) &= \varphi(\kappa(s_1) \wedge s_2) \overset{(B.35)}{=} (-1)^{k(k-1)/2} \cdot \varphi(s_1 \wedge s_2) \\ &\overset{(B.36)}{=} (-1)^{k(k-1)/2} \cdot (-1)^{(r-k)k} \cdot \varphi(s_1 \wedge (* * s_2)) \\ &= (-1)^{k(k+1)/2} \cdot (-1)^{rk} \cdot \varphi(\langle s_1, *s_2 \rangle \cdot \omega) \\ &= (-1)^{k(k+1)/2} \cdot (-1)^{rk} \cdot \langle s_1, *s_2 \rangle \ . \end{aligned}$$

□

B.6 The Principle of Triality

Triality is a specific phenomenon occurring in the case $\dim V = 8$ which exhibits a relationship between the vector representation $\chi|\mathrm{Spin}(V, \beta)$ on V and the spin representations $\rho_\pm$ on the spaces $S_\pm$ of even resp. odd half-spinors. The present description of triality closely follows the approach of [Che54], Chapter IV.[42]

We now suppose in the situation of Section B.5 that $n = \dim V = 8$ and hence $r = 4$ holds. We consider the non-degenerate bilinear form $\beta_S : S \times S \to \mathbb{C}$ of Proposition B.30, which here is symmetric by Proposition B.30(b)(v). It therefore induces a quadratic form

$$q_S : S \to \mathbb{C},\ s \mapsto \tfrac{1}{2}\beta_S(s, s)\ .$$

We put $\beta_+ := \beta_S|(S_+ \times S_+)$ and $\beta_- := \beta_S|(S_- \times S_-)$; these symmetric bilinear forms are still non-degenerate because S_+ and S_- are β_S-orthogonal to each other by Proposition B.30(c). Their corresponding quadratic forms are $q_+ := q_S|S_+$ resp. $q_- := q_S|S_-$.

Proposition B.29 shows that

$$\dim S_+ = \dim S_- = 8 = \dim V$$

holds. As we will show in the present section, the representations χ, ρ_+ and ρ_- on the spaces V , S_+ resp. S_- are in fact "intertwined" in the following way: There exists a Lie group automorphism $\vartheta : \mathrm{Spin}(V, \beta) \to \mathrm{Spin}(V, \beta)$ with $\vartheta^3 = \mathrm{id}_{\mathrm{Spin}(V,\beta)}$ and $\mathbb{C}$-linear isometries $T_{V+} : (V, \beta) \to (S_+, \beta_+)$, $T_{+-} : (S_+, \beta_+) \to (S_-, \beta_-)$ and $T_{-V} : (S_-, \beta_-) \to (V, \beta)$ with $T_{-V} \circ T_{+-} \circ T_{V+} = \mathrm{id}_V$, so that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Spin}(V,\beta) \times V \arrow[r, "\vartheta \times T_{V+}"] \arrow[d, "\chi"'] & \mathrm{Spin}(V,\beta) \times S_+ \arrow[r, "\vartheta \times T_{+-}"] \arrow[d, "\rho_+"'] & \mathrm{Spin}(V,\beta) \times S_- \arrow[r, "\vartheta \times T_{-V}"] \arrow[d, "\rho_-"'] & \mathrm{Spin}(V,\beta) \times V \arrow[d, "\chi"] \\ V \arrow[r, "T_{V+}"'] & S_+ \arrow[r, "T_{+-}"'] & S_- \arrow[r, "T_{-V}"'] & V\ . \end{tikzcd} \tag{B.37}$$

[42]However, when applying information from [Che54] it should be noted that [Che54] uses the "non-twisted" vector representation, see Remark B.13.

Here, we denote by χ and $\rho_\pm$ also the maps

$$\chi : \mathrm{Spin}(V,\beta) \times V \to V,\ (g,v) \mapsto \chi(g)v \quad \text{resp.} \quad \rho_\pm : \mathrm{Spin}(V,\beta) \times S_\pm \to S_\pm,\ (g,s) \mapsto \rho_\pm(g)s\ .$$

The fact of the existence of maps ϑ and $T_{\dots}$ so that Diagram (B.37) commutes is called the "principle of triality". This name reflects the relationship between the three representations χ, ρ_+ and ρ_- described by the diagram.

For the construction of the isomorphisms, we consider the "composite" 24-dimensional linear space $\mathfrak{T} := V \oplus S_+ \oplus S_-$. We will define a composition map $\diamond : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ so that $(\mathfrak{T}, \diamond)$ becomes a non-associative algebra. It will then turn out that the maps $T_{\dots}$ of Diagram (B.37) can be defined as restrictions of an automorphism T of the algebra $(\mathfrak{T}, \diamond)$. In this regard, the algebra $(\mathfrak{T}, \diamond)$ carries the information of triality.

First, we note that the Clifford group $\Gamma(V,\beta)$ acts on $\mathfrak{T}$ via the "composite" linear representation $\mu : \Gamma(V,\beta) \to \mathrm{GL}(\mathfrak{T})$ given by

$$\forall g \in \Gamma(V,\beta),\ v \in V,\ s_+ \in S_+,\ s_- \in S_-\ :\ \mu(g)(v+s_++s_-) := \chi(g)v+\rho(g)s_++\rho(g)s_-\ . \tag{B.38}$$

μ is injective because we have $\ker \mu = \ker \chi \cap \ker(\rho|\Gamma(V,\beta)) = \{1\}$.

Let $\beta_{\mathfrak{T}} : \mathfrak{T} \times \mathfrak{T} \to \mathbb{C}$ be the "composite" bilinear form characterized by

$$\beta_{\mathfrak{T}}(v + s_+ + s_-,\ v' + s'_+ + s'_-) = \beta(v,v') + \beta_+(s_+,s'_+) + \beta_-(s_-,s'_-)$$

for every $v, v' \in V,\ s_+, s'_+ \in S_+$ and $s_-, s'_- \in S_-$. Because β, β_+ and β_- are non-degenerate and symmetric, so is $\beta_{\mathfrak{T}}$. With respect to $\beta_{\mathfrak{T}}$, the spaces V, S_+ and S_- are pairwise orthogonal to each other.

The map $F : \mathfrak{T} \to \mathbb{C}$ defined by

$$\forall v \in V,\ s_+ \in S_+,\ s_- \in S_-\ :\ F(v + s_+ + s_-) = \beta_-(\rho(v)s_+, s_-) = \beta_+(s_+, \rho(v)s_-)$$

(for the second equality sign see Proposition B.30(b)(iv)) is a cubic form on $\mathfrak{T}$, meaning that $F(t\,X) = t^3\,F(X)$ holds for every $X \in \mathfrak{T}$ and $t \in \mathbb{C}$. Therefore there exists one and only one symmetric, trilinear form[43] $\gamma : \mathfrak{T} \times \mathfrak{T} \times \mathfrak{T} \to \mathbb{C}$ so that

$$\forall X \in \mathfrak{T}\ :\ \tfrac{1}{6} \cdot \gamma(X,X,X) = F(X)$$

holds; γ can be explicitly described in the following way: Let $X_1, X_2, X_3 \in \mathfrak{T}$ be given, say $X_k = v_k + s_{+,k} + s_{-,k}$ with $v_k \in V$ and $s_{\pm,k} \in S_\pm$ for $k \in \{1,2,3\}$. Then we have

$$\gamma(X_1,X_2,X_3) = \sum_{\sigma \in \mathfrak{S}_3} F(v_{\sigma(1)} + s_{+,\sigma(2)} + s_{-,\sigma(3)})\ . \tag{B.39}$$

We now define the composition map $\diamond : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$. Let $X, Y \in \mathfrak{T}$ be given. Then $\gamma(X,Y,\cdot)$ is a linear form on $\mathfrak{T}$. Because $\beta_{\mathfrak{T}}$ is non-degenerate, there exists one and only one element $X \diamond Y \in \mathfrak{T}$ so that

$$\gamma(X,Y,\cdot) = \beta_{\mathfrak{T}}(X \diamond Y, \cdot) \tag{B.40}$$

[43] This trilinear form should not be confused with the conjugation of the Clifford algebra $C(V,\beta)$, which we previously also denoted by γ.

holds. The composition map $\diamond$ so defined turns $\mathfrak{T}$ into a $\mathbb{C}$-algebra, which is commutative (because γ is in particular symmetric in its first two entries) but not associative and which does not have a unit element (see Remark B.32 below). We call $(\mathfrak{T}, \diamond)$ the *triality algebra*.

B.31 Proposition. *(a) (i)* $\forall v \in V,\ s_+ \in S_+,\ s_- \in S_- \ :\ \gamma(v, s_+, s_-) = F(v + s_+ + s_-)$.

(ii) We have $\gamma(X, Y, Z) = 0$ *in either of the following two situations:*

(1) all *of* X, Y, Z *are from one and the same of the spaces* $V \oplus S_+$ *,* $V \oplus S_-$ *or* $S_+ \oplus S_-$ *,*

(2) two *of* X, Y, Z *are from one and the same of the spaces* V *,* S_+ *or* S_- *.*

(b) Let $v, v' \in V$ *and* $s_\pm, s'_\pm \in S_\pm$ *be given. Then the composition* $\diamond$ *is described by the following composition table:*

$\diamond$	v'	s'_+	s'_-
v	0	$\rho(v)s'_+$	$\rho(v)s'_-$
s_+		0	$(\, v \mapsto \beta_-(\rho(v)s_+, s'_-)\,)^\sharp$
s_-			0

Here, for every $\alpha \in V^*$ *let* $\alpha^\sharp \in V$ *be the vector uniquely characterized by* $\beta(\alpha^\sharp, \cdot) = \alpha$ *. Note that* $\diamond$ *is commutative, and therefore completely specified by the above table. In particular, we have the following relations:*

$$V \diamond S_+ \subset S_- \ , \ V \diamond S_- \subset S_+ \quad \text{and} \quad S_+ \diamond S_- \subset V \ . \tag{B.41}$$

(c) For any $v, v_1, v_2 \in V$ *,* $s, s_1, s_2 \in S$ *we have*

(i) $v \diamond (v \diamond s) = q(v) \cdot s$

(ii) $\beta_S(v \diamond s_1, v \diamond s_2) = q(v) \cdot \beta_S(s_1, s_2)$

(iii) $\beta_S(v_1 \diamond s, v_2 \diamond s) = q_S(s) \cdot \beta(v_1, v_2)$.

(d) If $\sigma : \mathfrak{T} \to \mathfrak{T}$ *is a linear map which leaves both the bilinear form* $\beta_{\mathfrak{T}}$ *and the cubic form* F *invariant, then* σ *is an algebra automorphism of* $(\mathfrak{T}, \diamond)$ *.*

(e) Let $g \in \Gamma(V, \beta)$ *be given and put* $\varepsilon(g) := \alpha(g)\, g^{-1} \in \{\pm 1\}$ *(see Proposition B.12(f)). If* $\lambda(g) = \varepsilon(g)$ *holds, then* $\varepsilon(g) \cdot \mu(g)$ *is an automorphism of* $(\mathfrak{T}, \diamond)$ *which leaves* $\beta_{\mathfrak{T}}$ *and* F *invariant. It leaves* V *invariant; for* $\varepsilon(g) = 1$ *it also leaves* S_+ *and* S_- *invariant, whereas for* $\varepsilon(g) = -1$ *it exchanges* S_+ *and* S_- *.*

As a consequence of Proposition B.31(e), we see that $\mu(g)$ leaves the symmetric bilinear form $\beta_{\mathfrak{T}}$ invariant for every $g \in \Gamma(V, \beta)$. Consequently, μ is in fact a group representation $\mu : \Gamma(V, \beta) \to \mathrm{SO}(\mathfrak{T}, \beta_{\mathfrak{T}})$.

Proof of Proposition B.31. For (a). Notice Equation (B.39) and the fact that

$$\forall X \in (V \oplus S_+) \cup (S_+ \oplus S_-) \cup (S_- \oplus V) \ : \ F(X) = 0 \tag{B.42}$$

holds.

For (b). First, we show the correctness of the three zeros on the main diagonal of the table. Let $v, v' \in V$ be given. Then we have for every $X \in \mathfrak{T}$: $\beta_{\mathfrak{T}}(v \diamond v', X) = \gamma(v, v', X) = 0$ by (a)(ii)(2), and therefore $v \diamond v' = 0$ because of the non-degeneracy of $\beta_{\mathfrak{T}}$. Analogously, one shows $s_+ \diamond s'_+ = 0$ for any $s_+, s'_+ \in S_+$ and $s_- \diamond s'_- = 0$ for any $s_-, s'_- \in S_-$.

Now, let $v \in V$ and $s'_+ \in S_+$ be given. Then we have for every $X \in V \oplus S_+ \subset \mathfrak{T}$: $\beta_{\mathfrak{T}}(v \diamond s_+, X) = \gamma(v, s_+, X) = 0$ by (a)(ii)(1), and therefore $v \diamond s_+$ lies in the $\beta_{\mathfrak{T}}$-ortho-complement of $V \oplus S_+$ in $\mathfrak{T}$, i.e. in S_-. Now, we have for any $s_- \in S_-$

$$\beta_-(v \diamond s'_+, s_-) = \beta_{\mathfrak{T}}(v \diamond s'_+, s_-) = \gamma(v, s'_+, s_-) \overset{\text{(a)(i)}}{=} F(v + s'_+ + s_-) = \beta_-(\rho(v)s'_+, s_-) \; .$$

By the non-degeneracy of β_-, $v \diamond s'_+ = \rho(v)s'_+$ follows. Analogously, one shows $v \diamond s'_- = \rho(v)s'_-$ for every $v \in V$ and $s'_- \in S_-$.

Finally, let $s_+ \in S_+$ and $s'_- \in S_-$ be given. By an analogous argument as before, we see that $s_+ \diamond s'_- \in V$ holds. For any $v \in V$ we have

$$\beta(s_+ \diamond s'_-, v) = \beta_{\mathfrak{T}}(s_+ \diamond s'_-, v) = \gamma(s_+, s'_-, v) \overset{\text{(a)(i)}}{=} F(v + s_+ + s'_-) = \beta_-(\rho(v)s_+, s'_-) \; ,$$

whence $s_+ \diamond s'_- = (\, v \mapsto \beta_-(\rho(v)s_+, s'_-)\,)^\sharp$ follows.

For (c)(i). By (b) we have $v \diamond (v \diamond s) = \rho(v)(\rho(v)s) = \rho(v \cdot v)s = q(v) \cdot s$.

For (c)(ii). We have $\beta_S(v \diamond s_1, v \diamond s_2) = \beta_S(\rho(v)s_1, \rho(v)s_2) = q(v) \cdot \beta_S(s_1, s_2)$ by (b) and Proposition B.30(b)(i).

For (c)(iii). Because both sides of the equation (c)(iii) are bilinear and symmetric in (v_1, v_2), it suffices to show the equation for the case $v_1 = v_2 =: v$, and in that case, it follows from (c)(ii).

For (d). Let a linear map $\sigma : \mathfrak{T} \to \mathfrak{T}$ which leaves $\beta_{\mathfrak{T}}$ and F invariant be given. Because σ leaves $\beta_{\mathfrak{T}}$ invariant, it is a linear isomorphism, and because it leaves F invariant, it also leaves γ invariant. For any $X, Y, Z \in \mathfrak{T}$, we now have

$$\beta_{\mathfrak{T}}(\sigma X \diamond \sigma Y, \sigma Z) = \gamma(\sigma X, \sigma Y, \sigma Z) = \gamma(X, Y, Z) = \beta_{\mathfrak{T}}(X \diamond Y, Z) = \beta_{\mathfrak{T}}(\sigma(X \diamond Y), \sigma Z) \; .$$

Because σ is a linear isomorphism and $\beta_{\mathfrak{T}}$ is non-degenerate, it follows that $\sigma X \diamond \sigma Y = \sigma(X \diamond Y)$ holds.

For (e). It is clear that the linear map $\sigma := \varepsilon(g) \cdot \mu(g) : \mathfrak{T} \to \mathfrak{T}$ leaves V invariant. By Proposition B.27(b)(ii), σ leaves S_+ and S_- invariant for $\varepsilon(g) = 1$, whereas it exchanges S_+ and S_- for $\varepsilon(g) = -1$. To prove that σ is an algebra automorphism of $(\mathfrak{T}, \diamond)$, it suffices to show that it leaves $\beta_{\mathfrak{T}}$ and F invariant because of (d). We have for any $X_k = v_k + s_k \in \mathfrak{T}$ ($k \in \{1,2\}$, $v_k \in V$, $s_k \in S$)

$$\beta_{\mathfrak{T}}(\sigma(X_1), \sigma(X_2)) = \beta(\chi(g)v_1, \chi(g)v_2) + \beta_S(\rho(g)s_1, \rho(g)s_2) = \beta(v_1, v_2) + \underbrace{\varepsilon(g) \cdot \lambda(g)}_{=1} \cdot \beta_S(s_1, s_2) = \beta_{\mathfrak{T}}(X_1, X_2)$$

by Proposition B.15(e) and Proposition B.30(b)(ii).

For the proof of the F-invariance of σ, let $X = v + s_+ + s_- \in \mathfrak{T}$ with $v \in V$ and $s_\pm \in S_\pm$ be given. We have $\chi(g)v = \alpha(g)vg^{-1} = \varepsilon(g) \cdot gvg^{-1}$ and therefore

$$\rho(\chi(g)v) = \rho(\varepsilon(g)\, gvg^{-1}) = \varepsilon(g) \cdot \rho(g) \circ \rho(v) \circ \rho(g)^{-1} \; . \tag{B.43}$$

We have either $\varepsilon(g) = 1$ and then $\rho(g)s_\pm \in S_\pm$, or else $\varepsilon(g) = -1$ and then $\rho(g)s_\pm \in S_\mp$. In either case, we obtain

$$\begin{aligned} F(\sigma(X)) &= F(\, \varepsilon(g) \cdot (\chi(g)v + \rho(g)s_+ + \rho(g)s_-)\,) = \varepsilon(g) \cdot F(\chi(g)v + \rho(g)s_+ + \rho(g)s_-) \\ &= \varepsilon(g) \cdot \beta_S(\, \rho(\chi(g)v)\rho(g)s_+ \,,\, \rho(g)s_-\,) \\ &\overset{\text{(B.43)}}{=} \beta_S(\, \rho(g)\rho(v)\rho(g)^{-1}\rho(g)s_+ \,,\, \rho(g)s_-\,) \\ &= \beta_S(\, \rho(g)\rho(v)s_+ \,,\, \rho(g)s_-\,) = \varepsilon(g) \cdot \lambda(g) \cdot \beta_S(\rho(v)s_+, s_-) = F(X) \; , \end{aligned}$$

see also Proposition B.30(b)(ii). □

B.32 Remark. The algebra $(\mathfrak{T}, \diamond)$ is not associative. For example, for $v_1, v_2 \in V$ and $s_+ \in S_+$, we have by Proposition B.31(b) $(v_1 \diamond v_2) \diamond s_+ = 0$, but $v_1 \diamond (v_2 \diamond s_+) = \rho(v_1 \cdot v_2)s_+$, where the latter expression is generally non-zero.

Also, $(\mathfrak{T}, \diamond)$ does not have a unit element, as the following argument shows: Assuming to the contrary that $X = v + s_+ + s_- \in \mathfrak{T}$ satisfies $X \diamond X' = X'$ for every $X' \in \mathfrak{T}$, we have

$$V \ni v = X \diamond v = \underbrace{v \diamond v}_{=0} + \underbrace{s_+ \diamond v}_{\in S_-} + \underbrace{s_- \diamond v}_{\in S_+}$$

(see Proposition B.31(b)) and therefore $v = 0$. Similarly, one sees $s_+ = 0$ and $s_- = 0$, hence $X = 0$, which is a contradiction to the assumption $X \diamond X' = X'$ for every $X' \in \mathfrak{T}$.

B.33 Theorem. *Let us denote by* $\mathrm{Aut}'(\mathfrak{T})$ *the group of algebra automorphisms of* $(\mathfrak{T}, \diamond)$ *which leave the spaces* V, S_+ *and* S_- *invariant. Then* $\mu' := \mu|\mathrm{Spin}(V, \beta) : \mathrm{Spin}(V, \beta) \to \mathrm{Aut}'(\mathfrak{T})$ *is a group isomorphism.*

Proof. For any $g \in \mathrm{Spin}(V, \beta)$, we have $\varepsilon(g) = 1$, and therefore Proposition B.31(e) shows that $\mu(g) \in \mathrm{Aut}'(\mathfrak{T})$ holds. Hence μ' in fact maps into $\mathrm{Aut}'(\mathfrak{T})$. It is clear that μ' is an injective group homomorphism along with μ.

It remains to show the surjectivity of μ'. For this, let $\sigma \in \mathrm{Aut}'(\mathfrak{T})$ be given. Because $\rho : C(V, \beta) \to \mathrm{End}(S)$ is an isomorphism of algebras (see Theorem B.26), there exists $g \in C(V, \beta)$ with

$$\rho(g) = \sigma|S \ \in \ \mathrm{End}(S) \ ; \tag{B.44}$$

because $\sigma|S$ is invertible, we have $g \in C(V, \beta)^\times$. Next, we show $g \in C^+(V, \beta)$. For this, we write $g = g_+ + g_-$ with $g_\pm \in C^\pm(V, \beta)$; then we have for any $s_+ \in S_+$

$$S_+ \ni \sigma(s_+) = \rho(g)s_+ = \underbrace{\rho(g_+)s_+}_{\in S_+} + \underbrace{\rho(g_-)s_+}_{\in S_-}$$

(see Proposition B.27(b)(iii)) and therefore $\rho(g_-)|S_+ = 0$. Analogously one shows $\rho(g_-)|S_- = 0$, and thus we have $\rho(g_-) = 0$. Because $\rho : C(V, \beta) \to \mathrm{End}(S)$ is injective, we conclude $g_- = 0$ and therefore $g = g_+ \in C^+(V, \beta)$.

For $v \in V$ and $s \in S$ we now have by Proposition B.31(b) and the fact that $\sigma(V) \subset V$, $\sigma(S) \subset S$ holds

$$\begin{aligned} \rho(g \cdot v)s = \rho(g)(\rho(v)s) &\overset{(B.44)}{=} \sigma(\rho(v)s) = \sigma(v \diamond s) \\ = \sigma(v) \diamond \sigma(s) = \rho(\sigma(v))\sigma(s) &\overset{(B.44)}{=} \rho(\sigma(v))\rho(g)s = \rho(\sigma(v) \cdot g)s \end{aligned}$$

and therefore $\rho(g \cdot v) = \rho(\sigma(v) \cdot g)$. Because ρ is injective, we obtain $g \cdot v = \sigma(v) \cdot g$ and therefore $\sigma(v) = gvg^{-1} = \alpha(g)vg^{-1}$. Because we have $\sigma(V) \subset V$, this equation implies $g \in \Gamma(V, \beta)$ and

$$\sigma|V = \chi(g) \ . \tag{B.45}$$

Because of $g \in C^+(V, \beta)$, we in fact have $g \in \Gamma^+(V, \beta)$, and Equations (B.44) and (B.45) show that $\mu(g) = \sigma$ holds.

Thus, it only remains to prove $\lambda(g) = 1$. For this, we put $g' := \frac{1}{t} \cdot g \in \Gamma^+(V, \beta)$, where $t \in \mathbb{C}^\times$ is chosen such that $t^2 = \lambda(g)$ holds. Then we have $\lambda(g') = 1$, hence $g' \in \mathrm{Spin}(V, \beta)$, and therefore $\mu(g')$ also is an algebra automorphism of $(\mathfrak{T}, \diamond)$ by Proposition B.31(e). Note that we have $\mu(g)|V = \mu(g')|V$ by Proposition B.12(b) and $\mu(g)|S_\pm = t \cdot \mu(g')|S_\pm$ by Theorem B.26. For $s_+ \in S_+$ and $s_- \in S_-$ we therefore have

$$\begin{aligned} \mu(g)(s_+ \diamond s_-) &= \mu(g)s_+ \diamond \mu(g)s_- = (t\mu(g')s_+) \diamond (t\mu(g')s_-) = \lambda(g) \cdot (\mu(g')s_+ \diamond \mu(g')s_-) \\ &= \lambda(g) \cdot \mu(g') \underbrace{(s_+ \diamond s_-)}_{\in V} = \lambda(g) \cdot \mu(g)(s_+ \diamond s_-) \ . \end{aligned}$$

This equality implies $\lambda(g) = 1$, provided that there exist some $s_+ \in S_+$, $s_- \in S_-$ with $s_+ \diamond s_- \neq 0$. To show that this is indeed the case, fix $v \in V$ with $q(v) = 1$ and $s_- \in S_-$ with $q_-(s_-) \neq 0$, and put $s_+ := \rho(v)s_- \in S_+$. Then we have $\rho(v)s_+ = \rho(v^2)s_- = \rho(q(v)\,1_C)s_- = q(v)\,s_- = s_-$ and therefore by Proposition B.31(b)

$$\beta(s_+ \diamond s_-, v) = \beta_-(\rho(v)s_+, s_-) = \beta_-(s_-, s_-) = 2\,q_-(s_-) \neq 0 ,$$

whence $s_+ \diamond s_- \neq 0$ follows. □

B.34 Theorem. (Triality on $\mathfrak{T}$.) *Let $w_1 \in W$ and $w_1' \in W'$ be given so that $\beta(w_1, w_1') = 1$ holds.*[44]

We put $v_0 := w_1 + w_1' \in V$ and $s_0 := 1 + \omega \in S_+$. The linear map $\tau' : V \to S_-,\ v \mapsto s_0 \diamond v$ is an isomorphism of linear spaces. Let us consider the linear map $\tau : \mathfrak{T} \to \mathfrak{T}$ characterized by

$$\tau|V = \tau' , \quad \forall s_+ \in S_+ : \tau(s_+) = \beta_+(s_+, s_0)s_0 - s_+ \quad \textit{and} \quad \tau|S_- = (\tau')^{-1} .$$

Then $T := -\mu(v_0) \circ \tau$ leaves $\beta_{\mathfrak{T}}$ and F invariant and therefore is an algebra automorphism of $(\mathfrak{T}, \diamond)$. Moreover,

$$T^3 = \mathrm{id}_{\mathfrak{T}} , \quad T(V) = S_+ , \quad T(S_+) = S_- \quad \textit{and} \quad T(S_-) = V \tag{B.46}$$

holds. We call any automorphism of $\mathfrak{T}$ obtained by this construction a triality automorphism *of* $(\mathfrak{T}, \diamond)$.

T is described explicitly in the following way: Let $(w_1, \ldots, w_4)$ be an extension of w_1 to a basis of W such that $w_1 \wedge \ldots \wedge w_4 = \omega$ holds and denote by $(w_1', \ldots, w_4')$ the basis of W' uniquely determined by

$$\forall k, k' \in \{1, \ldots, 4\} \ : \ \beta(w_k, w_{k'}') = \delta_{k,k'} \tag{B.47}$$

(see Proposition B.23). Then T, T^2 and T^3 act on the basis $(w_1, \ldots, w_4, w_1', \ldots, w_4')$ of V in the following way:

$v \in V$	$Tv \in S_+$	$T^2 v \in S_-$	$T^3 v \in V$
w_1	-1_S	$w_2 \wedge w_3 \wedge w_4$	w_1
w_2	$-w_1 \wedge w_2$	$-w_2^S$	w_2
w_3	$-w_1 \wedge w_3$	$-w_3^S$	w_3
w_4	$-w_1 \wedge w_4$	$-w_4^S$	w_4
w_1'	$-w_1 \wedge w_2 \wedge w_3 \wedge w_4$	w_1^S	w_1'
w_2'	$w_3 \wedge w_4$	$w_1 \wedge w_3 \wedge w_4$	w_2'
w_3'	$-w_2 \wedge w_4$	$-w_1 \wedge w_2 \wedge w_4$	w_3'
w_4'	$w_2 \wedge w_3$	$w_1 \wedge w_2 \wedge w_3$	w_4'

.

Here, we denote w_k by w_k^S when we regard it as an element of $\bigwedge^1 W \subset S_-$ (rather than as an element of V).

[44] For any $w_1 \in W \setminus \{0\}$ there exist vectors $w_1' \in W \setminus \{0\}$ so that $\beta(w_1, w_1') = 1$ holds because of the non-degeneracy of β.

Proof. We have $v_0 \in \Gamma(V,\beta)$ by Proposition B.12(c), $\varepsilon(v_0) = -1$, and $\lambda(v_0) = -q(v_0) = -1$ by Proposition B.14(b). Therefore Proposition B.31(e) shows that $-\mu(v_0)$ is an automorphism of $(\mathfrak{T},\diamond)$ which leaves $\beta_{\mathfrak{T}}$ and F invariant, and which satisfies

$$-\mu(v_0)V = V\ , \quad -\mu(v_0)S_+ = S_- \quad \text{and} \quad -\mu(v_0)S_- = S_+\ . \tag{B.48}$$

Furthermore, we have $q_+(s_0) = 1$ and therefore for any $v_1, v_2 \in V$ by Proposition B.31(c)(iii)

$$\beta_-(\tau'(v_1),\tau'(v_2)) = \beta_-(s_0 \diamond v_1, s_0 \diamond v_2) = q_+(s_0)\cdot\beta(v_1,v_2) = \beta(v_1,v_2)\ . \tag{B.49}$$

Because of the non-degeneracy of β, this equation shows $\tau' : V \to S_-$ to be injective; because we have $\dim S_- = 8 = \dim V$, τ' is in fact an isomorphism of linear spaces.

We will now show that the linear map $\tau : \mathfrak{T} \to \mathfrak{T}$ leaves $\beta_{\mathfrak{T}}$ and F invariant and therefore is an isomorphism of the algebra $(\mathfrak{T},\diamond)$ by Proposition B.31(d).

For the $\beta_{\mathfrak{T}}$-invariance of τ: Because τ permutes the $\beta_{\mathfrak{T}}$-orthogonal spaces V, S_+ and S_-, it suffices to show that the restrictions of τ to these spaces are $\beta_{\mathfrak{T}}$-invariant. Equation (B.49) shows that $\tau|V = \tau'$ and $\tau|S_- = (\tau')^{-1}$ leave $\beta_{\mathfrak{T}}$ invariant. Now, let $s_+ \in S_+$ be given. Then we have

$$\begin{aligned} q_+(\tau(s_+)) &= \tfrac{1}{2}\beta_+(\,\beta_+(s_+,s_0)s_0 - s_+\ ,\ \beta_+(s_+,s_0)s_0 - s_+\,) \\ &= \tfrac{1}{2}\big(\,\beta_+(s_+,s_0)^2\beta_+(s_0,s_0) - 2\,\beta_+(s_+,s_0)\beta_+(s_0,s_+) + \beta_+(s_+,s_+)\,\big) = q_+(s_+)\ , \end{aligned}$$

and therefore $\tau|S_+$ also leaves $\beta_{\mathfrak{T}}$ invariant.

For the F-invariance of τ: Let $X = v + s_+ + s_- \in \mathfrak{T}$ with $v \in V$, $s_\pm \in S_\pm$ be given. We put $v' := \tau(s_-) \in V$, then $\rho(v')s_0 = s_0 \diamond v' = \tau(v') = s_-$ holds (see Proposition B.31(b)). For the following calculations, keep Propositions B.30(b) and B.31(b),(c) in mind. We have

$$\begin{aligned} F(\tau(X)) &= F(\underbrace{\tau(v)}_{\in S_-} + \underbrace{\tau(s_+)}_{\in S_+} + \underbrace{v'}_{\in V}) \\ &= \beta_-(\,\rho(v')(\tau(s_+))\ ,\ \tau(v)\,) = \beta_+(\,\tau(s_+)\ ,\ \rho(v')(\tau(v))\,) \\ &= \beta_+(\,\beta_+(s_+,s_0)s_0 - s_+\ ,\ \rho(v')\rho(v)s_0\,) \\ &= \beta_+(s_+,s_0)\cdot\beta_+(s_0,\rho(v')\rho(v)s_0) - \beta_+(s_+,\rho(v')\rho(v)s_0)\ . \end{aligned} \tag{B.50}$$

Now, we have

$$\beta_+(s_0,\rho(v')\rho(v)s_0) = \beta_+(\rho(v')s_0,\rho(v)s_0) = \beta_+(v' \diamond s_0, v \diamond s_0) = q_+(s_0)\cdot\beta(v',v) = \beta(v,v')\ .$$

Therefore, we can continue the calculation of (B.50) in the following way:

$$\begin{aligned} F(\tau(X)) &= \beta_+(s_+,s_0)\cdot\beta(v,v') - \beta_+(s_+,\rho(v')\rho(v)s_0) \\ &= \beta_+(\,s_+\ ,\ \beta(v,v')s_0 - \rho(v')\rho(v)s_0\,) \\ &= \beta_+(\,s_+\ ,\ \rho(\beta(v,v')\,1_C - v'\cdot v)s_0\,) \\ &= \beta_+(\,s_+\ ,\ \rho(v\cdot v')s_0\,) = \beta_+(\,s_+\ ,\ \rho(v)(\rho(v')s_0)\,) = \beta_+(\,s_+\ ,\ \rho(v)s_-\,) \\ &= \beta_-(\rho(v)s_+, s_-) = F(X)\ . \end{aligned}$$

Thus we have proved that $\tau : \mathfrak{T} \to \mathfrak{T}$ is an algebra automorphism. Also, we have

$$\tau(V) = S_-\ , \quad \tau(S_+) = S_+ \quad \text{and} \quad \tau(S_-) = V\ . \tag{B.51}$$

Therefore $T = -\mu(v_0)\circ\tau$ also is an algebra automorphism, and from Equations (B.48) and (B.51) we see that

$$T(V) = S_+\ , \quad T(S_+) = S_- \quad \text{and} \quad T(S_-) = V$$

holds.

Now, let an extension of w_1 to a basis $(w_1, \ldots, w_4)$ of W so that $\omega = w_1 \wedge \ldots \wedge w_4$ holds be given, and denote by $(w'_1, \ldots, w'_4)$ the basis of W' uniquely characterized by (B.47) (see Proposition B.23). One can then verify the table of values of T given in the theorem by explicitly calculating $T(X)$ for the elements $X \in \mathfrak{T}$ mentioned in that table. For example, one has for $k \in \{1, \ldots, 4\}$

$$\tau(w_k) = s_0 \diamond w_k = \rho(w_k)(1+\omega) = w_k \wedge (1+\omega) = w_k^S$$

and consequently

$$\begin{aligned} T(w_k) &= -\mu(v_0)(\tau(w_k)) = -\rho(v_0)w_k^S = -(\rho(w_1)w_k^S + \rho(w'_1)w_k^S) = -(w_1 \wedge w_k + \nu_{\beta(\cdot, w'_1)} w_k^S) \\ &= -w_1 \wedge w_k - \beta(w_k, w'_1) \cdot 1_S = \begin{cases} -1_S & \text{for } k = 1 \\ -w_1 \wedge w_k & \text{for } k \geq 2 \end{cases} . \end{aligned}$$

It also follows from the table that $T^3 = \mathrm{id}_{\mathfrak{T}}$ holds.[45] □

B.35 Theorem. (Triality on $\mathrm{Spin}(V,\beta)$.) *Let $T : \mathfrak{T} \to \mathfrak{T}$ be a triality automorphism. Then there exists one and only one automorphism $\vartheta : \mathrm{Spin}(V,\beta) \to \mathrm{Spin}(V,\beta)$ of Lie groups of order 3 (i.e. which satisfies $\vartheta^3 = \mathrm{id}_{\mathrm{Spin}(V,\beta)}$) so that*

$$\forall g \in \mathrm{Spin}(V,\beta) \ : \ T \circ \mu(g) = \mu(\vartheta(g)) \circ T \tag{B.52}$$

holds. We call ϑ the triality automorphism of $\mathrm{Spin}(V,\beta)$ corresponding to T.

Proof. Let $g \in \mathrm{Spin}(V,\beta)$ be given. By Proposition B.31(e), $\mu(g)$, and thus also $T \circ \mu(g) \circ T^{-1}$ is an automorphism of the algebra $(\mathfrak{T}, \diamond)$ which leaves the spaces V, S_+ and S_- invariant. Theorem B.33 therefore shows that there exists one and only one element $\vartheta(g) \in \mathrm{Spin}(V,\beta)$ so that $\mu(\vartheta(g)) = T \circ \mu(g) \circ T^{-1}$ and therefore Equation (B.52) holds. We have $\vartheta = (\mu|\mathrm{Spin}(V,\beta))^{-1} \circ f \circ (\mu|\mathrm{Spin}(V,\beta))$ with the group automorphism $f : \mathrm{Aut}'(\mathfrak{T}) \to \mathrm{Aut}'(\mathfrak{T}),\ A \mapsto T \circ A \circ T^{-1}$; because $\mu|\mathrm{Spin}(V,\beta) : \mathrm{Spin}(V,\beta) \to \mathrm{Aut}'(\mathfrak{T})$ also is an isomorphism of groups, we see that ϑ is an automorphism of the group $\mathrm{Spin}(V,\beta)$. Moreover, we have for $g \in \mathrm{Spin}(V,\beta)$: $\mu(\vartheta^3(g)) = T^3 \circ \mu(g) \circ T^{-3} = \mu(g)$ and thus because μ is injective $\vartheta^3(g) = g$.

For the differentiability of ϑ: $\mathrm{Aut}'(\mathfrak{T})$ is a closed subgroup of the Lie group $\mathrm{GL}(\mathfrak{T})$ and therefore inherits a Lie group structure in a canonical way (see [Var74], Theorem 2.12.6, p. 99). In this regard, f is differentiable (note that T is a linear isomorphism), and also $\mu|\mathrm{Spin}(V,\beta) : \mathrm{Spin}(V,\beta) \to \mathrm{Aut}'(\mathfrak{T})$ is differentiable. It follows that ϑ is an automorphism of Lie groups.

Note that the only property of T we used in the proof is the fact that it is an algebra automorphism of $(\mathfrak{T}, \diamond)$ which satisfies (B.46). □

Let $T : \mathfrak{T} \to \mathfrak{T}$ be a triality automorphism of $(\mathfrak{T}, \diamond)$ and $\vartheta : \mathrm{Spin}(V,\beta) \to \mathrm{Spin}(V,\beta)$ be the corresponding triality automorphism of $\mathrm{Spin}(V,\beta)$. Considering the way the representation μ is composed of the representations χ and ρ (see Equation (B.38)), we see that Equation (B.52) implies that we have for any $g \in \mathrm{Spin}(V,\beta)$:

$$\begin{aligned} (T|V) \circ \chi(g) &= \rho_+(\vartheta(g)) \circ (T|V) , \\ (T|S_+) \circ \rho_+(g) &= \rho_-(\vartheta(g)) \circ (T|S_+) , \\ \text{and} \quad (T|S_-) \circ \rho_-(g) &= \chi(\vartheta(g)) \circ (T|S_-) . \end{aligned} \tag{B.53}$$

[45] For a different proof for $T^3 = \mathrm{id}_{\mathfrak{T}}$ which does not involve calculations using the bases $(w_1, \ldots, w_4)$ and $(w'_1, \ldots, w'_4)$ see [Che54], p. 119f.

Thus we have attained the objective of "intertwining" the representations χ, ρ_+ and ρ_-, as was described at the beginning of the section. Indeed, the preceding equations show that with the linear isometries $T_{V+} := T|V : (V,\beta) \to (S_+,\beta_+)$, $T_{+-} := T|S_+ : (S_+,\beta_+) \to (S_-,\beta_-)$ and $T_{-V} := T|S_- : (S_-,\beta_-) \to (V,\beta)$ Diagram (B.37) commutes.

From Equations (B.53) we also see that in the present situation, the representations $\rho_+ : \mathrm{Spin}(V,\beta) \to \mathrm{GL}(S_+)$ and $\rho_- : \mathrm{Spin}(V,\beta) \to \mathrm{GL}(S_-)$ are irreducible, a fact that holds in the general situation of Section B.5 but was not proved there. Indeed, from the first equation of (B.53) it follows that $\rho_+ \circ \vartheta = (g \mapsto (T|V) \circ \chi(g) \circ (T|V)^{-1})$ holds. Because $\chi|\mathrm{Spin}(V,\beta) : \mathrm{Spin}(V,\beta) \to \mathrm{GL}(V)$ is irreducible (remember that $\chi(\mathrm{Spin}(V,\beta)) = \mathrm{SO}(V,\beta)$ holds by Proposition B.15(e)), ϑ is an automorphism of $\mathrm{Spin}(V,\beta)$ and $T|V : V \to S_+$ is a linear isomorphism, it follows that ρ_+ is irreducible. An analogous argument involving the equation $\rho_- \circ \vartheta = (g \mapsto (T|S_+) \circ \rho_+(g) \circ (T|S_+)^{-1})$ shows that the irreducibility of ρ_+ implies the irreducibility of ρ_-.

It is of interest to describe the kernels of the actions ρ_+ and ρ_- explicitly, analogously to the description of the kernel of $\chi|\mathrm{Spin}(V,\beta)$ in Proposition B.15(d). Besides the elements of these kernels, the elements of $(\chi|\mathrm{Spin}(V,\beta))^{-1}(\{-\mathrm{id}_V\})$, $\rho_+^{-1}(\{-\mathrm{id}_{S_+}\})$ and $\rho_-^{-1}(\{-\mathrm{id}_{S_-}\})$ play a special role. The following proposition is concerned with the mentioned elements.

B.36 Proposition. *(a) There exist elements* $g_+, g_- \in \mathrm{Spin}(V,\beta)$ *so that besides the already known equation* $\ker(\chi|\mathrm{Spin}(V,\beta)) = \{1,-1\}$ *(see Proposition B.15(d)) we also have*

$$\ker \rho_+ = \{1, g_+\} \quad \textit{and} \quad \ker \rho_- = \{1, g_-\}\ .$$

(b) The elements $1, -1, g_+, g_-$ *are pairwise unequal, and they are multiplied in the following way:*

$\cdot$	1	-1	g_+	g_-
1	1	-1	g_+	g_-
-1	-1	1	g_-	g_+
g_+	g_+	g_-	1	-1
g_-	g_-	g_+	-1	1

.

Therefore $G := \{1,-1,g_+,g_-\}$ *is a subgroup of* $\mathrm{Spin}(V,\beta)$ *isomorphic to the Klein four-group. Also, we have* $g_- = -g_+$.

(c) The elements of G *act via* χ *and* $\rho_\pm$ *in the following way:*

$g \in G$	$\chi(g)$	$\rho_+(g)$	$\rho_-(g)$
1	id_V	id_{S_+}	id_{S_-}
-1	id_V	$-\mathrm{id}_{S_+}$	$-\mathrm{id}_{S_-}$
g_+	$-\mathrm{id}_V$	id_{S_+}	$-\mathrm{id}_{S_-}$
g_-	$-\mathrm{id}_V$	$-\mathrm{id}_{S_+}$	id_{S_-}

.

(d) Let $\vartheta : \mathrm{Spin}(V,\beta) \to \mathrm{Spin}(V,\beta)$ *be any triality automorphism of* $\mathrm{Spin}(V,\beta)$ *. Then* ϑ *maps in the following way:*

$$-1 \overset{\vartheta}{\longmapsto} g_+ \overset{\vartheta}{\longmapsto} g_- \overset{\vartheta}{\longmapsto} -1 \; .$$

In particular, the subgroup G *is invariant under* ϑ *.*

(e) Let $(w_1, \dots, w_4)$ *be any basis of* W*, and let* $(w_1', \dots, w_4')$ *be the basis of* W' *characterized by* $\beta(w_k, w_\ell') = \delta_{k\ell}$ *. Then we have*

$$\begin{aligned} g_+ &= (w_1 + w_1') \cdot (w_1 - w_1') \cdots (w_4 + w_4') \cdot (w_4 - w_4') \\ &= (w_1' \cdot w_1 - w_1 \cdot w_1') \cdots (w_4' \cdot w_4 - w_4 \cdot w_4') \; . \end{aligned} \tag{B.54}$$

Proof. Let ϑ be any triality automorphism of $\mathrm{Spin}(V,\beta)$. Equations (B.53) show that we have $\ker \rho_+ = \vartheta(\ker \chi | \mathrm{Spin}(V,\beta)) = \vartheta(\{1,-1\}) = \{1, \vartheta(-1)\}$ and similarly $\ker \rho_- = \{1, \vartheta^2(-1)\}$. Therefore (a) is fulfilled with $g_+ := \vartheta(-1)$ and $g_- := \vartheta^2(-1)$, and there is no other way to define g_+ and g_-. Then (d) also holds; note that we have $\vartheta^3 = \mathrm{id}_{\mathrm{Spin}(V,\beta)}$.

We next verify the table in (c). The line for $g = 1$ is obvious, and the line for $g = -1$ follows from Proposition B.15(d) and Theorem B.26. The line for $g = g_+$ follows from the line for $g = -1$ via Equations (B.53) in the following way:

$$\begin{aligned} \rho_+(g_+) &= (T|V) \circ \chi(-1) \circ (T|V)^{-1} = \mathrm{id}_{S_+} \; , \\ \rho_-(g_+) &= (T|S_+) \circ \rho_+(-1) \circ (T|S_+)^{-1} = -\mathrm{id}_{S_-} \quad \text{and} \\ \chi(g_+) &= (T|S_-) \circ \rho_-(-1) \circ (T|S_-)^{-1} = -\mathrm{id}_V \; . \end{aligned}$$

The line for $g = g_-$ follows from the line for $g = g_+$ in an analogous way.

For (b), we first note that the table in (c) shows that the elements $1, -1, g_+, g_-$ are pairwise unequal. By (c) and the fact that $\rho : C(V,\beta) \to \mathrm{End}(S)$ is an algebra isomorphism (Theorem B.26), we have

$$\rho(g_+ \cdot g_+) = \rho(g_- \cdot g_-) = \mathrm{id}_S = \rho(1) \quad \text{and} \quad \rho(g_+ \cdot g_-) = \rho(g_- \cdot g_+) = -\mathrm{id}_S = \rho(-1) \; ,$$

whence by the injectivity of ρ we obtain

$$g_+ \cdot g_+ = g_- \cdot g_- = 1 \quad \text{and} \quad g_+ \cdot g_- = g_- \cdot g_+ = -1 \; . \tag{B.55}$$

Via calculations in the Clifford algebra $C(V,\beta)$ in which G is contained, we deduce from (B.55) first $g_\pm^{-1} = g_\pm$, then $g_- = -g_+$, and then the correctness of the table in (b).

For the proof of (e), let us put $g := (w_1 + w_1') \cdot (w_1 - w_1') \cdots (w_4 + w_4') \cdot (w_4 - w_4')$. Below, we show

$$\rho_+(g) = \mathrm{id}_{S_+} \quad \text{and} \quad \rho_-(g) = -\mathrm{id}_{S_-} \; . \tag{B.56}$$

By comparison with the table in (c), we see from Equations (B.56) that $\rho(g) = \rho(g_+)$ holds, whence the first equality $g = g_+$ in (B.54) follows because of the injectivity of ρ.

For the proof of (B.56): For $k \in \{1, \dots, 4\}$, the elements $v_k := w_k - w_k'$ and $\widetilde{v}_k := i\,(w_k + w_k')$ satisfy $q(v_k) = q(\widetilde{v}_k) = -1$ by Proposition B.25(a), and therefore we have $g_k := \widetilde{v}_k \cdot v_k \in \mathrm{Spin}(V,\beta)$ by Proposition B.15(c)(ii). We have

$$g_k = i\,(w_k + w_k') \cdot (w_k - w_k') = i\,(\underbrace{w_k \cdot w_k}_{=0} - w_k \cdot w_k' + \underbrace{w_k' \cdot w_k}_{=1 - w_k \cdot w_k'} - \underbrace{w_k' \cdot w_k'}_{=0}) = i\,(1 - 2\,w_k \cdot w_k') \; . \tag{B.57}$$

We now use the notation w_N of (B.21) with respect to the given basis $(w_1, \dots, w_4)$ of W, and let $N \subset \{1, \dots, 4\}$ be given. In order to calculate $\rho(g_k)w_N$, we put $\ell := \#\{\, k' \in N \,|\, k' < k \,\}$. Then we have

$$\begin{aligned} \rho(g_k)w_N &\overset{(B.57)}{=} \rho(i\,(1 - 2\,w_k \cdot w_k'))w_N = i\,(w_N - 2\,\rho(w_k)\,\rho(w_k')w_N) = i\,(w_N - 2\,w_k \wedge \nu_{\beta(\cdot,w_k')}(w_N)) \\ &= \begin{cases} i\,(w_N - 2\,(-1)^\ell\, w_k \wedge w_{N\setminus\{k\}}) = i\,(w_N - 2\,w_N) = -(i\,w_N) & \text{for } k \in N \\ i\,w_N & \text{for } k \notin N \end{cases} . \end{aligned} \tag{B.58}$$

Hence we see that $g = i^4\, g = g_1 \cdots g_4 \in \mathrm{Spin}(V,\beta)$ holds and that we have

$$\rho(g)w_N = \rho(g_1)\cdots\rho(g_4)\,w_N \overset{(B.58)}{=} i^4\,(-1)^{\#N}\,w_N = (-1)^{\#N}\,w_N\,,$$

whence Equations (B.56) follow.

The second equals sign in (B.54) now follows from the fact that we have for any $k \in \{1, \dots, 4\}$

$$(w_k + w_k') \cdot (w_k - w_k') = w_k\,w_k - w_k\,w_k' + w_k'\,w_k - w_k'\,w_k' = w_k'\,w_k - w_k\,w_k'\,.$$

□

B.37 Corollary. *ϑ does not descend to an automorphism of* $\mathrm{SO}(V,\beta)$*, more precisely: There exists* no *Lie group automorphism* $\Theta : \mathrm{SO}(V,\beta) \to \mathrm{SO}(V,\beta)$ *so that*

$$(\chi|\mathrm{Spin}(V,\beta)) \circ \vartheta = \Theta \circ (\chi|\mathrm{Spin}(V,\beta))$$

holds.

Proof. If such a Lie group automorphism Θ existed, $\ker(\chi|\mathrm{Spin}(V,\beta)) = \{\pm 1\}$ would be invariant under ϑ, which is a contradiction to Proposition B.36. □

B.38 Remark. The non-associative complex division algebra of *octonions with complex coefficients* $\mathbb{O}^{\mathbb{C}}$ can be obtained from the triality algebra by the following construction: Fix $v_0 \in V$ and $s_0 \in S_+$ with $q(v_0) = q_+(s_0) = 1$ and put $s_0' := v_0 \diamond s_0 \in S_-$. Then it can be shown that V becomes an 8-dimensional complex division algebra isomorphic to $\mathbb{O}^{\mathbb{C}}$ via the composition map

$$\star : V \times V \to V,\ (x, y) \mapsto x \star y := (x \diamond s_0') \diamond (y \diamond s_0)\ ;$$

its unit element is v_0. (See [Che54], Section IV.5, p. 123ff.) We remark that the automorphism group of $(V, \star)$ is isomorphic to the exceptional simple Lie group G_2.

If $w_1 \in W$ and $w_1' \in W'$ are given with $\beta(w_1, w_1') = 1$, T is the triality automorphism of $(\mathfrak{T}, \diamond)$ corresponding to this choice of w_1, w_1' and if we perform the above construction of the composition $\star$ with $v_0 = w_1 + w_1'$ and $s_0 = 1 + \omega$, then the triality automorphism ϑ of $\mathrm{Spin}(V,\beta)$ corresponding to T can be characterized by

$$\forall g \in \mathrm{Spin}(V,\beta),\ x, y \in V\ :\ \chi(g)(\overline{x} \star \overline{y}) = \overline{\chi(\vartheta^2 g)x}\ \star\ \overline{\chi(\vartheta g)y}\,, \tag{B.59}$$

see [Che54], p. 125. Here, $\overline{x}$ denotes the *conjugation* of $(V, \star)$, i.e. the linear map characterized by $\overline{v_0} = v_0$ and $\overline{x} = -x$ for any $x \in V$ with $\beta(x, v_0) = 0$.

It is possible to base the theory of triality on Equation (B.59); for an example of this approach, see [Por95], Chapter 24, in particular Theorem 24.13, p. 278.

Bibliography

[Art57] E. Artin, *Geometric algebra*, Interscience, New York, 1957.

[Ber97] J. Berndt, *Riemannian geometry of complex two-plane grassmannians*, Rend. Sem. Mat. Univ. Politec. Torino **55** (1997), 19–83.

[Boo86] W. M. Boothby, *An introduction to Differentiable Manifolds and Riemannian geometry*, second ed., Academic Press, Orlando, 1986.

[Bou67] N. Bourbaki, *Variétés différentielles et analytiques: Fascicule de résultats*, Hermann, Paris, 1967.

[Bri85] E. Brieskorn, *Lineare Algebra und analytische Geometrie II*, Vieweg, Braunschweig/Wiesbaden, 1985.

[BtD85] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, Springer, New York, 1985.

[Car14] É. Cartan, *Les groupes réels simples finis et continus*, Ann. Sci. École Norm. Sup. **31** (1914), 263–355.

[Car51] ______, *Géométrie des espaces de Riemann*, second ed., Gauthier-Villars, Paris, 1951.

[Che46] C. C. Chevalley, *Theory of Lie groups, I*, Princeton University Press, Princeton, 1946.

[Che54] ______, *The algebraic theory of spinors*, Columbia University Press, New York, 1954.

[CL75] B.-Y. Chen and H.-S. Lue, *Differential geometry of* $\mathrm{SO}(n+2)/\mathrm{SO}(2) \times \mathrm{SO}(n)$ *, I*, Geometriae Dedicata **4** (1975), 253–261.

[CN77] B.-Y. Chen and T. Nagano, *Totally geodesic submanifolds of symmetric spaces, I*, Duke Math. J. **44** (1977), 745–755.

[CN78] ______, *Totally geodesic submanifolds of symmetric spaces, II*, Duke Math. J. **45** (1978), 405–425.

[Esc84] J.-H. Eschenburg, *Freie isometrische Aktionen auf kompakten Lie-Gruppen mit positiv gekrümmten Orbiträumen*, Schriftenreihe des Mathematischen Instituts der Universität Münster, 2. Serie, Heft 32, 1984.

[GH78] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.

[Har90] F. R. Harvey, *Spinors and calibrations*, Academic Press, Boston, 1990.

[Hel78] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York, 1978.

[HL71] W. Hsiang and H. Lawson, *Minimal submanifolds of low cohomogeneity*, J. Diff. Geom. **5** (1971), 1–38.

[IT91] M. Ise and M. Takeuchi, *Theory of Lie groups I and II*, American Mathematical Society, Providence, Rhode Island, 1991.

[Jac43] N. Jacobson, *The theory of rings*, American Mathematical Society, New York, 1943.

[Jac64] ______, *Structure of rings*, revised ed., AMS Colloquium Publications, Providence, 1956, 1964.

[KN63] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. I, Interscience Publications, New York, 1963.

[KN69] ______, *Foundations of differential geometry*, vol. II, Interscience Publications, New York, 1969.

[Kob61] S. Kobayashi, *On compact Kähler manifolds with positive definite Ricci tensor*, Ann. Math. **74** (1961), 570–574.

[Kob72] ______, *Transformation groups in differential geometry*, Springer-Verlag, Berlin, 1972.

[KR05] S. Klein and H. Reckziegel, *Families of congruent submanifolds*, Banach Center Publ. **69** (2005), 119–132.

[Lan93] S. Lang, *Algebra*, third ed., Addison-Wesley, Reading, Mass., 1993.

[Lan99] ______, *Fundamentals of differential geometry*, Springer, New York, 1999.

[LM89] H. B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton University Press, Princeton, New Jersey, 1989.

[Loo69] O. Loos, *Symmetric spaces II: Compact spaces and classification*, W. A. Benjamin Inc., New York, 1969.

[Mye35] S. B. Myers, *Riemannian manifolds in the large*, Duke Math. J. **1** (1935), 39–49.

[Nai86] H. Naitoh, *Symmetric submanifolds of compact symmetric spaces*, Tsukuba J. Math. **10** (1986), 215–242.

[Nar68] R. Narasimhan, *Analysis on real and complex manifolds*, Masson & Cie, Editeur, Paris, 1968.

[Nom54] K. Nomizu, *Invariant affine connections on homogeneous spaces*, Amer. J. Math. **76** (1954), 33–65.

[NT89] H. Naitoh and M. Takeuchi, *Symmetric submanifolds of symmetric spaces*, Sugaku Exp. **2** (1989), 157–188.

[O'N83] B. O'Neill, *Semi-riemannian geometry with applications to relativity*, Academic Press, New York, 1983.

[Poo81] W. A. Poor, *Differential geometric structures*, McGraw-Hill, New York, 1981.

[Por95] I. R. Porteous, *Clifford algebras and the classical groups*, Cambridge University Press, Cambridge, 1995.

[Raw84] J. H. Rawnsley, *f-structures, f-twistor spaces and harmonic maps*, Lecture Notes in Mathematics (Geometry Seminar "Luigi Bianchi" II) **1164** (1984), 86–159.

[Rec85] H. Reckziegel, *Horizontal lifts of isometric immersions into the bundle space of a pseudo-riemannian submersion*, Lecture Notes in Mathematics (Global Differential Geometry and Global Analysis 1984) **1156** (1985), 264–279.

[Rec95] ———, *On the geometry of the complex quadric*, Geometry and Topology of Submanifolds **VIII** (1995), 302–315.

[Rec99] ———, *Definitionen, Formeln, Sätze zu den Vorlesungen über Differentialgeometrie im WS 1997/98 – WS 1998/99*, 1999, unpublished lecture scriptum.

[Rec04] ———, *Euklidische und komplexe Clifford-Algebren und Spingruppen*, 2004, unpublished exposition.

[Smy67] B. Smyth, *Differential geometry of complex hypersurfaces*, Ann. Math. **85** (1967), 246–266.

[Tit67] J. Tits, *Tabellen zu den einfachen Lie-Gruppen und ihren Darstellungen*, Springer-Verlag, Berlin, 1967.

[TT72] R. Takagi and T. Takahashi, *On the principal curvatures of homogeneous hypersurfaces in a sphere*, Differential Geometry in honor of K. Yano (S. Kobayashi, M. Obata, and T. Takahashi, eds.), Kinokuniya, Tokyo, 1972, pp. 469–481.

[Var74] V. S. Varadarajan, *Lie groups, Lie algebras and their representations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1974.

[Wal85] R. Walter, *Lineare Algebra und analytische Geometrie*, Vieweg, Braunschweig, 1985.

[Wei58] A. Weil, *Introduction à l'étude des variétés kählériennes*, Hermann, Paris, 1958.

[Wel80] R. O. Wells, *Differential analysis on complex manifolds*, Springer-Verlag, New York, 1980.

Zusammenfassung in deutscher Sprache

Die komplexen Hyperflächen eines komplex-projektiven Raums $\mathbb{P}^n$, die (abgesehen von den projektiven Unterräumen, deren Geometrie vollständig bekannt ist) die geringste Komplexität aufweisen, sind diejenigen, die durch eine nicht-entartete quadratische Gleichung bestimmt werden, die *komplexen Quadriken*. Diese sind vom algebraischen Standpunkt alle gleichwertig. Betrachtet man den $\mathbb{P}^n$ jedoch als Riemannsche Mannigfaltigkeit (mit der Fubini-Study-Metrik), so zeigt sich, dass bestimmte komplexe Quadriken besonders gut an diese Metrik angepasst sind, insbesondere handelt es sich bei ihnen um symmetrische Untermannigfaltigkeiten des Riemann-symmetrischen Raums $\mathbb{P}^n$. Diese Quadriken zeichnen sich auch dadurch aus, dass sie (abgesehen von den projektiven Unterräumen) die einzigen komplexen Hyperflächen im $\mathbb{P}^n$ sind, die Einstein-Mannigfaltigkeiten sind (siehe SMYTH, [Smy67]). Ist im Folgenden von komplexen Quadriken die Rede, so sind stets diejenigen Quadriken gemeint, die in der beschriebenen Weise an die Metrik von $\mathbb{P}^n$ angepasst sind.

Während das algebraische Verhalten der komplexen Quadrik Q gut bekannt ist, ist über die innere und äußere Riemannsche Geometrie der komplexen Quadrik noch einiges zu sagen; die vorliegende Dissertation liefert einen Beitrag hierzu. Im Einzelnen werden die folgenden Untersuchungen durchgeführt bzw. die folgenden Hauptergebnisse erzielt:

- Die Klassifikation der totalgeodätischen Untermannigfaltigkeiten der komplexen Quadrik.
- Die Untersuchung bestimmter Kongruenz-Familien von totalgeodätischen Untermannigfaltigkeiten in Q; diese werden in einem allgemeinen Kontext mit der Struktur eines natürlich reduktiven homogenen Raums versehen, und es wird untersucht, in welchen Fällen diese Struktur von der Struktur eines symmetrischen Raums herkommt.
- Es wird gezeigt, dass sich die Menge der in einer Quadrik enthaltenen k-dimensionalen „Unterquadriken“ (diese sind alle zueinander isometrisch) aus einer Ein-Parameter-Schar von Kongruenzklassen zusammensetzt; außerdem wird die extrinsische Geometrie dieser Unterquadriken untersucht.
- Bekanntlich bestehen die folgenden Isomorphien zwischen komplexen Quadriken niederer Dimension und Gliedern anderer Serien Riemann-symmetrischer Räume:

  $$Q^1 \cong \mathbb{S}^2, \quad Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1, \quad Q^3 \cong \mathrm{Sp}(2)/\mathrm{U}(2), \quad Q^4 \cong G_2(\mathbb{C}^4) \quad \text{und} \quad Q^6 \cong \mathrm{SO}(8)/\mathrm{U}(4)\,.$$

  Hierzu werden auf einem recht geometrischen Weg Isomorphismen explizit konstruiert.

Im Folgenden schildere ich mein Vorgehen zur Erzielung dieser Ergebnisse, und diskutiere diese genauer.

Für das Studium der Geometrie einer Riemannschen Mannigfaltigkeit spielt ihr Krümmungstensor eine wesentliche Rolle. Dies zeigt sich beispielsweise daran, dass zumindest falls der Krümmungstensor parallel ist, er schon alle Informationen über die lokale Struktur der betreffenden Riemannschen Mannigfaltigkeit enthält (wie die lokale Version des Theorems von Cartan/Ambrose/Hicks zeigt). Ein weiterer Grund liegt darin, dass die Tangentialräume der Mannigfaltigkeit durch den Krümmungstensor mit einer zusätzlichen Struktur versehen werden, die insbesondere für die Untermannigfaltigkeitsgeometrie der Mannigfaltigkeit von Bedeutung ist. Daher ist die algebraischen Struktur des Krümmungstensors für das Verständnis der Geometrie der Mannigfaltigkeit von großem Interesse.

In der Arbeit [Rec95] von Prof. H. Reckziegel, die den Ausgangspunkt für die Dissertation bildete, wird dieser Gedanke für die komplexe Quadrik durchgeführt. Die Kapitel 1–3 der Dissertation (mit Ausnahme von Abschnitt 3.4) stellen eine erweiterte, ausführliche Ausarbeitung der Arbeit von Reckziegel dar.

Der folgende in [Rec95] eingeführte Begriff spielt für die gesamte Dissertation eine entscheidende Rolle: Ist $\mathbb{V}$ ein unitärer Vektorraum und A eine Konjugation[46] auf $\mathbb{V}$, so nennen wir, [Rec95] folgend, den „Kreis von Konjugationen“ $\mathfrak{A} := \{\, \lambda\, A \,|\, \lambda \in \mathbb{S}^1 \,\}$ eine $\mathbb{C}$Q-Struktur und das Paar $(\mathbb{V}, \mathfrak{A})$ einen $\mathbb{C}$Q-Raum.

Die große Bedeutung des Begriffs der $\mathbb{C}$Q-Struktur für die Untersuchung komplexer Quadriken hat zwei Ursachen: Die eine ist, dass die Menge der $\mathbb{C}$Q-Strukturen auf einem unitären Vektorraum $\mathbb{V}$ in eineindeutiger Beziehung zu der Menge der (im oben erläuterten Sinne) an die Metrik von $\mathbb{P}(\mathbb{V})$ angepassten komplexen Quadriken in $\mathbb{P}(\mathbb{V})$ steht.

Die zweite, noch wesentlichere Ursache für die Bedeutung von $\mathbb{C}$Q-Strukturen für die Untersuchung der komplexen Quadrik ergibt sich aus dem folgenden Ergebnis, das schon in [Rec95] zentral ist: Ist $Q \subset \mathbb{P}(\mathbb{V})$ eine komplexe Quadrik und bezeichnen wir für $p \in Q$ mit $\perp^1_p Q$ die Menge der Einheitsnormalenvektoren an Q in p, und für $\eta \in \perp^1_p Q$ mit A_η den Formoperator von Q bezüglich η, so ist die Menge $\mathfrak{A}(Q,p) := \{\, A_\eta \,|\, \eta \in \perp^1_p Q \,\}$ eine $\mathbb{C}$Q-Struktur auf dem Tangentialraum T_pQ. Weil es aufgrund der Gaußschen Ableitungsgleichung zweiter Ordnung möglich ist, den Krümmungstensor von Q in p mit Hilfe dieser $\mathbb{C}$Q-Struktur $\mathfrak{A}(Q,p)$ (sowie der Riemannschen Metrik und der komplexen Struktur von Q) auszudrücken, werden durch die $\mathbb{C}$Q-Räume $(T_pQ, \mathfrak{A}(Q,p))_{p\in Q}$ die lokalen Informationen über die komplexe Quadrik in Gänze widergegeben. In diesem Sinne erscheint es sinnvoll, die Riemannsche Metrik von Q, die komplexe Struktur von Q, und die durch den Formoperator induzierte Familie $(\mathfrak{A}(Q,p))_{p\in Q}$ von $\mathbb{C}$Q-Strukturen als die „fundamentalen geometrischen Objekte“ der komplexen Quadrik Q anzusehen; die Dissertation ist von dieser Sichtweise geprägt.

[46] Sei $\mathbb{V}$ ein unitärer Raum, dessen komplexe Struktur wir mit $J : \mathbb{V} \to \mathbb{V},\ v \mapsto i \cdot v$ und dessen komplexes Skalarprodukt wir mit $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ bezeichnen. Dann heißt eine $\mathbb{R}$-lineare Abbildung $A : \mathbb{V} \to \mathbb{V}$ eine *Konjugation* auf $\mathbb{V}$, wenn sie bezüglich des reellen Skalarprodukts $\mathrm{Re}(\langle\cdot,\cdot\rangle_{\mathbb{C}})$ selbstadjungiert und orthogonal ist, und außerdem $A \circ J = -J \circ A$ gilt.

Man beachte, dass zwei $\mathbb{C}$Q-Räume gleicher Dimension zueinander isomorph sind. Aus diesem Grunde kann man viele Informationen über die beiden beschriebenen Situationen schon durch das abstrakte Studium von $\mathbb{C}$Q-Räumen erhalten. Dies geschieht in Kapitel 2 der Dissertation. Zwei der dort hergeleiteten Tatsachen sind für die weitere Arbeit mit $\mathbb{C}$Q-Räumen von besonders großer Bedeutung:

(1) Die Gruppe $\mathrm{Aut}(\mathfrak{A})$ der $\mathbb{C}$Q-Automorphismen von $(\mathbb{V}, \mathfrak{A})$ (d.h. derjenigen unitären Transformationen $B : \mathbb{V} \to \mathbb{V}$, für die $B \circ A \circ B^{-1} \in \mathfrak{A}$ für alle $A \in \mathfrak{A}$ gilt) operiert nicht transitiv auf der Einheitssphäre $\mathbb{S}(\mathbb{V})$ (und somit sind in einem $\mathbb{C}$Q-Raum, anders als in einem unitären Raum, nicht alle Einheitsvektoren „gleichwertig"), und zwar gibt es eine surjektive, stetige Funktion $\varphi_{\mathfrak{A}} : \mathbb{S}(\mathbb{V}) \to [0, \frac{\pi}{4}]$, die auf $\varphi_{\mathfrak{A}}^{-1}(]0, \frac{\pi}{4}[)$ submersiv ist, so dass die Orbits der Operation von $\mathrm{Aut}(\mathfrak{A})$ auf $\mathbb{S}(\mathbb{V})$ gerade die Niveauflächen von $\varphi_{\mathfrak{A}}$ sind. Dieser Tatbestand ist schon in [Rec95] zu finden; neu ist jedoch die einfache Beschreibung von $\varphi_{\mathfrak{A}}$ durch die Gleichung $2\cos(\varphi_{\mathfrak{A}}(v)) = |\langle v, Av\rangle_{\mathbb{C}}|$ mit einem beliebigen $A \in \mathfrak{A}$ (siehe Theorem 2.28(a)).

(2) Wie oben schon gesagt wurde, läßt sich der Krümmungstensor einer komplexen Quadrik Q in $p \in Q$ allein durch die Größen des $\mathbb{C}$Q-Raums $(T_pQ, \mathfrak{A}(Q,p))$ beschreiben. Aus diesem Grunde läßt sich ein diesem Krümmungstensor entsprechender Tensor auf einem beliebigen $\mathbb{C}$Q-Raum $(\mathbb{V}, \mathfrak{A})$ einführen, wir nennen ihn den Krümmungstensor R des $\mathbb{C}$Q-Raums. Es werden die (schon in [Rec95] zu findenden) Eigenwerte und -räume des Jacobi-Operators $R(\,\cdot\,, w)w : \mathbb{V} \to \mathbb{V}$ (Abschnitt 2.7) sowie die bezüglich R flachen Unterräume von $\mathbb{V}$ (Abschnitt 2.8) angegeben. Diese Informationen sind für das Folgende von entscheidendem Nutzen.

Die Erkenntnisse über $\mathbb{C}$Q-Räume werden in Kapitel 3 auf komplexe Quadriken angewandt. Abschnitt 3.1 zeigt, auf welche Weise $\mathbb{C}$Q-(Anti-)Automorphismen eines $\mathbb{C}$Q-Raums $(\mathbb{V}, \mathfrak{A})$ (anti-)holomorphe Isometrien der durch die $\mathbb{C}$Q-Struktur $\mathfrak{A}$ bestimmten komplexen Quadrik $Q(\mathfrak{A}) \subset \mathbb{P}(\mathbb{V})$ induzieren. Der grundsätzliche Tatbestand, der im Wesentlichen schon in [Rec95] zu finden ist, wird hier ergänzt durch eine Beschreibung der „Beweglichkeit" von Basen in T_pQ in der Sprache der $\mathbb{C}$Q-Theorie (Theorem 3.5). Daraus folgt auch die wohlbekannte Tatsache, dass eine m-dimensionale komplexe Quadrik Q ein zu $\mathrm{SO}(m+2)/(\mathrm{SO}(2) \times \mathrm{SO}(m))$ isomorpher Hermitesch-symmetrischer Raum ist; die durch die symmetrische Struktur induzierte Spaltung $\mathfrak{o}(m+2) = \mathfrak{k} \oplus \mathfrak{m}$ wird explizit beschrieben. Die Informationen aus den Abschnitten 2.7 und 2.8 über den Krümmungstensor kann man nun als Beschreibung der Cartan-Unteralgebren, der Wurzeln und der Wurzelräume des symmetrischen Raums Q deuten; diese Sichtweise wird hier bedeutend stärker als in [Rec95] genutzt. Während die Struktur des Wurzelsystems von Q natürlich wohlbekannt ist, ist die hier vorliegende explizite Beschreibung der Cartan-Unteralgebren und der Wurzelräume, die allein die Größen des $\mathbb{C}$Q-Raums $(T_pQ, \mathfrak{A}(Q,p))$ verwendet (und die insbesondere ohne „künstliche" Koordinaten auskommt) an anderer Stelle nicht zu finden, und für die folgenden Untersuchungen wesentlich.

Die bisher beschriebenen Ergebnisse bilden das Fundament der vorliegenden Untersuchung der Geometrie komplexer Quadriken.

Als erste Anwendung werden in Abschnitt 3.3 die Isometrien der komplexen Quadrik Q klassifiziert. Das wesentliche Ergebnis, dass nämlich (a) jede (anti-)holomorphe Isometrie $Q \to Q$

von einem $\mathbb{C}$Q-(Anti-)Automorphismus herrührt, und dass (b) für $\dim Q \neq 2$ jede Isometrie $f : Q \to Q$ entweder holomorph oder anti-holomorph ist (Theorem 3.23), ist zwar schon in [Rec95] zu finden; mir ist jedoch ein wesentlich kürzerer Beweis möglich, bei dem ich ausnutze, dass für jede Isometrie $f : Q \to Q$ und jedes $p \in Q$ gilt: $\varphi_{\mathfrak{A}(Q,f(p))} \circ (f_*|\mathbb{S}(T_pQ)) = \varphi_{\mathfrak{A}(Q,p)}$ (wie sich aus der Äquivarianz des Krümmungsoperators unter f_* ergibt).

Inhalt der Kapitel 4 und 5 ist die Klassifikation der totalgeodätischen Untermannigfaltigkeiten der komplexen Quadrik Q.

Schon Chen und Nagano haben sich in ihren Arbeiten [CN77] und [CN78] mit der Klassifikation totalgeodätischer Untermannigfaltigkeiten in symmetrischen Räumen befasst. Die Arbeit [CN77] gibt eine Klassifikation der totalgeodätischen Untermannigfaltigkeiten komplexer Quadriken mit Hilfe von „ad-hoc-Methoden“ an. Allerdings enthält diese mehrere Lücken, die dazu führen, dass zwei Typen von totalgeodätischen Untermannigfaltigkeiten übersehen werden. Auch sind die in [CN77] benutzten Argumente nicht immer stichhaltig. — War [CN77] noch ausschließlich mit der Untersuchung der komplexen Quadrik befasst, so ist die in der Anschlußarbeit [CN78] eingeführte (M_+, M_-)-Methode ein Hilfsmittel zur Bestimmung totalgeodätischer Untermannigfaltigkeiten in allgemeinen symmetrischen Räumen von kompaktem Typ. Jedoch handelt es sich nur um ein notwendiges Kriterium für die Existenz einer totalgeodätischen Einbettung von einem symmetrischen Raum in einen anderen. Man erhält durch die (M_+, M_-)-Methode also weder Beweise für die Existenz totalgeodätischer Untermannigfaltigkeiten in einem symmetrischen Raum, noch Informationen über deren Lage. Deshalb ergeben die zitierten Arbeiten keine zufriedenstellende Untersuchung der totalgeodätischen Untermannigfaltigkeiten der komplexen Quadrik, und auch sonst ist mir eine solche Untersuchung nicht bekannt.

Für die detailliertere Diskussion der Arbeiten [CN77] und [CN78], sowie der älteren Arbeit [CL75] von Chen und Lue, in der die reell-2-dimensionalen totalgeodätischen Untermannigfaltigkeiten von Q untersucht werden, verweise ich auf Bemerkung 4.13.

Bei der von mir durchgeführten Klassifikation der totalgeodätischen Untermannigfaltigkeiten von Q verwende ich weder die in [CN77] benutzten Mittel noch die (M_+, M_-)-Methode. Stattdessen gehe ich wie folgt vor: Bekanntlich sind die zusammenhängenden, vollständigen, totalgeodätischen Untermannigfaltigkeiten des symmetrischen Raums Q genau dessen symmetrische Unterräume, und die durch einen Punkt $p \in Q$ verlaufenden symmetrischen Unterräume stehen in bijektiver Beziehung zu den krümmungsinvarianten Unterräumen des Tangentialraums T_pQ. Das Problem der Klassifikation der totalgeodätischen Untermannigfaltigkeiten von Q zerfällt also in zwei Teile: (1) Die Klassifikation der krümmungsinvarianten Unterräume von T_pQ und (2) Die Beschreibung des globalen Isometrietyps und der Lage in Q der zu den im ersten Teil gefundenen krümmungsinvarianten Unterräumen gehörenden totalgeodätischen Untermannigfaltigkeiten.

Die Lösung des ersten Teilproblems beruht auf der Verbindung der allgemeinen Wurzelraumtheorie symmetrischer Räume mit den durch die Theorie der $\mathbb{C}$Q-Räume erhaltenen und beschriebenen konkreten Resultaten für die komplexe Quadrik. Zunächst leite ich in Abschnitt 4.2 für einen allgemeinen symmetrischen Raum M von kompaktem Typ Beziehungen zwischen den

Wurzeln bzw. Wurzelräumen von M und den Wurzeln bzw. Wurzelräumen seiner symmetrischen Unterräume her. Dank der expliziten Darstellung der Wurzeln und Wurzelräume von Q in Abschnitt 3.2 erhält man durch die Anwendung der Beziehungen auf $M = Q$ Bedingungen für die mögliche Lage von krümmungsinvarianten Unterräumen in T_pQ, welche eine Klassifikation dieser Unterräume ermöglichen; dies ist in den Abschnitten 4.3 und 4.4 ausgeführt. Der Klassifikationsbeweis wird durch Symmetrieeigenschaften der Wurzelsysteme vereinfacht und strukturiert; zur Nutzung dieser Symmetrieeigenschaften bin ich durch einen Hinweis von Prof. J.-H. Eschenburg (Augsburg) angeregt worden.

Das zweite Teilproblem wird in Kapitel 5 angegangen: Hier werden für die zuvor gefundenen krümmungsinvarianten Unterräume U von T_pQ (mit Ausnahme eines bestimmten Kongruenztyps von 2-dimensionalen Unterräumen) totalgeodätische, injektive isometrische Immersionen in Q angegeben, deren Bild jeweils tangential zu U verläuft. Damit ist die Klassifikation der totalgeodätischen Untermannigfaltigkeiten der komplexen Quadrik abgeschlossen.

Unter den totalgeodätischen Untermannigfaltigkeiten einer m-dimensionalen komplexen Quadrik $Q \subset \mathbb{P}(\mathbb{V})$ verdienen gewisse Typen besondere Erwähnung (eine vollständige Liste ist in Theorem 5.1 zu finden): (1) Für jedes $k < m$ gibt es totalgeodätische Untermannigfaltigkeiten Q' von Q, die isometrisch zu einer k-dimensionalen komplexen Quadrik sind. Diese sind „Unterquadriken“ von Q, das soll heißen: Es existiert jeweils ein komplex-$(k+1)$-dimensionaler projektiver Unterraum $\Lambda \subset \mathbb{P}(\mathbb{V})$, so dass Q' eine komplexe Quadrik in Λ im bisherigen Sinne ist. (2) Für jedes $k \leq \frac{m}{2}$ gibt es komplex-k-dimensionale projektive Unterräume von $\mathbb{P}(\mathbb{V})$, die ganz in Q enthalten und daher totalgeodätische Untermannigfaltigkeiten von Q sind. (3) Ist $m \geq 3$, so gibt es in Q totalgeodätische Untermannigfaltigkeiten, die isometrisch zu einer 2-Sphäre vom Radius $\frac{1}{2}\sqrt{10}$ sind; diese Untermannigfaltigkeiten sind weder komplex noch total-reell. Ihr Durchmesser $\frac{\pi}{2}\sqrt{10}$ ist größer als der Durchmesser $\frac{\pi}{\sqrt{2}}$ der Quadrik Q.

Es stellt sich die Frage, ob es neben den in (1) genannten, totalgeodätischen k-dimensionalen Unterquadriken von Q noch weitere (nicht totalgeodätische) gibt. Wie ich in Kapitel 6 zeige, ist diese Frage für $k \leq \frac{m}{2} - 1$ positiv zu beantworten. Für diese k gibt es unendlich viele Kongruenzklassen von k-dimensionalen Unterquadriken von Q, die Menge dieser Kongruenzklassen wird durch einen „Winkel“ $t \in [0, \frac{\pi}{4}]$ parametrisiert (der in enger Beziehung zu der Funktion $\varphi_{\mathfrak{A}} : \mathbb{S}(\mathbb{V}) \to [0, \frac{\pi}{4}]$ steht), und eine Unterquadrik Q' ist genau dann eine totalgeodätische Untermannigfaltigkeit von Q, wenn sie zur Kongruenzklasse mit $t = 0$ gehört. Ich zeige auch, dass die zweite Fundamentalform der Inklusion $Q' \hookrightarrow Q$ genau dann parallel ist, wenn Q' entweder zur Kongruenzklasse mit $t = 0$ oder zur Kongruenzklasse mit $t = \frac{\pi}{4}$ gehört. Die Elemente der letzteren Kongruenzklasse sind genau diejenigen Unterquadriken von Q, deren umgebender projektiver Unterraum $\Lambda \subset \mathbb{P}(\mathbb{V})$ ganz in Q enthalten ist.

Ist für $t \in [0, \frac{\pi}{4}]$ Q'_t eine Unterquadrik von Q, die zur Kongruenzklasse mit dem Parameter t gehört, so ist die gesamte Kongruenzklasse von Unterquadriken zu diesem Parameter definitionsgemäß durch $\{\, f(Q'_t) \,|\, f \in I(Q) \,\}$ gegeben, wobei $I(Q)$ die Isometriegruppe von Q bezeichnet. In der allgemeinen Situation, wo M ein beliebiger Riemann-symmetrischer Raum und N_0 eine Untermannigfaltigkeit von M ist, nenne ich die Menge $\mathfrak{F}(N_0, M) := \{\, f(N_0) \,|\, f \in I(M) \,\}$ die von N_0 induzierte „Familie von kongruenten Untermannigfaltigkeiten“ oder „Kongruenzfamilie“. Die in Kapitel 7 durchgeführte Untersuchung solcher Kongruenzfamilien stellte ich an, nachdem mich Prof. M. Rapoport (Bonn) auf die Untersuchung der in einer m-dimensionalen komplexen Quadrik enthaltenen projektiven Unterräume in [GH78], S. 735f hinwies, dort werden jedoch keine metrischen Gesichtspunkte berücksichtigt. Die Ergebnisse sind mittlerweile als [KR05] veröffentlicht worden.

In Abschnitt 7.1 wird zunächst in einer allgemeinen Situation gezeigt, wie man eine Kongruenzfamilie mit der Struktur einer Riemannschen Mannigfaltigkeit versehen kann, und dass sie dadurch zu einem natürlich reduktiven Riemannsch homogenen Raum wird. Anschließend untersuche ich spezielle Beispiele von Kongruenzfamilien. Zum einen (in Abschnitt 7.2) zwei Beispiele im komplex-projektiven Raum $\mathbb{P}(\mathbb{V})$: die von einem projektiven Unterraum erzeugte und die von einer k-dimensionalen komplexen Quadrik erzeugte Kongruenzfamilie; zum anderen (in Abschnitt 7.3) zwei Beispiele in einer komplexen Quadrik $Q \subset \mathbb{P}(\mathbb{V})$: die von einer totalgeodätischen Unterquadrik von Q erzeugte und die von einem in Q enthaltenen projektiven Unterraum der Dimension $\leq \frac{m}{2}$ erzeugte Kongruenzfamilie. (Die zuletzt genannte Kongruenzfamilie ist die in [GH78] behandelte.) Es zeigt sich, dass für gewisse, aber nicht alle der betrachteten Beispiele die reduktive Struktur der Kongruenzfamilie von einer symmetrischen Struktur erzeugt wird. Beispielsweise gilt für die von einem k-dimensionalen, in der m-dimensionalen Quadrik Q enthaltenen projektiven Unterraum erzeugte Kongruenzfamilie $\mathfrak{F}(\mathbb{P}^k, Q)$ (siehe Theorem 7.11): Ist $2k = m$, so besitzt $\mathfrak{F}(\mathbb{P}^k, Q)$ genau zwei Zusammenhangskomponenten und diese lassen sich derart mit der Struktur eines zu $\mathrm{SO}(m+2)/\mathrm{U}(k+1)$ isomorphen Hermitesch-symmetrischen Räumen versehen, dass die symmetrische Struktur die ursprüngliche natürlich reduktive Struktur erzeugt. Ist hingegen $2k < m$, so ist $\mathfrak{F}(\mathbb{P}^k, Q)$ zusammenhängend, und die natürlich reduktive Struktur von $\mathfrak{F}(\mathbb{P}^k, Q)$ wird nicht von einer symmetrischen Struktur erzeugt.

Wie zuerst von E. Cartan bemerkt wurde und wohlbekannt ist, sind die komplexen Quadriken Q^m von Dimension $m \in \{1, 2, 3, 4, 6\}$ (und keine weiteren) als Riemann-symmetrische Räume isomorph zu Mitgliedern anderer Reihen von Riemann-symmetrischen Räumen (siehe auch [Hel78], S. 519f.). An den Dynkin-Diagrammen der irreduziblen symmetrischen Räume kann man ablesen (siehe [Loo69], Theorem VII.3.9(a), S. 145 und Tabelle 4 auf S. 119), dass die folgenden Isomorphien gelten:

$$Q^1 \cong \mathbb{S}^2, \quad Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1, \quad Q^3 \cong \mathrm{Sp}(2)/\mathrm{U}(2), \quad Q^4 \cong G_2(\mathbb{C}^4) \quad \text{und} \quad Q^6 \cong \mathrm{SO}(8)/\mathrm{U}(4)\ .$$

(Dass es sich nicht nur um lokale Isomorphien handelt, ergibt sich daraus, dass alle genannten Räume einfach zusammenhängend sind.) Diese Betrachtung liefert jedoch kein Verfahren zur Konstruktion von Isomorphismen zwischen den jeweiligen Räumen. Es gelingt aber in der Arbeit (Abschnitt 3.4 und Kapitel 8), auf recht geometrische Weise Konstruktionen der Isomorphismen anzugeben: Die Segre-Einbettung führt zu einem Isomorphismus zwischen Q^2 und $\mathbb{P}^1 \times \mathbb{P}^1$; insbesondere ist Q^2 (im Unterschied zu den komplexen Quadriken anderer Dimension) reduzibel. — Die Plücker-Einbettung führt zu einem Isomorphismus zwischen der komplexen Graßmann-Mannigfaltigkeit $G_2(\mathbb{C}^4)$ und einer 4-dimensionalen komplexen Quadrik $Q(*) \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^4)$; hierbei wird die Quadrik $Q(*)$ durch den Hodge-Operator $* : \bigwedge^2 \mathbb{C}^4 \to \bigwedge^2 \mathbb{C}^4$ beschrieben. — Schränkt man den genannten Isomorphismus $G_2(\mathbb{C}^4) \to Q(*)$ auf einen geeigneten, totalgeodätischen $\mathrm{Sp}(2)$-Orbit in $G_2(\mathbb{C}^4)$ ein, so erhält man einen Isomorphismus zwischen dem

Hermitesch-symmetrischen Raum $\mathrm{Sp}(2)/\mathrm{U}(2)$ und einer 3-dimensionalen, totalgeodätischen Unterquadrik von $Q(*)$. — Mit Hilfe der Theorie der Spingruppen, ihrer Darstellungen und des Prinzips der Trialität läßt sich zeigen, dass Q^6 isomorph zum Hermitesch-symmetrischen Raum $\mathrm{SO}(8)/\mathrm{U}(4)$ ist. Der letztere Raum besitzt mehrere geometrische Realisierungen. Beispielsweise ist er isomorph zu den Zusammenhangskomponenten der Kongruenzfamilie $\mathfrak{F}(\mathbb{P}^3, Q^6)$ der 3-dimensionalen projektiven Unterräume, die in Q^6 enthalten sind; diese Tatsache wird auch bei der Konstruktion des Isomorphismus $Q^6 \to \mathrm{SO}(8)/\mathrm{U}(4)$ ausgenutzt. Eine andere geometrische Realisierung von $\mathrm{SO}(8)/\mathrm{U}(4)$ ist der Raum der orthogonalen komplexen Strukturen auf $\mathbb{R}^8$ mit fester Orientierung; mit Hilfe dieser Realisierung kann der Isomorphismus zwischen $\mathrm{SO}(8)/\mathrm{U}(4)$ und den Zusammenhangskomponenten von $\mathfrak{F}(\mathbb{P}^3, Q^6)$ konstruiert werden.

Es soll gesagt werden, dass wir auf die Existenz der Isomorphie $Q^4 \cong G_2(\mathbb{C}^4)$ erstmals durch Prof. M. Guest (Metropolitan University of Tokyo) aufmerksam gemacht wurden. Die Einsichten, die sich bei der Konstruktion dieser Isomorphie ergeben haben, waren auch für das allgemeine Verständnis komplexer Quadriken sehr fruchtbar.

Die Anhänge enthalten überwiegend reproduktive Darstellungen zu bestimmten Themen, soweit sie für die vorliegende Arbeit von Bedeutung sind. Die zugrundeliegenden Quellen sind im Folgenden und in der Einleitung des jeweiligen Anhangs, ggfs. auch bei einzelnen Sätzen und Beweisen angegeben.

In Anhang A werden die für die vorliegende Arbeit relevanten Aspekte der Theorie symmetrischer Räume dargestellt. Bei der in den Abschnitten A.1, A.2 und A.3 dargelegten Betrachtungsweise symmetrischer Räume habe ich von einer unveröffentlichten Ausarbeitung von Prof. H. Reckziegel profitiert; bei der in Abschnitt A.4 dargestellten Wurzelraumtheorie für symmetrische Räume war mir das Skriptum einer Vorlesung von Prof. G. Thorbergsson von Nutzen.

Der Gegenstand von Anhang B ist die Theorie der Clifford-Algebren, Spingruppen, ihrer Darstellungen, und des Prinzips der Trialität. Sie spielt bei der Konstruktion der Isomorphie zwischen Q^6 und den Zusammenhangskomponenten von $\mathfrak{F}(\mathbb{P}^3, Q^6)$ eine wesentliche Rolle. Hier sind als Quellen das Buch [LM89] von Lawson/Michelsohn (für Clifford-Algebren, Spingruppen und ihre Darstellungen) und das Buch [Che54] von Chevalley (für das Prinzip der Trialität) zu nennen. Außerdem waren mir die Diskussionen mit Prof. H. Reckziegel zu diesen Themen, aus denen auch die Ausarbeitung [Rec04] entstanden ist, hilfreich.

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit — einschließlich Tabellen, Karten und Abbildungen — die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie — abgesehen von unten angegebenen Teilpublikationen — noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde.

Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Professor Dr. H. Reckziegel betreut worden.

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